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A Study on Bi-Univalent Functions of Complex Order Arising from the q-Fibonacci Analogue

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Abstract. This paper introduces new subclasses of bi-univalent functions of complex order linked with shell-like domains, formulated via the subordination principle and the q analog of Fibonacci numbers. Motivated by recent advances in q-calculus and its applications in geometric function theory, we construct and examine two distinct families of analytic bi-univalent functions. For these subclasses, coefficient estimates are derived for the initial Taylor–Maclaurin coefficients, together with sharp bounds for the Fekete–Szegö functional expressed in terms of the parameters involved. The results of this study extend and unify earlier results in the theory of bi-univalent functions, while providing new insights into the interaction between the theory of bi-univalent functions, the q-Fibonacci framework, and shell-like geometries. Furthermore, the subclasses established here may serve as a foundation for future studies on analytic function spaces, special functions, and their operator-theoretic connections.

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1. Introduction and Preliminaries

Let $\mathscr{A}$ denote the class of analytic functions in the open unit disk

$$\mathscr{O} = \{z = \mathfrak{a} + i\mathfrak{b} \in \mathbb{C} : \mathfrak{a}, \mathfrak{b} \in \mathbb{R},\ |z| < 1\},$$

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the interior of the unit circle centered at the origin. Every member $f \in \mathscr{A}$ is assumed to satisfy the standard normalization

$$f(0) = 0, \qquad f'(0) = 1,$$

so that translation and dilation effects at the origin are eliminated.

Each function in $\mathscr{A}$ can be represented by a Maclaurin series of the form

$$f(z) = z + \sum_{\mathsf{s}=2}^{\infty} \delta_{\mathsf{s}} z^{\mathsf{s}}, \qquad z \in \mathscr{O}, \tag{1}$$

where the coefficients δ_{s} encode its higher-order analytic structure. In particular, normalization ensures that the leading term is exactly z.

A function f is called a *Schwarz function* if it is analytic in $\mathscr{O}$, vanishes at the origin and satisfies $|f(z)| < 1$ throughout $\mathscr{O}$. Such functions are central to geometric function theory because of their role in univalent and conformal mapping problems.

For two analytic functions $f_1, f_2 \in \mathscr{A}$, we write $f_1 \prec f_2$ whenever there exists a Schwarz function η with

$$f_1(z) = f_2(\eta(z)), \qquad z \in \mathscr{O}.$$

This concept of subordination provides a powerful framework for analyzing inclusion, growth, and distortion properties.

Let $\mathsf{S} \subset \mathscr{A}$ denote the class of univalent functions in $\mathscr{O}$. If $f \in \mathsf{S}$, then f admits an analytic inverse f^{-1} defined in a disk of radius at least $1/4$, with series expansion

$$f^{-1}(\xi) = \xi - \delta_2\xi^2 + (2\delta_2^2 - \delta_3)\xi^3 - (5\delta_2^3 + \delta_4 - 5\delta_2\delta_3)\xi^4 + \cdots. \tag{2}$$

A function is termed *bi-univalent* if both f and f^{-1} are univalent in $\mathscr{O}$. The set of such functions is denoted by Σ.

Another class of interest is P, which consists of analytic functions in $\mathscr{O}$ with a positive real part. Each $\psi \in \mathsf{P}$ admits the series

$$\psi(z) = 1 + \sum_{\mathsf{s}=1}^{\infty} p_{\mathsf{s}} z^{\mathsf{s}}, \qquad z \in \mathscr{O}, \tag{3}$$

where the sharp estimate

$$|p_{\mathsf{s}}| \le 2, \qquad \mathsf{s} \ge 1, \tag{4}$$

follows from Carathéodory's lemma [1]. Moreover, it is well known that $\psi \in \mathsf{P}$ if and only if

$$\psi(z) \prec \frac{1+z}{1-z}, \qquad z \in \mathscr{O}.$$

The class of starlike functions, denoted S^*, admits elegant formulations in terms of subordination. Ma and Minda [2] introduced the generalized family

$$\mathsf{S}^*(\Phi) = \left\{ f \in \mathscr{A} : \frac{zf'(z)}{f(z)} \prec \Phi(z), \quad \Phi \in \mathsf{P} \right\},$$

where Φ is analytic in $\mathscr{O}$ and satisfies $\Re e\,(\Phi(z)) > 0$.

Prominent subclasses of S^* emerge from specific selections of $\Phi(z)$. For example, Janowski [3, 4] employed $\Phi(z) = \frac{1+z}{1-z}$. Robertson [5] examined the choice $\Phi(z) = \frac{1+(1-2\vartheta)z}{1-z}$, where $0 \le \vartheta < 1$. Sokól [6] studied $\Phi(z) = \frac{1+\vartheta^2 z^2}{1-\vartheta z-\vartheta^2 z^2}$ with $\vartheta = \frac{1-\sqrt{5}}{2}$, and later introduced the case $\Phi(z) = \frac{3}{3+(\vartheta-3)z-\vartheta^2 z^2}$ for $\vartheta \in (-3, 1]$ [7]. These examples underscore the diversity of subclasses obtained from suitable choices of the function Φ.

Quantum calculus (or q-calculus) generalizes classical calculus by introducing a parameter $q \in (0, 1)$, providing a natural deformation with deep links to physics, quantum mechanics, and geometric function theory. A fundamental tool is the q-difference operator $\eth_q$. Seminal references include the monograph by Gasper and Rahman [8] and analytic function applications studied by Seoudy and Aouf [9]. Further developments appear in [10–31]. Thus, polynomials bridge the gap between abstract complex analysis and computational modeling, allowing deeper exploration of geometric mappings and their analytic behavior [32–36].

Definition 1 ([17])**.** *The q-bracket is defined by*

$$\lceil \kappa \rfloor_q = \begin{cases} \dfrac{1-q^\lambda}{1-q}, & 0<q<1,\ \lambda \in \mathbb{C}^*, \\ 1, & q \to 0^+, \\ \lambda, & q \to 1^-, \\ \sum\limits_{\mathsf{s}=0}^{\gamma-1} q^{\mathsf{s}}, & 0<q<1,\ \lambda = \gamma \in \mathbb{N}. \end{cases}$$

Definition 2 ([17])**.** *The q-derivative (or q-difference operator) of f is*

$$\eth_q\langle f(z)\rangle = \begin{cases} \dfrac{f(z)-f(qz)}{z-qz}, & 0<q<1,\ z \neq 0, \\ f'(0), & z = 0, \\ f'(z), & q \to 1^-. \end{cases}$$

Remark 1. *If f has the form* (1)*, then*

$$\eth_q\langle f(z)\rangle = 1 + \sum_{\mathsf{s}=2}^{\infty} \lceil \mathsf{s} \rfloor_q\, \delta_{\mathsf{s}}\, z^{\mathsf{s}-1},$$

while for its inverse f^{-1} given by (2)*,*

$$\eth_q\langle f^{-1}(\xi)\rangle = 1 - \lceil 2 \rfloor_q \delta_2 \xi + \lceil 3 \rfloor_q (2\delta_2^2 - \delta_3)\xi^2 - \lceil 4 \rfloor_q (5\delta_2^3 + \delta_4 - 5\delta_2\delta_3)\xi^3 + \cdots.$$

Alsoboh et al. [37] introduced the q-starlike class

$$\mathsf{SL}_q = \left\{ f \in \mathscr{A} : \frac{z\eth_q\langle f(z)\rangle}{f(z)} \prec \Upsilon(z;q) \right\}, \tag{5}$$

where

$$\Upsilon(z;q) = \frac{1 + q\vartheta_q^2 z^2}{1 - \vartheta_q z - q\vartheta_q^2 z^2}, \qquad \vartheta_q = \frac{1 - \sqrt{4q+1}}{2q}, \tag{6}$$

with ϑ_q representing the q analog of the Fibonacci numbers.

Moreover, they established connections between ϑ_q and the q-Fibonacci polynomials $\varphi_{\mathsf{s}}(q)$, with the recurrence

$$\widehat{p}_{\mathsf{s}} = \begin{cases} \vartheta_q, & \mathsf{s} = 1, \\ (2q+1)\vartheta_q^2, & \mathsf{s} = 2, \\ (3q+1)\vartheta_q^3, & \mathsf{s} = 3, \\ (\varphi_{\mathsf{s}+1}(q) + q\varphi_{\mathsf{s}-1}(q))\vartheta_q^{\mathsf{s}}, & \mathsf{s} \geq 4. \end{cases} \tag{7}$$

The initial terms of the q-Fibonacci sequence are listed in Table 1, reducing to the classical Fibonacci numbers as $q \to 1^-$.

Table 1: Classical Fibonacci numbers and their q-analogues.

Classical Fibonacci	q-Fibonacci
$\varphi_0 = 0$	$\varphi_0(q) = 0$
$\varphi_1 = 1$	$\varphi_1(q) = 1$
$\varphi_2 = 1$	$\varphi_2(q) = 1$
$\varphi_3 = 2$	$\varphi_3(q) = 1 + q$
$\varphi_4 = 3$	$\varphi_4(q) = 1 + 2q$

In the limit $q \to 1^-$, the class SL_q recovers the classical starlike family associated with the Fibonacci generating function:

$$\mathsf{SL} = \left\{ f \in \mathscr{A} : \tfrac{zf'(z)}{f(z)} \prec \Upsilon(z) \right\}, \qquad \Upsilon(z) = \frac{1 + \vartheta^2 z^2}{1 - \vartheta z - \vartheta^2 z^2},$$

where $\vartheta = \frac{1-\sqrt{5}}{2}$.

Furthermore, Alsoboh et al. [17] defined the q-convex class KSL_q via

$$1 + \frac{z\eth_q^2\langle f(z)\rangle}{\eth_q\langle f(z)\rangle} \prec \Upsilon(z;q), \qquad z \in \mathscr{O}. \tag{8}$$

This generalization involves a higher-order q-difference operator, capturing refined geometric structures.

2. Definition and Examples

Motivated by the theory of q-Fibonacci numbers, we introduce a new subclass of bi-univalent functions associated with shell-like curves.

Definition 3. *Let $\beta \in [0,1]$ and $\rho \in \mathbb{C} \setminus \{0\}$. A bi-univalent function f of the form (1) is said to belong to the class $\mathsf{SLM}_\Sigma(\beta, \rho; q)$ if and only if*

$$1 + \frac{1}{\rho}\left[(1-\beta)\frac{z\,\eth_q\langle f(z)\rangle}{f(z)} + \beta\frac{\eth_q(z\,\eth_q\langle f(z)\rangle)}{\eth_q\langle f(z)\rangle} - 1\right] \prec \Upsilon(z;q), \qquad z \in \mathscr{O}, \tag{9}$$

and simultaneously

$$1 + \frac{1}{\rho}\left[(1-\beta)\frac{\xi\,\eth_q\langle \chi(\xi)\rangle}{\chi(\xi)} + \beta\frac{\eth_q(\xi\,\eth_q\langle \chi(\xi)\rangle)}{\eth_q\langle \chi(\xi)\rangle} - 1\right] \prec \Upsilon(\xi;q), \qquad \xi \in \mathscr{O}, \tag{10}$$

where $\chi = f^{-1}$ is given by (2), and the functions $\Upsilon(z;q)$ and ϑ_q are defined in (6).

Changing the parameters $\rho \in \mathbb{C}^*$, $\beta \in [0,1]$, and $q \in (0,1)$, a family of subclasses of Σ is generated, exhibiting diverse geometric characteristics.

Example 1. *If $\rho = 1$, the class reduces to $\mathsf{SLM}_\Sigma(\beta; q)$, where each $f \in \Sigma$ satisfies*

$$(1-\beta)\frac{z\,\eth_q\langle f(z)\rangle}{f(z)} + \beta\frac{\eth_q(z\,\eth_q\langle f(z)\rangle)}{\eth_q\langle f(z)\rangle} \prec \Upsilon(z;q),$$

and

$$(1-\beta)\frac{\xi\,\eth_q\langle \chi(\xi)\rangle}{\chi(\xi)} + \beta\frac{\eth_q(\xi\,\eth_q\langle \chi(\xi)\rangle)}{\eth_q\langle \chi(\xi)\rangle} \prec \Upsilon(\xi;q),$$

where $\chi = f^{-1}$.

Example 2. *If $\beta = 0$ and $\rho = 1$, we obtain $\mathsf{SL}_\Sigma(\Upsilon(z;q))$, consisting of $f \in \Sigma$ such that*

$$\frac{z\,\eth_q\langle f(z)\rangle}{f(z)} \prec \Upsilon(z;q), \qquad \frac{\xi\,\eth_q\langle \chi(\xi)\rangle}{\chi(\xi)} \prec \Upsilon(\xi;q).$$

Example 3. *If $\beta = 1$ and $\rho = 1$, the class reduces to $\mathsf{KL}_\Sigma(\Upsilon(z;q))$, consisting of $f \in \Sigma$ such that*

$$1 + \frac{z\,\eth_q^2\langle f(z)\rangle}{\eth_q\langle f(z)\rangle} \prec \Upsilon(z;q), \qquad 1 + \frac{\xi\,\eth_q^2\langle \chi(\xi)\rangle}{\eth_q\langle \chi(\xi)\rangle} \prec \Upsilon(\xi;q).$$

Example 4. *If $q \to 1^-$ and $\rho = 1$, we recover the classical class $\mathsf{SLM}_\Sigma(\beta)$, where $f \in \Sigma$ satisfies*

$$(1-\beta)\frac{zf'(z)}{f(z)} + \beta\frac{zf''(z)}{f'(z)} \prec \Upsilon(z), \qquad (1-\beta)\frac{\xi\chi'(\xi)}{\chi(\xi)} + \beta\frac{\xi\chi''(\xi)}{\chi'(\xi)} \prec \Upsilon(\xi),$$

with $\vartheta = \frac{1-\sqrt{5}}{2}$.

Example 5. *If $q \to 1^-$, $\beta = 0$, and $\rho = 1$, we obtain $\mathsf{SL}_\Sigma(\Upsilon(z))$, where $f \in \Sigma$ satisfies*

$$\frac{zf'(z)}{f(z)} \prec \Upsilon(z), \qquad \frac{\xi\chi'(\xi)}{\chi(\xi)} \prec \Upsilon(\xi).$$

Example 6. *If $q \to 1^-$, $\beta = 1$, and $\rho = 1$, the class reduces to $\mathsf{KL}_\Sigma(\Upsilon(z))$, consisting of $f \in \Sigma$ such that*

$$1 + \frac{zf''(z)}{f'(z)} \prec \Upsilon(z), \qquad 1 + \frac{\xi\chi''(\xi)}{\chi'(\xi)} \prec \Upsilon(\xi).$$

3. Main Results

In this section, we establish coefficient bounds for the initial Taylor coefficients $|\delta_2|$ and $|\delta_3|$ of functions belonging to the class $\mathsf{SLM}_\Sigma(\beta, \rho; q)$, as introduced in Definition 3.

Firstly, let us

$$\mathscr{P}(z) = 1 + \mathscr{P}_1 z + \mathscr{P}_2 z^2 + \mathscr{P}_3 z^3 + \cdots, \qquad \mathscr{P}(z) \prec \Upsilon(z; q).$$

Then there exists a Schwarz function $\psi \in \mathsf{P}$, with $|\psi(z)| < 1$ for all $z \in \mathscr{O}$, such that

$$\mathscr{P}(z) = \Upsilon(\psi(z); q).$$

In this setting, we may define the following.

$$\hbar(z) = \frac{1 + \psi(z)}{1 - \psi(z)} = 1 + \ell_1 z + \ell_2 z^2 + \cdots, \qquad z \in \mathscr{O}, \tag{11}$$

which belongs to the class P.

Consequently, since $\psi(z)$ is analytic in $\mathscr{O}$ and subordinate to $\Upsilon(z; q)$, it admits the Taylor expansion

$$\psi(z) = \frac{\ell_1}{2} z + \frac{1}{2}\left(\ell_2 - \tfrac{\ell_1^2}{2}\right) z^2 + \frac{1}{2}\left(\ell_3 - \ell_1\ell_2 - \tfrac{\ell_1^3}{4}\right) z^3 + \cdots, \tag{12}$$

and

$$\begin{aligned}
\Upsilon(\psi(z); q) = 1 &+ \widehat{\mathscr{P}_1}\left[\frac{\ell_1 z}{2} + \left(\ell_2 - \frac{\ell_1^2}{2}\right)\frac{z^2}{2} + \left(\ell_3 - \ell_1\ell_2 - \frac{\ell_1^3}{4}\right)\frac{z^3}{2} + \cdots\right] \\
&+ \widehat{\mathscr{P}_2}\left[\frac{\ell_1 z}{2} + \left(\ell_2 - \frac{\ell_1^2}{2}\right)\frac{z^2}{2} + \left(\ell_3 - \ell_1\ell_2 - \frac{\ell_1^3}{4}\right)\frac{z^3}{2} + \cdots\right]^2 \\
&+ \widehat{\mathscr{P}_3}\left[\frac{\ell_1 z}{2} + \left(\ell_2 - \frac{\ell_1^2}{2}\right)\frac{z^2}{2} + \left(\ell_3 - \ell_1\ell_2 - \frac{\ell_1^3}{4}\right)\frac{z^3}{2} + \cdots\right]^3 + \cdots \\
&= 1 + \frac{\widehat{\mathscr{P}_1}\ell_1}{2} z + \frac{1}{2}\left[\left(\ell_2 - \frac{\ell_1^2}{2}\right)\widehat{\mathscr{P}_1} + \frac{\ell_1^2}{2}\widehat{\mathscr{P}_2}\right] z^2 \\
&+ \frac{1}{2}\left[\left(\ell_3 - \ell_1\ell_2 + \frac{\ell_1^3}{4}\right)\widehat{\mathscr{P}_1} + \ell_1\left(\ell_2 - \frac{\ell_1^2}{2}\right)\widehat{\mathscr{P}_2} + \frac{\ell_1^3}{4}\widehat{\mathscr{P}_3}\right] z^3 + \cdots.
\end{aligned} \tag{13}$$

Similarly, one can find an analytic function ν in $\mathscr{O}$, with $|\nu(\xi)| < 1$, such that $\mathscr{P}(\xi) = \Upsilon(\nu(\xi); q)$. Accordingly, we may express the associated function

$$\kappa(\xi) = (1 + \nu(\xi))(1 - \nu(\xi))^{-1} = 1 + \tau_1\xi + \tau_2\xi^2 + \cdots \in \mathsf{P}. \tag{14}$$

As a result, the Taylor expansion of $\nu(\xi)$ takes the form:

$$\nu(\xi) = \frac{\tau_1\xi}{2} + \left(\tau_2 - \frac{\tau_1^2}{2}\right)\frac{\xi^2}{2} + \left(\tau_3 - \tau_1\tau_2 - \frac{\tau_1^3}{4}\right)\frac{\xi^3}{2} + \cdots, \tag{15}$$

and, accordingly, the composition $\Upsilon(\nu(\xi); q)$ expands as:

$$\begin{aligned}\Upsilon(\nu(\xi); q) = 1 + \frac{\widehat{\mathscr{P}_1}\tau_1}{2}\xi + \frac{1}{2}\left[\left(\tau_2 - \frac{\tau_1^2}{2}\right)\widehat{\mathscr{P}_1} + \frac{\tau_1^2}{2}\widehat{\mathscr{P}_2}\right]\xi^2 \\ + \frac{1}{2}\left[\left(\tau_3 - \tau_1\tau_2 + \frac{\tau_1^3}{4}\right)\widehat{\mathscr{P}_1} + \tau_1\left(\tau_2 - \frac{\tau_1^2}{2}\right)\widehat{\mathscr{P}_2} + \frac{\tau_1^3}{4}\widehat{\mathscr{P}_3}\right]\xi^3 + \cdots.\end{aligned} \tag{16}$$

Having established the necessary groundwork and auxiliary results, we are now prepared to derive coefficient bounds for functions belonging to the newly introduced class $\mathsf{SLM}_\Sigma(\beta, \rho; q)$. These estimates provide valuable insights into the geometric behavior of such bi-univalent functions and, moreover, emphasize the role of the deformation parameter q and the weighting parameter β in shaping the coefficient structure. The next theorem provides sharp bounds for the initial coefficients $|\alpha_2|$ and $|\alpha_3|$.

Theorem 1. *For $\rho \in \mathbb{C}^*$ and $\beta \in [0, 1]$, let $f \in \mathsf{SLM}_{\sum}(\beta, \rho; q)$. Then*

$$\left|\alpha_2\right| \leq \frac{|\rho||\vartheta_q|}{\sqrt{\left|\rho\vartheta_q(K - X) + \big(1 - (2q + 1)\vartheta_q\big)C\right|}}. \tag{17}$$

$$\left|\alpha_3\right| \leq \frac{|\rho||\vartheta_q|\left\{\left|\big(K - X\big)\rho\vartheta_q + \big(1 - (2q + 1)\vartheta_q\big)C\right| + |\rho||\vartheta_q|K\right\}}{K\left|\big(K - X\big)\rho\vartheta_q + \big(1 - (2q + 1)\vartheta_q\big)C\right|}, \tag{18}$$

where

$$K = q\lceil 2\rfloor_q\big(1 + q\lceil 2\rfloor_q\beta\big), \tag{19}$$

$$X = q\Big[1 + \beta\left(\lceil 2\rfloor_q^2 - 1\right)\Big], \tag{20}$$

$$C = q^2(1 + q\beta)^2. \tag{21}$$

Proof. Let $f \in \mathsf{SL}_{\sum}(\Upsilon(z))$ and $\xi = f^{-1}$. Taking into account (9) and (10), we have

$$1 + \frac{1}{\rho}\left((1 - \beta)\frac{z\eth_q\langle f(z)\rangle}{f(z)} + \beta\frac{\eth_q\left(z\,\eth_q\langle f(z)\rangle\right)}{\eth_q\langle f(z)\rangle} - 1\right) = \Upsilon(\psi(z); q), \qquad (z \in \mathscr{O}), \tag{22}$$

and

$$1 + \frac{1}{\rho}\left((1 - \beta)\frac{\xi\eth_q\langle\chi(\xi)\rangle}{\chi(\xi)} + \beta\frac{\eth_q\left(\xi\eth_q\langle\chi(\xi)\rangle\right)}{\eth_q\langle\chi(\xi)\rangle} - 1\right) = \Upsilon(\nu(\xi); q), \qquad (\xi \in \mathscr{O}). \tag{23}$$

Since

$$\frac{1}{\rho}\left((1-\beta)\frac{z\eth_q\langle f(z)\rangle}{f(z)}+\beta\frac{\eth_q\left(z\,\eth_q\langle f(z)\rangle\right)}{\eth_q\langle f(z)\rangle}-1\right)=1+\frac{q(1+q\beta)}{\rho}\alpha_2 z$$
$$+\left(\frac{q\lceil 2\rfloor_q(1+q\lceil 2\rfloor_q\beta)\alpha_3-q\left(1+\beta(\lceil 2\rfloor_q^2-1)\right)\alpha_2^2}{\rho}\right)z^2+\boldsymbol{O}\left(z^3\right) \tag{24}$$
$$=1+\frac{q(1+q\beta)}{\rho}\alpha_2 z+\left(\frac{K\alpha_3-X\alpha_2^2}{\rho}\right)z^2+\boldsymbol{O}\left(z^3\right),$$

and

$$\frac{1}{\rho}\left((1-\beta)\frac{\xi\eth_q\langle\chi(\xi)\rangle}{\chi(\xi)}+\beta\frac{\eth_q\left(\xi\eth_q\langle\chi(\xi)\rangle\right)}{\eth_q\langle\chi(\xi)\rangle}-1\right)=1-\frac{q(1+q\beta)}{\rho}\alpha_2\xi$$
$$+\left(\frac{2q\lceil 2\rfloor_q(1+q\lceil 2\rfloor_q\beta)-q\left(1+\beta(\lceil 2\rfloor_q^2-1)\right)}{\rho}\alpha_2^2-\frac{q\lceil 2\rfloor_q(1+q\lceil 2\rfloor_q\beta)}{\rho}\alpha_3\right)\xi^2+\boldsymbol{O}\left(\xi^3\right)$$
$$=1-\frac{q(1+q\beta)}{\rho}\alpha_2\,\xi+\left(\frac{2K-X}{\rho}\alpha_2^2-\frac{K}{\rho}\alpha_3\right)\xi^2+\boldsymbol{O}\left(\xi^3\right). \tag{25}$$

Compared with (22) and (24), along (13), yields

$$\frac{q(1+q\beta)}{\rho}\alpha_2 z+\left(\frac{K\alpha_3-X\alpha_2^2}{\rho}\right)z^2+\boldsymbol{O}\left(z^3\right)=\frac{\widehat{\mathscr{P}_1}\ell_1}{2}z+\frac{1}{2}\left[\left(\ell_2-\frac{\ell_1^2}{2}\right)\widehat{\mathscr{P}_1}+\frac{\ell_1^2}{2}\widehat{\mathscr{P}_2}\right]z^2+\boldsymbol{O}\left(z^3\right). \tag{26}$$

Besied that by comparing (23) and (25), along (16), yields

$$-\frac{q(1+q\beta)}{\rho}\alpha_2\,\xi+\left(\frac{2K-X}{\rho}\alpha_2^2-\frac{K}{\rho}\alpha_3\right)\xi^2+\boldsymbol{O}\left(\xi^3\right)$$
$$+\cdots=\frac{\widehat{\mathscr{P}_1}\tau_1}{2}\xi+\frac{1}{2}\left[\left(\tau_2-\frac{\tau_1^2}{2}\right)\widehat{\mathscr{P}_1}+\frac{\tau_1^2}{2}\widehat{\mathscr{P}_2}\right]\xi^2+\cdots. \tag{27}$$

Equating the pertinent coefficient in (26) and (27), using (19) and (20), we obtain

$$\frac{q(1+q\beta)}{\rho}\alpha_2=\frac{\widehat{\mathscr{P}_1}\ell_1}{2} \tag{28}$$
$$-\frac{q(1+q\beta)}{\rho}\alpha_2=\frac{\widehat{\mathscr{P}_1}\tau_1}{2} \tag{29}$$
$$\frac{K\alpha_3-X\alpha_2^2}{\rho}=\frac{1}{2}\left[\left(\ell_2-\frac{\ell_1^2}{2}\right)\widehat{\mathscr{P}_1}+\frac{\ell_1^2}{2}\widehat{\mathscr{P}_2}\right] \tag{30}$$
$$\frac{2K-X}{\rho}\alpha_2^2-\frac{K}{\rho}\alpha_3=\frac{1}{2}\left[\left(\tau_2-\frac{\tau_1^2}{2}\right)\widehat{\mathscr{P}_1}+\frac{\tau_1^2}{2}\widehat{\mathscr{P}_2}\right] \tag{31}$$

From (28) and (29), we have

$$\ell_1 = -\tau_1 \quad \Longleftrightarrow \quad \ell_1^2 = \tau_1^2, \tag{32}$$

and

$$\alpha_2^2 = \frac{\rho^2 \vartheta_q^2}{8q^2(1+q\beta)^2}\big(\ell_1^2 + \tau_1^2\big) \quad \Longleftrightarrow \quad \ell_1^2 + \tau_1^2 = \frac{8q^2(1+q\beta)^2}{\rho^2 \vartheta_q^2}\alpha_2^2. \tag{33}$$

Now, by summing (30) and (31), we obtain

$$\begin{aligned}\frac{2(K-X)}{\rho}\alpha_2^2 &= \frac{1}{2}\left[(\ell_2+\tau_2)\widehat{\mathscr{P}_1} - \frac{\ell_1^2+\tau_1^2}{2}\widehat{\mathscr{P}_1} + \frac{\ell_1^2+\tau_1^2}{2}\widehat{\mathscr{P}_2}\right] \\ &= \frac{1}{2}\left[(\ell_2+\tau_2)\widehat{\mathscr{P}_1} + \frac{\ell_1^2+\tau_1^2}{2}(\widehat{\mathscr{P}_2} - \widehat{\mathscr{P}_1})\right].\end{aligned} \tag{34}$$

Since from the expansion of $\Upsilon(z;q)$ we have

$$\widehat{\mathscr{P}_1} = \vartheta_q, \qquad \widehat{\mathscr{P}_2} = (2q+1)\vartheta_q^2,$$

we can express

$$\widehat{\mathscr{P}_2} - \widehat{\mathscr{P}_1} = \vartheta_q\big((2q+1)\vartheta_q - 1\big).$$

Substituting these into (34) yields

$$\frac{2(K-X)}{\rho}\alpha_2^2 = \frac{\vartheta_q}{2}(\ell_2+\tau_2) + \left[\frac{(2q+1)\vartheta_q^2}{4} - \frac{\vartheta_q}{4}\right](\ell_1^2+\tau_1^2). \tag{35}$$

Now, substituting (33) into (35), we obtain

$$\begin{aligned}\frac{2(K-X)}{\rho}\alpha_2^2 &= \frac{\vartheta_q}{2}(\ell_2+\tau_2) + \left[\frac{(2q+1)\vartheta_q^2}{4} - \frac{\vartheta_q}{4}\right]\left(\frac{8q^2(1+q\beta)^2}{\rho^2\vartheta_q^2}\alpha_2^2\right) \\ &= \frac{\vartheta_q}{2}(\ell_2+\tau_2) + \frac{2q^2(1+q\beta)^2}{\rho^2}\left((2q+1) - \frac{1}{\vartheta_q}\right)\alpha_2^2.\end{aligned} \tag{36}$$

Rearranging terms, we get

$$\left[\frac{2(K-X)}{\rho} - \frac{2q^2(1+q\beta)^2}{\rho^2}\left((2q+1) - \frac{1}{\vartheta_q}\right)\right]\alpha_2^2 = \frac{\vartheta_q}{2}(\ell_2+\tau_2). \tag{37}$$

Finally, defining $C = q^2(1+q\beta)^2$, we can rewrite (37) as

$$\alpha_2^2 = \frac{(\ell_2+\tau_2)\,\rho^2\vartheta_q^2}{4\left[(K-X)\rho\vartheta_q + \big(1-(2q+1)\vartheta_q\big)C\right]}. \tag{38}$$

Since for functions $\psi, \nu \in \mathscr{P}$, we have $|\ell_2|, |\tau_2| \le 2$, it follows that $|\ell_2+\tau_2| \le 4$. Therefore, we obtain the bound

$$|\alpha_2| \le \frac{|\rho|\,|\vartheta_q|}{\sqrt{\left|\,(K-X)\rho\vartheta_q + \big(1-(2q+1)\vartheta_q\big)C\,\right|}}.$$

Now, to find the bound of $|\alpha_3|$, subtract (31) from (30). On the left-hand side we get

$$\frac{K\alpha_3 - X\alpha_2^2}{\rho} - \frac{(2K-X)\alpha_2^2 - K\alpha_3}{\rho} = \frac{2K}{\rho}\big(\alpha_3 - \alpha_2^2\big).$$

On the right-hand side,

$$\frac{1}{2}\left[\left(\ell_2 - \frac{\ell_1^2}{2}\right)\widehat{\mathscr{P}_1} + \frac{\ell_1^2}{2}\widehat{\mathscr{P}_2}\right] - \frac{1}{2}\left[\left(\tau_2 - \frac{\tau_1^2}{2}\right)\widehat{\mathscr{P}_1} + \frac{\tau_1^2}{2}\widehat{\mathscr{P}_2}\right] = \frac{1}{2}\left[(\ell_2-\tau_2)\widehat{\mathscr{P}_1} + \frac{\ell_1^2-\tau_1^2}{2}\left(\widehat{\mathscr{P}_2} - \widehat{\mathscr{P}_1}\right)\right].$$

Using (28)–(29) we have $\ell_1 = -\tau_1$, and hence $\ell_1^2 = \tau_1^2$ and the last term vanishes. Therefore,

$$\frac{2K}{\rho}\big(\alpha_3 - \alpha_2^2\big) = \frac{1}{2}\,(\ell_2 - \tau_2)\,\widehat{\mathscr{P}_1}.$$

Recalling $\widehat{\mathscr{P}_1} = \vartheta_q$, we obtain the identity

$$\alpha_3 = \alpha_2^2 + \frac{\rho\,\vartheta_q}{4K}\,(\ell_2 - \tau_2). \tag{39}$$

Taking absolute values and using the Carathéodory bounds $|\ell_2|, |\tau_2| \le 2$ (that is, $|\ell_2 - \tau_2| \le 4$), we get

$$|\alpha_3| \le |\alpha_2|^2 + \frac{|\rho|\,|\vartheta_q|}{K}. \tag{40}$$

Finally, substituting the estimate for $|\alpha_2|$ from (39) (or (38)) yields

$$|\alpha_3| \le \frac{|\rho|\,|\vartheta_q|\left\{\,\left|\,\rho\vartheta_q(K-X) + \big(1-(2q+1)\vartheta_q\big)C\,\right| + |\rho|\,|\vartheta_q|\,K\right\}}{K\,\left|\,\rho\vartheta_q(K-X) + \big(1-(2q+1)\vartheta_q\big)C\,\right|}.$$

Theorem 2. *For $\rho \in \mathbb{C}^*$ and $\beta \in [0,1]$, let $f \in \mathsf{SLM}_{\sum}(\beta, \rho; q)$. Then*

$$\left|\alpha_3 - \mu\alpha_2^2\right| \le \begin{cases} \frac{|\rho||\vartheta_q|}{K}, & \left|1-\mu\right| \le \frac{\left|\rho\vartheta_q(K-X)+(1-(2q+1)\vartheta_q)C\right|}{|\rho||\vartheta_q|K} \\ \frac{|1-\mu||\rho|^2|\vartheta_q|^2}{\left|(K-X)\rho\vartheta_q+(1-(2q+1)\vartheta_q)C\right|}, & \left|1-\mu\right| \ge \frac{\left|\rho\vartheta_q(K-X)+(1-(2q+1)\vartheta_q)C\right|}{|\rho||\vartheta_q|K} \end{cases} \tag{41}$$

where K, X, C are given by (19), (20) *and* (21), *respectively.*

Proof. Let $f \in \mathsf{SLM}_{\sum}(\beta, \rho; q)$, from (36) and (39) we have

$$\begin{aligned} \alpha_3 - \mu\alpha_2^2 &= \frac{(1-\mu)\rho^2\vartheta_q^2}{4\Big(\big(K-X\big)\rho\vartheta_q + \big(1-(2q+1)\vartheta_q\big)C\Big)}(\ell_2+\tau_2) + \frac{\rho\vartheta_q}{4K}\big(\ell_2-\tau_2\big) \\ &= \left(\mathscr{K}(\mu) + \frac{\rho\vartheta_q}{4K}\right)\ell_2 + \left(\mathscr{K}(\mu) - \frac{\rho\vartheta_q}{4K}\right)\tau_2, \end{aligned} \tag{42}$$

where

$$\mathscr{K}(\mu) = \frac{(1-\mu)\rho^2\vartheta_q^2}{4\Big(\big(K-X\big)\rho\vartheta_q + \big(1-(2q+1)\vartheta_q\big)C\Big)}. \tag{43}$$

Then, taking the modulus of (42), we conclude that

$$\left|\alpha_3 - \mu\alpha_2^2\right| \leq \begin{cases} \frac{|\rho||\vartheta_q|}{K}, & 0 \leq \left|\mathscr{K}(\mu)\right| \leq \frac{|\rho||\vartheta_q|}{4K} \\ 4\left|\mathscr{K}(\mu)\right|, & \left|\mathscr{K}(\mu)\right| \geq \frac{|\rho||\vartheta_q|}{4K} \end{cases}$$

If $\rho = 1$, we obtain the following results for the class $\mathsf{SLM}_{\sum}(\beta; q)$ defined in Example (1)

Corollary 1. *For $\rho \in \mathbb{C}^*$ and $\beta \in [0, 1]$, let $f \in \mathsf{SLM}_{\sum}(\beta, \rho; q)$. Then*

$$\begin{aligned} \left|\alpha_2\right| &\leq \frac{|\vartheta_q|}{\sqrt{\left|\vartheta_q(K-X) + \big(1-(2q+1)\vartheta_q\big)C\right|}}, \\ \left|\alpha_3\right| &\leq \frac{|\vartheta_q|\left\{\left|\big(K-X\big)\vartheta_q + \big(1-(2q+1)\vartheta_q\big)C\right| + |\vartheta_q|K\right\}}{K\left|\big(K-X\big)\vartheta_q + \big(1-(2q+1)\vartheta_q\big)C\right|}, \end{aligned}$$

and

$$\left|\alpha_3 - \mu\alpha_2^2\right| \leq \begin{cases} \frac{|\vartheta_q|}{K}, & \left|1-\mu\right| \leq \frac{\left|\vartheta_q(K-X)+(1-(2q+1)\vartheta_q)C\right|}{|\vartheta_q|K} \\ \frac{|1-\mu||\vartheta_q|^2}{\left|(K-X)\vartheta_q+(1-(2q+1)\vartheta_q)C\right|}, & \left|1-\mu\right| \geq \frac{\left|\vartheta_q(K-X)+(1-(2q+1)\vartheta_q)C\right|}{|\vartheta_q|K} \end{cases}$$

where K, X, C is given by (19), (20), *and* (21), *respectively.*

If $\beta = 0$ and $\rho = 1$, we obtain the following results for the class $\mathsf{SL}_{\sum}(\Upsilon(z; q))$ defined in Example (2)

Corollary 2. *[19] Let f given by* (1) *be in the class $\mathsf{SL}_{\sum}(\Upsilon(z); q)$. Then*

$$\left|\alpha_2\right| \leq \frac{\left|\vartheta_q\right|}{q\sqrt{1-2q\vartheta_q}}. \tag{44}$$

$$\left|\alpha_3\right| \le \frac{\left|\vartheta_q\right|\left(q-(1+q+2q^2)\vartheta_q\right)}{q^2(1+q)\left(1-2q\vartheta_q\right)}. \tag{45}$$

$$\left|\alpha_3-\mu\alpha_2^2\right| \le \begin{cases} \frac{|\vartheta_q|}{q(1+q)}, & \left|1-\mu\right| \le \frac{q\left(1-2q\vartheta_q\right)}{(1+q)|\vartheta_q|} \\ \frac{|1-\mu|\vartheta_q^2}{q^2\left(1-2q\vartheta_q\right)}, & \left|1-\mu\right| \ge \frac{q\left(1-2q\vartheta_q\right)}{(1+q)|\vartheta_q|} \end{cases} \tag{46}$$

If $\beta=1$ and $\rho=1$, we obtain the following results for the class $\mathsf{KL}_{\sum}(\Upsilon(z;q))$ defined in Example (3)

Corollary 3. *Let f given by* (1) *be in the class* $\mathsf{KL}_{\sum}(\Upsilon(z);q)$. *Then*

$$\left|\alpha_2\right| \le \frac{\left|\vartheta_q\right|}{\sqrt{\lceil 2\rfloor_q\left(\lceil 2\rfloor_q-\left(\lceil 3\rfloor_q+2q\right)\vartheta_q\right)}}$$

$$\left|\alpha_3\right| \le \frac{\left|\vartheta_q\right|\left(\lceil 2\rfloor_q-2(\lceil 3\rfloor_q+q)\vartheta_q\right)}{\lceil 2\rfloor_q\lceil 3\rfloor_q\left(\lceil 2\rfloor_q-\left(\lceil 3\rfloor_q+2q\right)\vartheta_q\right)},$$

and

$$\left|\alpha_3-£\alpha_2^2\right| \le \begin{cases} \frac{|\vartheta_q|}{\lceil 2\rfloor_q\lceil 3\rfloor_q}, & \left|1-£\right| \le \frac{\lceil 2\rfloor_q-\left(\lceil 3\rfloor_q+2q\right)\vartheta_q}{\lceil 3\rfloor_q|\vartheta_q|} \\ \frac{|1-£|\vartheta_q^2}{\lceil 2\rfloor_q\left(\lceil 2\rfloor_q-\left(\lceil 3\rfloor_q+2q\right)\vartheta_q\right)}, & \left|1-£\right| \ge \frac{\lceil 2\rfloor_q-\left(\lceil 3\rfloor_q+2q\right)\vartheta_q}{\lceil 3\rfloor_q|\vartheta_q|} \end{cases}$$

If $q\mapsto 1^-$ and $\rho=1$, we obtain the following results for the class $\mathsf{SLM}_{\sum}(\beta)$ defined in Example (4)

Corollary 4. *For $q\mapsto 1^-$ and $\rho=1$, let $f\in\mathsf{SLM}_{\sum}(\beta)$. Then*

$$\left|\alpha_2\right| \le \frac{|\vartheta|}{\sqrt{\left|\vartheta(K-X)+\left(1-3\vartheta\right)C\right|}},$$

$$\left|\alpha_3\right| \le \frac{|\vartheta|\left\{\left|\left(K-X\right)\vartheta+\left(1-3\vartheta\right)C\right|+|\vartheta|K\right\}}{K\left|\left(K-X\right)\vartheta+\left(1-3\vartheta\right)C\right|},$$

and

$$\left|\alpha_3-\mu\alpha_2^2\right| \le \begin{cases} \frac{|\vartheta|}{K}, & \left|1-\mu\right| \le \frac{\left|\vartheta(K-X)+(1-3\vartheta)C\right|}{|\vartheta|K} \\ \frac{|1-\mu||\vartheta|^2}{\left|(K-X)\vartheta+(1-3\vartheta)C\right|}, & \left|1-\mu\right| \ge \frac{\left|\vartheta(K-X)+(1-3\vartheta)C\right|}{|\vartheta|K} \end{cases}$$

where K, X, C are given by (19), (20) *and* (21), *respectively.*

If $q \mapsto 1^-$, $\beta = 0$ and $\rho = 1$, we obtain the following results for the class $\mathsf{SL}_{\sum}(\Upsilon(z))$ defined in Example (5)

Corollary 5. *[38] Let f given by* (1) *be in class* $\mathsf{SL}_{\sum}(\Upsilon(z))$. *Then*

$$\left|\alpha_2\right| \leq \frac{\left|\vartheta\right|}{\sqrt{1-2\vartheta}}, \qquad \left|\alpha_3\right| \leq \frac{\left|\vartheta\right|\left(1-4\vartheta\right)}{2\left(1-2\vartheta\right)}.$$

and

$$\left|\alpha_3 - \mu\alpha_2^2\right| \leq \begin{cases} \frac{|\vartheta|}{2}, & \left|1-\mu\right| \leq \frac{1-2\vartheta}{2|\vartheta|} \\ \frac{(1-\mu)\vartheta^2}{1-2\vartheta}, & \left|1-\mu\right| \geq \frac{1-2\vartheta}{2|\vartheta|} \end{cases}$$

If $q \mapsto 1^-$, $\beta = 1$ and $\rho = 1$, we obtain the following results for the class $\mathsf{KL}_{\sum}(\Upsilon(z))$ defined in Example (6)

Corollary 6. *[38] Let f given by* (1) *be in the class* $\mathsf{KL}_{\sum}(\Upsilon(z))$. *Then*

$$\left|\alpha_2\right| \leq \frac{\left|\vartheta\right|}{\sqrt{4-10\,\vartheta}}, \qquad \left|\alpha_3\right| \leq \frac{\left|\vartheta\right|\left(1-4\,\vartheta\right)}{3\left(1-2\,\vartheta\right)}.$$

and

$$\left|\alpha_3 - \pounds\alpha_2^2\right| \leq \begin{cases} \frac{|\vartheta|}{6}, & \left|1-\pounds\right| \leq \frac{2-5\vartheta}{3|\vartheta|} \\ \frac{|1-\pounds|\vartheta^2}{2\left(2-5\vartheta\right)}, & \left|1-\pounds\right| \geq \frac{2-5\vartheta}{3|\vartheta|} \end{cases}$$

4. Conclusion

In this paper, we study new subclasses of complex order bi-univalent functions that are associated with shell-like curves through the subordination principle and the use of the q-analogue of Fibonacci numbers. Motivated by recent developments in the q-calculus and its fruitful applications in geometric function theory, we construct and analyze two distinct families of analytic and bi-univalent functions. For these families, we establish coefficient estimates for the initial Taylor–Maclaurin terms and we derive sharp bounds for the Fekete–Szegö functional in terms of the relevant parameters. The results obtained in this investigation not only extend and generalize several previous contributions in the theory of bi-univalent functions, but also provide new insights into the interplay between bi-univalent function theory, the q-Fibonacci numbers and the shell-like geometries. Furthermore, the subclasses introduced here may serve as a useful platform for future research in analytic function spaces, special functions, and their associated operator-theoretic properties.

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