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Left Ideals and $\mathcal{L}$-classes in the Finite Direct Product of Semigroups

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Abstract. Let S_i be a semigroup for all $i \in \{1, 2, \dots, n\}$. Then the Cartesian product of $S_1, S_2, \dots, S_n$ becomes a semigroup under componentwise multiplication. Let $(s_1, s_2, \dots, s_n) \in S_1 \times S_2 \cdots \times S_n$. In this paper, we give necessary and sufficient condition when the Cartesian product of principal left ideals $L(s_1) \times L(s_2) \times \cdots \times L(s_n)$ is the principal left ideal $L((s_1, s_2, \dots, s_n))$ and the Cartesian product of $\mathcal{L}$-classes $L_{s_1} \times L_{s_2} \times \cdots \times L_{s_n}$ is an $\mathcal{L}$-class $L_{(s_1, s_2, \dots, s_n)}$ in a semigroup $S_1 \times S_2 \times \cdots \times S_n$.

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1. Introduction

Let S and T be semigroups. The Cartesian product $S \times T$ becomes a semigroup under a binary operation on $S \times T$ defined by

$$(s, t)(s', t') = (ss', tt')$$

for all $(s, t), (s', t') \in S \times T$. This semigroup is referred to as the direct product of S and T. A nonempty subset A of S is a *left ideal* of S if $SA \subseteq A$. For any $a \in S$, the *principal left ideal* of S generated by a, denoted by $L(a)$, is the smallest left ideal of S containing a. It is well-known that

$$L(a) = a \cup Sa.$$

A relation $\mathcal{L}$ on S is then defined by the rule that $a\mathcal{L}b$ if and only if $L(a) = L(b)$, i.e., if and only if $a \cup Sa = b \cup Sb$. It is one of Green's equivalence relations. Then the $\mathcal{L}$-class of S containing element a will be written by L_a. In [1], Fabrici considered a principal left ideal and a relation $\mathcal{L}$ on the direct product of two semigroups $S \times T$. Let $(s, t) \in S \times T$.

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Necessary and sufficient condition when $L(s) \times L(t) = L((s,t))$ were provided. Moreover, the author showed necessary and sufficient condition when $L_{(s,t)} = L_s \times L_t$ in $S \times T$. The principal (two-sided) ideals on the direct product of two semigroups were considered in the same way [2].

Let m, n be nonnegative integers. A subsemigroup A of S is called an (m,n)-*ideal* of S if $A^m S A^n \subseteq A$ [3]. Here, $A^0 S = S A^0 = S$. This definition is a generalized form of left ideals, right ideals, and bi-ideals. For any element a in S, the smallest (m,n)-ideal of S containing a is denoted by $[a]_{(m,n)}$. Luangchaisri and Changphas [4] provided necessary and sufficient condition for $[s]_{(m,n)} \times [t]_{(m,n)} = [(s,t)]_{(m,n)}$. Moreover, they determined an equivalence class on a semigroup S by for any $x \in S$, $J_{(m,n),x} = \{y \in S \mid [x]_{(m,n)} = [y]_{(m,n)}\}$. Then they provided the conditions for $J_{(m,n),a} \times J_{(m,n),b} = J_{(m,n),(a,b)}$.

A nonempty subset Q of S is called a *quasi-ideal* of S if $QS \cap SQ \subseteq Q$. The concept of quasi-ideals was introduced by Steinfeld [5]. For each $a \in S$, the principal quasi-ideal of S generated by a is denoted by $Q(a)$. Luangchaisri et al. [6] considered necessary and sufficient condition when $Q(s) \times Q(t) = Q((s,t))$. Moreover, they characterized when the Cartesian product of $\mathcal{H}$-classes $H_s \times H_t$ is an $\mathcal{H}$-class of $S \times T$.

According to the above examples, we can observe the research line to study various kinds of ideals and equivalence relations on the direct product of two semigroups. In this paper, we consider these concepts and extend to the finite direct product of semigroups. The principal left ideals and $\mathcal{L}$-classes are investigated. Moreover, we give an example to show that the Cartesian product of principal left ideals need not be the principal left ideal. In addition, an example for $\mathcal{L}$-classes is also provided.

2. Main Results

Let $\{S_i \mid i \in I\}$ be a family of semigroups indexed by the set $I = \{1, 2, \ldots, n\}$. Then $S_1 \times S_2 \times \cdots \times S_n$ becomes a semigroup under a componentwise multiplication, which is defined by

$$(s_1, s_2, \ldots, s_n)(s_1', s_2', \ldots, s_n') = (s_1 s_1', s_2 s_2', \ldots, s_n s_n')$$

for all $(s_1, s_2, \ldots, s_n), (s_1', s_2', \ldots, s_n') \in S_1 \times S_2 \times \cdots \times S_n$. This semigroup is called the direct product of $\{S_i \mid i \in I\}$. Note that the direct product of S_1 is trivially the semigroup S_1. Therefore, we assume throughout that the indexed set I is not a singleton.

If L_i is a left ideal of S_i for all $i \in I$, then the Cartesian product $L_1 \times L_2 \times \cdots \times L_n$ is a left ideal of $S_1 \times S_2 \times \cdots \times S_n$. However, the Cartesian product of principal left ideals need not be the principal left ideal. This is clarified by the following example:

Example 1. *Let $S = \{s_1, s_2, s_3, s_4\}$ be a semigroup under the following binary operation:*

$*$	s_1	s_2	s_3	s_4
s_1	s_1	s_1	s_1	s_1
s_2	s_1	s_2	s_2	s_4
s_3	s_1	s_2	s_2	s_4
s_4	s_1	s_4	s_4	s_2

This semigroup is applied from [7]. Since $(s_3, s_1) \in L(s_3) \times L(s_3)$ *and* $(s_3, s_1) \notin L((s_3, s_3))$*, this shows that* $L(s_3) \times L(s_3) \neq L((s_3, s_3))$*. In addition, since* $(s_3, s_3) \in L(s_3) \times L(s_3)$ *and* $(s_3, s_3) \notin L((x, y))$ *for all* $(x, y) \in S \times S \setminus \{(s_3, s_3)\}$*, it follows that* $L(s_3) \times L(s_3) \neq L((x, y)$ *for all* $(x, y) \in S \times S$*. Hence,* $L(s_3) \times L(s_3)$ *is not a principal left ideal of a semigroup* $S \times S$*.*

We begin with Lemma 1 to mention about the inclusion of $L((s_1, s_2, \ldots, s_n))$ and $L(s_1) \times L(s_2) \times \cdots \times L(s_n)$. Then, in Theorem 1, we give a necessary and sufficient condition when $L((s_1, s_2, \ldots, s_n)) = L(s_1) \times L(s_2) \times \cdots \times L(s_n)$.

Lemma 1. *Let* S_i *be a semigroup and let* $s_i \in S_i$ *where* $i \in \{1, 2, \ldots n\}$*. Then*

$$L((s_1, s_2, \ldots, s_n)) \subseteq L(s_1) \times L(s_2) \times \cdots \times L(s_n)$$

Proof. Let $x \in L((s_1, s_2, \ldots, s_n))$. Then

$$\begin{aligned} x &\in (s_1, s_2, \ldots, s_n) \cup (S_1 s_1 \times S_2 s_2 \times \cdots \times S_n s_n) \\ &\subseteq (s_1 \cup S_1 s_1) \times (s_2 \cup S_2 s_2) \times \cdots \times (s_n \cup S_n s_n) \\ &= L(s_1) \times L(s_2) \times \cdots \times L(s_n). \end{aligned}$$

Thus, $L((s_1, s_2, \ldots, s_n)) \subseteq L(s_1) \times L(s_2) \times \cdots \times L(s_n)$.

Theorem 1. *Let* S_i *be a semigroup and let* $s_i \in S_i$*, where* $i \in \{1, 2, \ldots, n\}$*. Then*

$$L((s_1, s_2, \ldots, s_n)) = L(s_1) \times L(s_2) \times \cdots \times L(s_n)$$

if and only if at least one of the following conditions is satisfied:

(i) $s_i \in S_i s_i$ *for all* $i \in \{1, 2, \ldots, n\}$*;*

(ii) *there exists* $i \in \{1, 2, \ldots, n\}$ *such that* $S_j s_j = \{s_j\}$ *for all* $j \in \{1, 2, \ldots, n\} \setminus \{i\}$*.*

Proof. Suppose (i) and (ii) do not hold. Then there exists $i \in \{1, 2, \ldots, n\}$ such that $s_i \notin S_i s_i$. This implies that $S_i s_i \neq \{s_i\}$. Since (ii) does not hold, there exists $j \in \{1, 2, \ldots, n\}$ such that $j \neq i$ and $S_j s_j \neq \{s_j\}$. Without loss of generality, we let

$$s = (s_1, s_2, \ldots, s_i, \ldots, s'_j, \ldots, s_n)$$

where $s'_j \in S_j s_j \setminus \{s_j\}$. Then $s \in L(s_1) \times L(s_2) \times \cdots \times L(s_n)$. Since $s \notin S_1 s_1 \times S_2 s_2 \times \cdots \times S_n s_n$ and $s \neq (s_1, s_2, \ldots, s_n)$, we have $s \notin L((s_1, s_2, \ldots, s_n))$. Thus, $L((s_1, s_2, \ldots, s_n)) \neq L(s_1) \times L(s_2) \times \cdots \times L(s_n)$.

Conversely, assume that (i) or (ii) holds. If (i) holds, then we have

$$\begin{aligned} L(s_1) \times L(s_2) \times \cdots \times L(s_n) &= S_1 s_1 \times S_2 s_2 \times \cdots \times S_n s_n \\ &\subseteq L((s_1, s_2, \ldots, s_n)) \\ &\subseteq L(s_1) \times L(s_2) \times \cdots \times L(s_n). \end{aligned}$$

Thus, $L(s_1) \times L(s_2) \times \cdots \times L(s_n) = L((s_1, s_2, \ldots, s_n))$. Meanwhile, if (ii) holds, let $i \in \{1, 2, \ldots, n\}$ be the index such that $S_j s_j = \{s_j\}$ for all $j \in \{1, 2, \ldots, n\} \setminus \{i\}$. Then

$$\begin{aligned} L(s_1) \times L(s_2) \times \cdots \times L(s_n) &= \{s_1\} \times \{s_2\} \times \cdots \times (\{s_i\} \cup S_i s_i) \times \cdots \times \{s_n\} \\ &= (s_1, s_2, \ldots, s_n) \cup (S_1 s_1 \times S_2 s_2 \times \cdots \times S_i s_i \times \cdots \times S_n s_n) \\ &= L((s_1, s_2, \ldots, s_n)). \end{aligned}$$

By these two cases, we conclude that $L((s_1, s_2, \ldots, s_n)) = L(s_1) \times L(s_2) \times \cdots \times L(s_n)$.

Remark 1. *Let S be a semigroup defined as in Example 1. We have $s_3 \notin \{s_1, s_2, s_4\} = Ss_3$. Thus, we immediately obtain from Theorem 1 that $L(s_3) \times L(s_3) \neq L((s_3, s_3))$.*

In Theorem 1, we establish the sufficient and necessary condition when $L((s_1, s_2, \ldots, s_n)) = L(s_1) \times L(s_2) \times \cdots \times L(s_n)$. However, the negation of such condition does not ensure that $L(s_1) \times L(s_2) \times \cdots \times L(s_n)$ is not a principal left ideal. This assumption can be confirmed by the following theorem.

Theorem 2. *Let S_i be a semigroup and let $s_i \in S_i$, where $i = 1, 2, \ldots, n$. If $L((s_1, s_2, \ldots, s_n)) \neq L(s_1) \times L(s_2) \times \cdots \times L(s_n)$, then $L(s_1) \times L(s_2) \times \cdots \times L(s_n)$ is not a principal left ideal.*

Proof. Assume that $L((s_1, s_2, \ldots, s_n)) \neq L(s_1) \times L(s_2) \times \cdots \times L(s_n)$. Suppose that $L(s_1) \times L(s_2) \times \cdots \times L(s_n)$ is a principal left ideal of $S_1 \times S_2 \times \cdots \times S_n$. Then

$$L(s_1) \times L(s_2) \times \cdots \times L(s_n) = L((t_1, t_2, \ldots, t_n))$$

for some $(t_1, t_2, \ldots, t_n) \in S_1 \times S_2 \times \cdots \times S_n$. We observe that

$$\begin{aligned} (s_1, s_2, \ldots, s_n) &\in L(s_1) \times L(s_2) \times \cdots \times L(s_n) \\ &= L((t_1, t_2, \ldots, t_n)) \\ &\subseteq L(t_1) \times L(t_2) \times \cdots \times L(t_n). \end{aligned}$$

On the same way, we also obtain $(t_1, t_2, \ldots, t_n) \in L(s_1) \times L(s_2) \times \cdots \times L(s_n)$. These imply that

$$\begin{aligned} S_1 s_1 \times S_2 s_2 \times \cdots \times S_n s_n &= (S_1 \times S_2 \times \cdots \times S_n)(s_1, s_2, \ldots, s_n) \\ &\subseteq (S_1 \times S_2 \times \cdots \times S_n)(L(t_1) \times L(t_2) \times \cdots \times L(t_n)) \\ &= S_1 t_1 \times S_2 t_2 \times \cdots \times S_n t_n \\ &= (S_1 \times S_2 \times \cdots \times S_n)(t_1, t_2, \ldots, t_n) \\ &\subseteq (S_1 \times S_2 \times \cdots \times S_n)(L(s_1) \times L(s_2) \times \cdots \times L(s_n)) \\ &= S_1 s_1 \times S_2 s_2 \times \cdots \times S_n s_n. \end{aligned}$$

Thus, $S_1 s_1 \times S_2 s_2 \times \cdots \times S_n s_n = S_1 t_1 \times S_2 t_2 \times \cdots \times S_n t_n$. Since $(s_1, s_2, \ldots, s_n) \neq (t_1, t_2, \ldots, t_n)$ and $(s_1, s_2, \ldots, s_n) \in L((t_1, t_2, \ldots, t_n))$, we have that

$$(s_1, s_2, \ldots, s_n) \in S_1 t_1 \times S_2 t_2 \times \cdots \times S_n t_n = S_1 s_1 \times S_2 s_2 \times \cdots \times S_n s_n.$$

By Theorem 1(i), $L((s_1, s_2, \ldots, s_n)) = L(s_1) \times L(s_2) \times \cdots \times L(s_n)$. This contradicts to assumption. Therefore, $L(s_1) \times L(s_2) \times \cdots \times L(s_n)$ is not a principal left ideal.

The following example shows that the Cartesian product of $\mathcal{L}$-classes need not be an $\mathcal{L}$-class.

Example 2. *From the definition of the semigroup S in Example 1, we have $L_{s_2} \times L_{s_3} \neq L_{(s_2,s_3)}$. Indeed: we have that $L_{(s_2,s_3)} = \{(s_2, s_3)\}$ and $L_{s_2} \times L_{s_3} = \{s_2, s_4\} \times \{s_3\} = \{(s_2, s_3), (s_4, s_3)\}$. Since $(s_4, s_3) \in L_{s_2} \times L_{s_3}$ and $(s_4, s_3) \notin L_{(s_2,s_3)}$, we get $L_{s_2} \times L_{s_3} \neq L_{(s_2,s_3)}$. This shows that $L_{s_2} \times L_{s_3}$ is not an $\mathcal{L}$-class.*

Next, we present Theorem 3 to mention the conclusion of $L_{s_1} \times L_{s_2} \times \cdots \times L_{s_n}$ and $L_{(s_1,s_2,\ldots,s_n)}$. Then we give a necessary and sufficient condition when $L_{s_1} \times L_{s_2} \times \cdots \times L_{s_n} = L_{(s_1,s_2,\ldots,s_n)}$ in Theorem 4. Furthermore, we provide the relation between the Cartesian product of principal left ideals and the Cartesian product of $\mathcal{L}$-classes in Theorem 5.

Theorem 3. *Let S_i be a semigroup and let $s_i \in S_i$ where $i = 1, 2, \ldots, n$. Then the following statements hold:*

(i) $L_{(s_1,s_2,\ldots,s_n)} \subseteq L_{s_1} \times L_{s_2} \times \cdots \times L_{s_n}$;

(ii) *if $L_{(s_1,s_2,\ldots,s_n)} \neq L_{s_1} \times L_{s_2} \times \cdots \times L_{s_n}$, then $L_{s_1} \times L_{s_2} \times \cdots \times L_{s_n}$ contains at least two $\mathcal{L}$-classes in $S_1 \times S_2 \times \cdots \times S_n$.*

Proof. (i) Let $(t_1, t_2, \ldots, t_n) \in L_{(s_1,s_2,\ldots,s_n)}$. Then $L((t_1, t_2, \ldots, t_n)) = L((s_1, s_2, \ldots, s_n))$. This implies that

$$(s_1, s_2, \ldots, s_n) \in L((t_1, t_2, \ldots, t_n)) \subseteq L(t_1) \times L(t_2) \times \cdots \times L(t_n)$$

and

$$(t_1, t_2, \ldots, t_n) \in L((s_1, s_2, \ldots, s_n)) \subseteq L(s_1) \times L(s_2) \times \cdots \times L(s_n).$$

Thus, $s_i \in L(t_i)$ and $t_i \in L(s_i)$ for all $i = 1, 2, \ldots, n$. It follows that

$$L(s_i) = s_i \cup S_i s_i \subseteq L(t_i) \cup S_i L(t_i) = L(t_i).$$

Similarly, we obtain $L(t_i) \subseteq L(s_i)$. Thus, $L(s_i) = L(t_i)$ for all $i = 1, 2, \ldots, n$. Therefore,

$$(t_1, t_2, \ldots, t_n) \in L_{s_1} \times L_{s_2} \times \cdots \times L_{s_n}.$$

(ii) Assume that $L_{(s_1,s_2,\ldots,s_n)} \neq L_{s_1} \times L_{s_2} \times \cdots \times L_{s_n}$. By (i), there exists $(t_1, t_2, \ldots, t_n) \in L_{s_1} \times L_{s_2} \times \cdots \times L_{s_n}$ such that $(t_1, t_2, \ldots, t_n) \notin L_{(s_1,s_2,\ldots,s_n)}$. Then $L_{t_i} = L_{s_i}$ for all $i \in \{1, 2, \ldots, n\}$. Thus,

$$L_{(t_1,t_2,\ldots,t_n)} \subseteq L_{t_1} \times L_{t_2} \times \cdots \times L_{t_n} = L_{s_1} \times L_{s_2} \times \cdots \times L_{s_n}.$$

Since $L_{(s_1,s_2,\ldots,s_n)}$ and $L_{(t_1,t_2,\ldots,t_n)}$ are difference, we obtain that $L_{s_1} \times L_{s_2} \times \cdots \times L_{s_n}$ contains at least two $\mathcal{L}$-classes of $S_1 \times S_2 \times \ldots \times S_n$.

Theorem 4. *Let S_i be a semigroup and let $s_i \in S_i$ where $i = 1, 2, \ldots, n$. Then*

$$L_{(s_1,s_2,\ldots,s_n)} = L_{s_1} \times L_{s_2} \times \cdots \times L_{s_n}$$

if and only if at least one of the following conditions is satisfied:

(i) $L_{s_i} = \{s_i\}$ *for all* $i \in \{1, 2, \ldots, n\}$*;*

(ii) $s_i \in S_i s_i$ *for all* $i \in \{1, 2, \ldots, n\}$*.*

Proof. Assume that $L_{(s_1,s_2,\ldots,s_n)} = L_{s_1} \times L_{s_2} \times \cdots \times L_{s_n}$. If $L_{(s_1,s_2,\ldots,s_n)} = \{(s_1, s_2, \ldots, s_n)\}$, then by assumption, we have

$$L_{s_1} \times L_{s_2} \times \cdots \times L_{s_n} = \{(s_1, s_2, \ldots, s_n)\}.$$

Thus, $L_{s_i} = \{s_i\}$ for all $i \in \{1, 2, \ldots, n\}$. Suppose that there exists $(t_1, t_2, \ldots, t_n) \in L_{(s_1,s_2,\ldots,s_n)}$ such that $(t_1, t_2, \ldots, t_n) \neq (s_1, s_2, \ldots, s_n)$. Then $L((t_1, t_2, \ldots, t_n)) = L((s_1, s_2, \ldots, s_n))$. Since $(t_1, t_2, \ldots, t_n) \in L((s_1, s_2, \ldots, s_n))$ and $(t_1, t_2, \ldots, t_n) \neq (s_1, s_2, \ldots, s_n)$, we have that

$$(t_1, t_2, \ldots, t_n) \in S_1 s_1 \times S_2 s_2 \times \cdots \times S_n s_n.$$

Similarly, we get $(s_1, s_2, \ldots, s_n) \in S_1 t_1 \times S_2 t_2 \times \cdots \times S_n t_n$. Thus,

$$\begin{aligned}
(s_1, s_2, \ldots, s_n) &\in S_1 t_1 \times S_2 t_2 \times \cdots \times S_n t_n \\
&= (S_1 \times S_2 \times \cdots \times S_n)(t_1, t_2, \ldots, t_n) \\
&\subseteq (S_1 \times S_2 \times \cdots \times S_n)(S_1 s_1 \times S_2 s_2 \times \cdots \times S_n s_n) \\
&= S_1 S_1 s_1 \times S_2 S_2 s_2 \times \cdots \times S_n S_n s_n \\
&\subseteq S_1 s_1 \times S_2 s_2 \times \cdots \times S_n s_n.
\end{aligned}$$

Therefore, $s_i \in S_i s_i$ for all $i \in \{1, 2, \ldots, n\}$.

Conversely, assume that (i) or (ii) holds. If (i) holds, then we get

$$\begin{aligned}
L_{s_1} \times L_{s_2} \times \cdots \times L_{s_n} &= \{(s_1, s_2, \ldots, s_n)\} \\
&\subseteq L_{(s_1,s_2,\ldots,s_n)} \\
&\subseteq L_{s_1} \times L_{s_2} \times \cdots \times L_{s_n}.
\end{aligned}$$

Thus, $L_{(s_1,s_2,\ldots,s_n)} = L_{s_1} \times L_{s_2} \times \cdots \times L_{s_n}$. Assume that (ii) holds. Then we have

$$L((s_1, s_2, \ldots, s_n)) = S_1 s_1 \times S_2 s_2 \times \cdots \times S_n s_n.$$

Let $(t_1, t_2, \ldots, t_n) \in L_{s_1} \times L_{s_2} \times \cdots \times L_{s_n}$. For each $i \in \{1, 2, \ldots, n\}$, we have

$$S_i t_i \subseteq L(t_i) = L(s_i) = S_i s_i \subseteq S_i(L(t_i)) = S_i t_i.$$

Thus, $S_i s_i = S_i t_i$. That is $t_i \in L(t_i) = L(s_i) = S_i s_i = S_i t_i$ for all $i \in \{1, 2, \ldots, n\}$. This implies

$$L((t_1, t_2, \ldots, t_n)) = (t_1, t_2, \ldots, t_n) \cup (S_1 t_1 \times S_2 t_2 \times \cdots \times S_n t_n)$$

$$= S_1t_1 \times S_2t_2 \times \cdots \times S_nt_n$$
$$= S_1s_1 \times S_2t_2 \times \cdots \times S_ns_n$$
$$= L((s_1, s_2, \ldots, s_n)).$$

Thus, $(t_1, t_2, \ldots, t_n) \in L_{(s_1,s_2,\ldots,s_n)}$. Hence $L_{s_1} \times L_{s_2} \times \cdots \times L_{s_n} \subseteq L_{(s_1,s_2,\ldots,s_n)}$. The opposite inclusion is obtained by Theorem 3(i). Therefore, $L_{(s_1,s_2,\ldots,s_n)} = L_{s_1} \times L_{s_2} \times \cdots \times L_{s_n}$.

Theorem 5. *Let S_i be a semigroup and let $s_i \in S_i$ where $i \in \{1, 2, \ldots, n\}$. If $L((s_1, s_2, \ldots, s_n)) = L(s_1) \times L(s_2) \times \cdots \times L(s_n)$, then $L_{(s_1,s_2,\ldots,s_n)} = L_{s_1} \times L_{s_2} \times \cdots \times L_{s_n}$.*

Proof. Assume that $L((s_1, s_2, \ldots, s_n)) = L(s_1) \times L(s_2) \times \cdots \times L(s_n)$. By Theorem 1, there are two possible cases, as follows.
Case 1: $s_i \in S_is_i$ for all $i \in \{1, 2, \ldots, n\}$. We obtain from Theorem 4(ii) that $L_{(s_1,s_2,\ldots,s_n)} = L_{s_1} \times L_{s_2} \times \cdots \times L_{s_n}$.
Case 2: There exists $i \in \{1, 2, \ldots, n\}$ such that for each $j \in \{1, 2, \ldots, n\} \setminus \{i\}$, $S_js_j = \{s_j\}$, which yields $L(s_j) = \{s_j\}$ and $L_{s_j} = \{s_j\}$. By focusing on the index i, if $s_i \in S_is_i$, we obtain from Theorem 4(ii) that $L_{(s_1,s_2,\ldots,s_n)} = L_{s_1} \times L_{s_2} \times \cdots \times L_{s_n}$. On the other hand, we suppose that $s_i \notin S_is_i$. We will prove that $L_{s_i} = \{s_i\}$ by a contradiction. Suppose that there exists $t_i \in S_i$ such that $t_i \neq s_i$ and $L(t_i) = L(s_i)$, which implies $S_it_i = S_is_i$. Since $s_i \in L(s_i) = L(t_i)$ and $s_i \neq t_i$, we get $s_i \in S_it_i = S_is_i$, which is a contradiction. Therefore, $L_{s_i} = \{s_i\}$. We thus conclude by Theorem 4(i) that $L_{(s_1,s_2,\ldots,s_n)} = L_{s_1} \times L_{s_2} \times \cdots \times L_{s_n}$ as required.

Let S be a semigroup. Since the relation $\mathcal{L}$ on S is defined in terms of left ideals of S, the order among these left ideals induces a partial order among their equivalence classes, that is,

$$L_a \leq L_b \text{ if } L(a) \subseteq L(b)$$

for all $a, b \in S$. Then an $\mathcal{L}$-class L_s of S is maximal if there is no $u \in S$ such that $L(s) \subsetneq L(u)$.

Lemma 2. *Let S be a semigroup and let $s, u \in S$. If $L(s) \subsetneq L(u)$, then the following statements hold:*

(i) $u \notin L(s)$;

(ii) $L(s) \subseteq Su$.

Proof. Assume that $L(s) \subsetneq L(u)$. To prove (i), suppose $u \in L(s)$. Then

$$L(u) \subseteq L(s) \subsetneq L(u).$$

This contradiction implies $u \notin L(s)$. To prove (ii), let $t \in L(s)$. It follows that $t \in L(u) = u \cup Su$. If $t = u$, then $L(s) \subsetneq L(u) = L(t)$. By (i), we get that $t \notin L(s)$, which is a contradiction. Thus, $t \in Su$.

Theorem 6. *Let S_i be a semigroup and let $s_i \in S_i$ where $i \in \{1, 2, \dots, n\}$. If $(s_1, s_2, \dots, s_n) \in S_1s_1 \times S_2s_2 \times \cdots \times S_ns_n$, then $L_{(s_1,s_2,\dots,s_n)}$ is a maximal $\mathcal{L}$-class if and only if L_{s_i} is a maximal $\mathcal{L}$-class for all $i \in \{1, 2, \dots, n\}$.*

Proof. Assume that $(s_1, s_2, \dots, s_n) \in S_1s_1 \times S_2s_2 \times \cdots \times S_ns_n$. By Theorem 1(i), we have

$$L((s_1, s_2, \dots, s_n)) = L(s_1) \times L(s_2) \times \cdots \times L(s_n).$$

Suppose that L_{s_i} is not a maximal $\mathcal{L}$-class for some $i \in \{1, 2, \dots, n\}$. Then there exists $u_i \in S_i$ such that

$$L(s_i) \subsetneq L(u_i).$$

By Lemma 2(i), $u_i \notin S_is_i$ and $S_is_i \subseteq S_iu_i$. Thus,

$$\begin{aligned} L((s_1, s_2, \dots, s_i, \dots, s_n)) &= L(s_1) \times L(s_2) \times \cdots \times L(s_i) \times \cdots \times L(s_n) \\ &= S_1s_1 \times S_2s_2 \times \cdots \times S_is_i \times \cdots \times S_ns_n \\ &\subsetneq (s_1, s_2, \dots, u_i, \dots, s_n) \cup (S_1s_1 \times S_2s_2 \times \cdots \times S_iu_i \times \cdots \times S_ns_n) \\ &= L((s_1, s_2, \dots, u_i, \dots, s_n)). \end{aligned}$$

Therefore, $L_{(s_1,s_2,\dots,s_n)}$ is not a maximal $\mathcal{L}$-class.

Conversely, assume that $L_{(s_1,s_2,\dots,s_n)}$ is not a maximal $\mathcal{L}$-class. Then there exists $(u_1, u_2, \dots, u_n) \in S_1 \times S_2 \times \cdots \times S_n$ such that

$$L((s_1, s_2, \dots, s_n)) \subsetneq L((u_1, u_2, \dots, u_n)).$$

By Lemma 2(i), we obtain

$$\begin{aligned} (u_1, u_2, \dots, u_n) &\notin L((s_1, s_2, \dots, s_n)) \\ &= (s_1, s_2, \dots, s_n) \cup (S_1s_1 \times S_2s_2 \times \cdots \times S_ns_n) \\ &= S_1s_1 \times S_2s_2 \times \cdots \times S_ns_n. \end{aligned}$$

This implies that $u_i \notin S_is_i$ for some $i \in \{1, 2, \dots, n\}$. It follows by assumption that $u_i \neq s_i$. Therefore, $L(s_i) \subseteq L(u_i)$ and $u_i \notin s_i \cup S_is_i = L(s_i)$. These imply $L(s_i) \subsetneq L(u_i)$. Thus, L_{s_i} is not a maximal $\mathcal{L}$-class.

Corollary 1. *Let S_i be a semigroup and let $s_i \in S_i$ where $i \in \{1, 2, \dots, n\}$. If $(s_1, s_2, \dots, s_n) \in S_1s_1 \times S_2s_2 \times \cdots \times S_ns_n$, then $L_{s_1} \times L_{s_2} \times \cdots \times L_{s_i}$ is a maximal $\mathcal{L}$-class if and only if L_{s_i} is a maximal $\mathcal{L}$-class for all $i \in \{1, 2, \dots, n\}$.*

Proof. The proof is obtained directly from Theorem 4 and Theorem 6

Definition 1. *Let S be a semigroup. An element $s \in S$ is decomposable if $s \in S^2$. If an element $s \in S$ is not decomposable, we say that s is indecomposable.*

In [1], the author remarked that if an element s of a semigroup S is indecomposable, then $L_s = \{s\}$. In the direct product of $\{S_i \mid i \in \{1, 2, \ldots, n\}\}$ we have that if $(s_1, s_2, \ldots, s_n) \in S_1 \times S_2 \times \cdots \times S_n$ is indecomposable, then $L_{(s_1,s_2,\ldots,s_n)} = \{(s_1, s_2, \ldots, s_n)\}$ and s_i is indecomposable for some $i \in \{1, 2, \ldots, n\}$. On the other hand, if there exists $i \in \{1, 2, \ldots, n\}$ such that s_i is indecomposable, then we get that $(s_1, s_2, \ldots, s_n)$ is indecomposable. Next, we consider relationships between indecomposable elements and maximal $\mathcal{L}$-classes.

Theorem 7. *Let S_i be a semigroup and let $s_i \in S_i$ where $i \in \{1, 2, \ldots, n\}$.*

(i) *If $(s_1, s_2, \ldots, s_n)$ is indecomposable, then $L_{(s_1,s_2,\ldots,s_n)}$ is a maximal $\mathcal{L}$-class.*

(ii) *If $L_{(s_1,s_2,\ldots,s_n)}$ is a maximal $\mathcal{L}$-class of $S_1 \times S_2 \times \cdots \times S_n$ and $(s_1, s_2, \ldots, s_n) \notin S_1s_1 \times S_2s_2 \times \cdots \times S_ns_n$, then $(s_1, s_2, \ldots, s_n)$ is indecomposable.*

Proof. (i) Assume that $(s_1, s_2, \ldots, s_n)$ is indecomposable. Suppose that $L_{(s_1,s_2,\ldots,s_n)}$ is not a maximal $\mathcal{L}$-class. Then there exists $(u_1, u_2, \ldots, u_n) \in S_1 \times S_2 \times \cdots \times S_n$ such that

$$L((s_1, s_2, \ldots, s_n)) \subsetneq L((u_1, u_2, \ldots, u_n)).$$

By Lemma 2(ii), we obtain that

$$(s_1, s_2, \ldots, s_n) \in L((s_1, s_2, \ldots, s_n)) \subseteq S_1u_1 \times S_2u_2 \times \cdots \times S_nu_n \subseteq S_1^2 \times S_2^2 \times \cdots \times S_n^2.$$

This contradicts to an assumption. Therefore, $L_{(s_1,s_2,\ldots,s_n)}$ is a maximal $\mathcal{L}$-class.

(ii) Assume that $L_{(s_1,s_2,\ldots,s_n)}$ is a maximal $\mathcal{L}$-class and $(s_1, s_2, \ldots, s_n) \notin S_1s_1 \times S_2s_2 \times \cdots \times S_ns_n$. Suppose that $(s_1, s_2, \ldots, s_n)$ is decomposable. Then there exists $(t_1, t_2, \ldots, t_n)$ such that $(s_1, s_2, \ldots, s_n) \in (S_1 \times S_2 \times \cdots \times S_n)(t_1, t_2, \ldots, t_n)$. This implies that

$$L((s_1, s_2, \ldots, s_n)) \subseteq L((t_1, t_2, \ldots, t_n)).$$

We observe that $(t_1, t_2, \ldots, t_n) \neq (s_1, s_2, \ldots, s_n)$. Suppose that $(t_1, t_2, \ldots, t_n) \in (S_1 \times S_2 \times \cdots \times S_n)(s_1, s_2, \ldots, s_n)$, it follows that

$$(s_1, s_2, \ldots, s_n) \in S_1t_1 \times S_2t_2 \times \cdots \times S_nt_n = S_1s_1 \times S_2s_2 \times \cdots \times S_ns_n.$$

This contradicts to our assumption. Thus, $(t_1, t_2, \ldots, t_n) \notin L((s_1, s_2, \ldots, s_n))$. Therefore, $L((s_1, s_2, \ldots, s_n)) \subsetneq L((t_1, t_2, \ldots, t_n))$. This contradicts to maximality of $L_{(s_1,s_2,\ldots,s_n)}$. Hence, $(s_1, s_2, \ldots, s_n)$ is indecomposable.

According to the observation of indecomposable elements on a direct product of semigroups together with Theorem 7, we have the following corollary.

Corollary 2. *Let S_i be a semigroup and let $s_i \in S_i$ where $i \in \{1, 2, \ldots, n\}$.*
If $(s_1, s_2, \ldots, s_n) \in S_1s_1 \times S_2s_2 \times \cdots \times S_ns_n$, then $L_{(s_1,s_2,\ldots,s_n)}$ is a maximal $\mathcal{L}$-class in $S_1 \times S_2 \times \cdots \times S_n$ if and only if $s_i \in S_i$ is indecomposable for some $i \in \{1, 2, \ldots, n\}$.

3. Conclusions

In this paper, we studied the Cartesian product of principal left ideals and the Cartesian product of $\mathcal{L}$-classes in the direct product of n semigroups ($n \geq 2$). We gave an explicit counterexample showing that the Cartesian product of principal left ideals is not necessarily a principal left ideal. Similarly, an explicit counterexample for the Cartesian product of $\mathcal{L}$-classes was provided. We established the two main results, consisting of a necessary and sufficient condition for the Cartesian product of principal left ideals to be a principal left ideal, and a necessary and sufficient condition for the Cartesian product of L-classes to be an $\mathcal{L}$-class in the direct product of n semigroups. In addition, the condition when the Cartesian product of maximal $\mathcal{L}$-classes is maximal was also investigated.

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