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Analytic Subclass Conditions for Operators Involving the Generalized Imaginary Error Function

A. Alameer[1,*], Tariq Al-Hawary[2,*], Basem Aref Frasin[3], Feras Yousef[4]

[1] *Department of Mathematics, University of Hafr Al-Batin, Hafr Al Batin 31991, Saudi Arabia*
[2] *Department of Applied Science, Ajloun College, Al-Balqa Applied University, Ajloun 26816, Jordan*
[3] *Faculty of Science, Department of Mathematics, Al al-Bayt University, Mafraq, Jordan*
[4] *Department of Mathematics, The University of Jordan, Amman 11942, Jordan*

Abstract. In this paper, we establish necessary and sufficient criteria for the generalized imaginary error function $\varpi i_n(z)$ to belong to the class $\mathbf{F}(\hbar_1, \hbar_2)$. We also derive an inclusion relation between the classes $\mathbf{G}^{\tau}(A_1, A_2)$ and $\mathbf{F}(\hbar_1, \hbar_2)$. Furthermore, we provide a necessary and sufficient condition for the integral operator $\mathcal{G}i_m(z) := \int_0^z \frac{\varpi i_n(z)}{t} dt$ to lie in the class $\mathbf{F}(\hbar_1, \hbar_2)$.

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1. Introduction and preliminaries

Let AN indicate the class of analytic functions of the form

$$g(z) = z + \sum_{\rho=2}^{\infty} a_\rho z^\rho; \quad z \in \Delta = \{z \in \mathbb{C} : |z| < 1\}. \tag{1}$$

Also, consider $\mathcal{S}$ to be the subclass of AN consisting of functions which are univalent in Δ.

According to Kamali et al. [1], let $\mathbf{F}(\hbar_1, \hbar_2)$ be a subclass of AN, containing the functions of the form (1) that satisfy the following analytical requirement:

$$\operatorname{Re}\left(\frac{\hbar_2 z^3 g'''(z) + (2\hbar_2 + 1) z^2 g''(z) + z g'(z)}{\hbar_2 z^2 g''(z) + z g'(z)}\right) > \hbar_1; \quad (\hbar_1, \hbar_2 \in [0, 1), \ \ |z| < 1).$$

*Corresponding author.
*Corresponding author.
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Email addresses: `aamalameer@uhb.edu.sa` (A. Alameer), `tariq_amh@bau.edu.jo` (T. Al-Hawary), `bafrasin@yahoo.com` (B. A. Frasin), `fyousef@ju.edu.jo` (F. Yousef)

Example 1. [2] *For some* $\hbar_1 \in [0,1)$*, by choosing* $\hbar_2 = 0$ *and taking* $g(z)$ *of the form* (1)*, let the subclass* $\mathbf{F}(\hbar_1, 0) \equiv \mathcal{K}(\hbar_1)$ *consists of functions in* AN *that satisfy the inequality:*

$$Re\left(\frac{zg''(z)}{g'(z)} + 1\right) > \hbar_1; \quad |z| < 1.$$

Also, if $\hbar_1 = \hbar_2 = 0$*, the subclass* $\mathbf{F}(0,0) \equiv \mathcal{K}(0) \equiv \mathcal{K}$ *consists of functions in* AN *satisfying the inequality:*

$$Re\left(\frac{zg''(z)}{g'(z)} + 1\right) > 0; \quad |z| < 1.$$

Both subclasses $\mathcal{K}(\hbar_1)$ and $\mathcal{K}$ are well known subclasses of convex functions of order $\hbar_1$ and convex functions (convex functions of order zero), respectively (see [2]).

For functions of the previously described subclass, we require the following sufficient and necessary conditions.

Lemma 1. [1] *A function* $g \in \mathbf{F}(\hbar_1, \hbar_2)$ *if and only if*

$$\sum_{\rho=2}^{\infty} \rho(\rho - \hbar_1)\left(\hbar_2(\rho - 1) + 1\right)|a_\rho| \le 1 - \hbar_1. \tag{2}$$

Definition 1. [3] *A function* $g \in AN$ *is said to be in the class* $\mathbf{G}^\tau(A_1, A_2)$*,* $\tau \in \mathbb{C}\backslash\{0\}$*,* $-1 \le A_2 < A_1 \le 1$*, if it satisfies the inequality:*

$$\left|\frac{g'(z) - 1}{(A_1 - A_2)\tau - A_2[g'(z) - 1]}\right| < 1; \qquad z \in \Delta.$$

Special functions are widely acknowledged as being crucial to geometric function theory and are employed in numerous applications in applied mathematics, statistics, engineering, and physics [4–8]. Because these functions are widely used, many academics are interested in obtaining the geometric features of special functions; see [9–15].

Abramowitz and Stegun [16] defined the following error function

$$erg\,(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \sum_{\rho=0}^{\infty} \frac{(-1)^\rho\, z^{2\rho+1}}{(2\rho+1)\,\rho!}, \qquad (z \in \mathbb{C}), \tag{3}$$

whereas the imaginary error function

$$ergi\,(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{t^2} dt = \frac{2}{\sqrt{\pi}} \sum_{\rho=0}^{\infty} \frac{z^{2\rho+1}}{(2\rho+1)\,\rho!}, \qquad (z \in \mathbb{C}). \tag{4}$$

The physics of partial differential equations, probability theory, statistics, and applied mathematics make extensive use of the error function. The error function is crucial to estimating the likelihood of seeing a particle in a given area in quantum mechanics. Alzer [17] and Coman [18] demonstrated several characteristics and inequalities of the error function, whereas Elbert et. al [19] studied the properties of complementary error function.

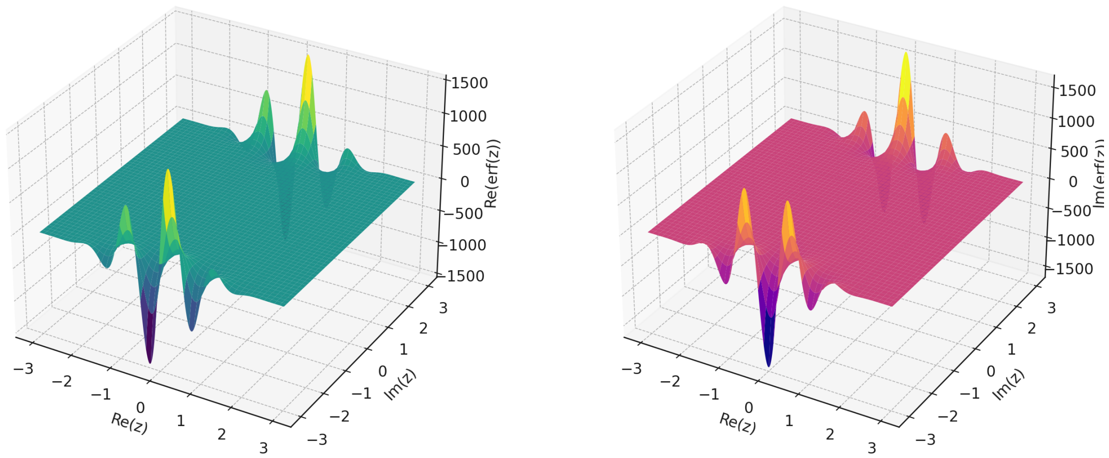

Figure 1: Real (left) and imaginary (right) parts of the error function $erq\,(z)$ over the complex plane.

The behavior of the real and imaginary components of $erg\,(z)$ in the complex plane is depicted in Figure 1 below. Its relevance in the geometric characterization of subclasses of analytic functions is motivated by the rich geometric structure it displays, including symmetry and curvature.

A generalization of the error function (3) is given by

$$\begin{aligned} erg_n\,(z) &= \frac{n!}{\sqrt{\pi}} \int_0^z e^{-t^n} dt, \quad n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \\ &= \frac{n!}{\sqrt{\pi}} \sum_{\rho=0}^{\infty} \frac{(-1)^{\rho} z^{n\rho+1}}{(n\rho+1)\,\rho!}, \qquad (z \in \mathbb{C}). \end{aligned} \tag{5}$$

In addition, a generalization of the imaginary error function (4) is given by

$$\begin{aligned} ergi_n\,(z) &= \frac{n!}{\sqrt{\pi}} \int_0^z e^{t^n} dt, \quad n \in \mathbb{N}_0 \\ &= \frac{n!}{\sqrt{\pi}} \sum_{\rho=0}^{\infty} \frac{z^{n\rho+1}}{(n\rho+1)\,\rho!}, \qquad (z \in \mathbb{C}). \end{aligned} \tag{6}$$

From (5) and (6), we get

$$erg_0\,(z) = \frac{z}{e\sqrt{\pi}}, erg_1\,(z) = \frac{1-e^z}{\sqrt{\pi}} = -ergi_1\,(z)\,, erg_2\,(z) = erg\,(z) \text{ and } ergi_2\,(z) = ergi\,(z)\,.$$

It is clear that the functions $erg_n\,(z)$ and $ergi_n\,(z)$ are not members of the class AN. Thus, it is natural to consider the following functions given by Al-Hawary et. al [20]

$$\mathcal{E}_n\,(z) = \frac{\sqrt{\pi}}{n!} z_n^{\left(1-\frac{1}{n}\right)} erg_n\left(z^{1/n}\right) = z + \sum_{\rho=2}^{\infty} \frac{(-1)^{\rho-1}}{((\rho-1)\,n+1)\,(\rho-1)!} z^{\rho}, \qquad (n \in \mathbb{N}), \tag{7}$$

and

$$\mathcal{E}i_n\left(z\right)=\frac{\sqrt{\pi}}{n!}z^{\left(1-\frac{1}{n}\right)}\operatorname{erf}i_n\left(z^{1/n}\right)=z+\sum_{\rho=2}^{\infty}\frac{1}{\left(\left(\rho-1\right)n+1\right)\left(\rho-1\right)!}z^{\rho},\qquad\left(n\in\mathbb{N}\right).\quad(8)$$

Specifically, we get the normalizations $\mathcal{E}_2 g = Erg$ and $\mathcal{E}i_2 g = Ergi$ given by Ramachandran et al. [21] and is the normalization given by Mohammed et al. [22].

From (7) and (8), we have

$$\mathcal{E}_1 g\left(z\right)=\sqrt{\pi}erg_1\left(z\right)=1-e^{z},\mathcal{E}i_1 g\left(z\right)=\sqrt{\pi}ergi_1\left(z\right)=e^{z}-1$$

and

$$\mathcal{E}_2 g\left(z\right)=\frac{\sqrt{\pi z}}{2}erg_2\left(\sqrt{z}\right)\text{ and }\mathcal{E}i_1 g\left(z\right)=\frac{\sqrt{\pi z}}{2}ergi_2\left(\sqrt{z}\right).$$

Let the function $\varpi i_n\left(z\right)$ be defined as

$$\varpi i_n\left(z\right)=2z-\mathcal{E}i_n\left(z\right)=z-\sum_{\rho=2}^{\infty}\frac{1}{\left(\left(\rho-1\right)n+1\right)\left(\rho-1\right)!}z^{\rho};\qquad z\in\Delta,\quad(9)$$

and consider the linear operator

$$\mathcal{I}i_n: AN\to AN$$

defined by the convolution or Hadamard product

$$\mathcal{I}i_n(z)=\mathcal{E}i_n\left(z\right)*g(z)=z+\sum_{\rho=2}^{\infty}\frac{1}{\left(\left(\rho-1\right)n+1\right)\left(\rho-1\right)!}a_\rho z^{\rho}.\quad(10)$$

Recently, several academics determined the necessary and sufficient conditions using Rabotnov functions [23], generalized Bessel functions [24, 25], Struve functions [26, 27], hypergeometric functions [28], Pascal distribution series [29, 30], Poisson distribution series [31], normalized Wright functions [32], ad Touchard polynomials [33].

The remainder of the document is structured as follows. In Section 2, we derive the necessary and sufficient conditions for the function ϖi_n to be in the class $\mathbf{F}(\hbar_1,\hbar_2)$. In Section 3, we will study the action of the function $\mathcal{I}i_n(z)$ on the class $\mathbf{F}(\hbar_1,\hbar_2)$. Finally, in Section 4, we give a necessary and sufficient condition for an integral operator $\mathcal{G}i_n(z):=\int_0^z\frac{\varpi i_n(t)}{t}dt$ to be in the class $\mathbf{F}(\hbar_1,\hbar_2)$.

We recall the following lemma, which will be helpful in derive our result in Section 3.

Lemma 2. [3] *If $g\in\mathbf{G}^{\tau}(A_1,A_2)$ is of the form* (1)*, then*

$$|a_\rho|\leq(A_1-A_2)\frac{|\tau|}{\rho};\qquad\rho\in\mathbb{N}-\{1\}.$$

The result is sharp.

We employ the following well-known series sums throughout the sequel.

$$\sum_{\rho=2}^{\infty} \frac{1}{(\rho-1)\,2^{\rho}} = \frac{1}{2}\ln 2, \tag{11}$$

$$\sum_{\rho=3}^{\infty} \frac{1}{2^{\rho}\,(\rho-1)} = \frac{1}{2}\ln 2 - \frac{1}{4}, \tag{12}$$

and

$$\sum_{\rho=4}^{\infty} \frac{1}{2^{\rho}\,(\rho-1)} = \frac{1}{2}\ln 2 - \frac{1}{4} - \frac{1}{16}. \tag{13}$$

Note that

$$\sum_{\rho=j}^{\infty} \frac{1}{(\rho-1)\,2^{\rho}} = \frac{1}{2}\ln 2 - \sum_{\rho=2}^{j-1} \frac{1}{(\rho-1)\,2^{\rho}},\ j = 3, 4, \cdots. \tag{14}$$

In addition, we need the following inequalities.

$$(\rho-1)n + 1 > (\rho-1)n \quad (\rho, n \in \mathbb{N}) \tag{15}$$

and

$$\rho! \geq 2^{\rho-1} \quad (\rho \in \mathbb{N}). \tag{16}$$

2. Necessary and sufficient conditions

In this section, we give the necessary and sufficient conditions for the function ϖi_n to be in the class $\mathbf{F}(\hbar_1, \hbar_2)$.

Theorem 1. *If* $\hbar_1, \hbar_2 \in [0, 1)$ *and* $n \in \mathbb{N}$, *then* $\varpi i_n(z)$ *is in the class* $\mathbf{F}(\hbar_1, \hbar_2)$ *if and only if*

$$[72\hbar_2 - 16\hbar_1\hbar_2 - 6\hbar_1 + 22]\ln 2 - [30\hbar_2 + 4(1 - \hbar_1\hbar_2)] \leq n\,(1 - \hbar_1). \tag{17}$$

Proof. Since

$$\varpi i_n(z) = z - \sum_{\rho=2}^{\infty} \frac{1}{((\rho-1)\,n+1)\,(\rho-1)!} z^{\rho},$$

by virtue of Lemma 1 it suffices to show that $L(\gamma, \beta) \leq 1 - \hbar_1$, where

$$L(\hbar_1, \hbar_2) := \sum_{\rho=2}^{\infty} \rho(\rho - \hbar_1)\,(\hbar_2(\rho-1)+1)\ \frac{1}{((\rho-1)\,n+1)\,(\rho-1)!}.$$

Writing

$$\rho = (\rho-1) + 1, \tag{18}$$

$$\rho^2 = (\rho-1)(\rho-2) + 3(\rho-1) + 1, \tag{19}$$

$$\rho^3 = (\rho-1)(\rho-2)(\rho-3) + 6(\rho-1)(\rho-2) + 7(\rho-1) + 1, \tag{20}$$

we get

$$\begin{aligned} L(\hbar_1,\hbar_2) &= \sum_{\rho=2}^{\infty}[\hbar_2\rho^3 + (1-\hbar_2(\hbar_1+1))\rho^2 + \hbar_1(\hbar_2-1))\rho]\frac{1}{((\rho-1)\,n+1)\,(\rho-1)!} \\ &= \hbar_2\sum_{\rho=2}^{\infty}(\rho-1)(\rho-2)(\rho-3)\frac{1}{((\rho-1)\,n+1)\,(\rho-1)!} \\ &+ (\hbar_2(5-\hbar_1)+1)\sum_{\rho=2}^{\infty}(\rho-1)(\rho-2)\frac{1}{((\rho-1)\,n+1)\,(\rho-1)!} \\ &+ (2\hbar_2(2-\hbar_1)+3-\hbar_1)\ \sum_{\rho=2}^{\infty}(\rho-1)\frac{1}{((\rho-1)\,n+1)\,(\rho-1)!} \\ &+ (1-\hbar_1)\sum_{\rho=2}^{\infty}\frac{1}{((\rho-1)\,n+1)\,(\rho-1)!} \end{aligned}$$

$$\begin{aligned} &= \hbar_2\sum_{\rho=4}^{\infty}\frac{1}{((\rho-1)\,n+1)\,(\rho-4)!} + (\hbar_2(5-\hbar_1)+1)\ \sum_{\rho=3}^{\infty}\frac{1}{((\rho-1)\,n+1)\,(\rho-3)!} \\ &+ (2\hbar_2(2-\hbar_1)+3-\hbar_1)\sum_{\rho=2}^{\infty}\frac{1}{((\rho-1)\,n+1)\,(\rho-2)!} + (1-\hbar_1)\sum_{\rho=2}^{\infty}\frac{1}{((\rho-1)\,n+1)\,(\rho-1)!}. \end{aligned}$$

By (15), we get

$$\begin{aligned} L(\hbar_1,\hbar_2) \le \frac{\hbar_2}{n}\sum_{\rho=4}^{\infty}\frac{1}{(\rho-1)\,(\rho-4)!} + \frac{\hbar_2(5-\hbar_1)+1}{n}\ \sum_{\rho=3}^{\infty}\frac{1}{(\rho-1)\,(\rho-3)!} \\ + \frac{2\hbar_2(2-\hbar_1)+3-\hbar_1}{n}\sum_{\rho=2}^{\infty}\frac{1}{(\rho-1)\,(\rho-2)!} + \frac{1-\hbar_1}{n}\sum_{\rho=2}^{\infty}\frac{1}{(\rho-1)\,(\rho-1)!}. \end{aligned}$$

By (16), we get

$$\begin{aligned} L(\hbar_1,\hbar_2) \le \frac{32\hbar_2}{n}\sum_{\rho=4}^{\infty}\frac{1}{(\rho-1)\,2^\rho} + \frac{16\,(\hbar_2(5-\hbar_1)+1)}{n}\ \sum_{\rho=3}^{\infty}\frac{1}{(\rho-1)\,2^\rho} \\ + \frac{8\,(2\hbar_2(2-\hbar_1)+3-\hbar_1)}{n}\ \sum_{\rho=2}^{\infty}\frac{1}{(\rho-1)\,2^\rho} + \frac{4\,(1-\hbar_1)}{n}\sum_{\rho=2}^{\infty}\frac{1}{(\rho-1)\,2^\rho}. \end{aligned}$$

Using the series sums (11), (12) and (13), we get

$$L(\hbar_1,\hbar_2) = \frac{\hbar_2}{n}\,(16\ln 2 - 8 - 2) + \frac{\hbar_2(5-\hbar_1)+1}{n}\,(8\ln 2 - 4)$$

$$
\begin{aligned}
&+\frac{2\hbar_2(2-\hbar_1)+3-\hbar_1}{n}(4\ln 2)+\frac{1-\hbar_1}{n}(2\ln 2)\\
&=\left(\frac{72\hbar_2-16\hbar_1\hbar_2-6\hbar_1+22}{n}\right)\ln 2-\frac{30\hbar_2+4(1-\hbar_1\hbar_2)}{n}.
\end{aligned}
$$

But this expression is bounded above by $1-\hbar_1$ if and only if (17) holds.

3. An inclusion property

Making use of Lemma 2, we will study the action of the function $\mathcal{I}i_n(z)$ on the class $\mathbf{F}(\hbar_1, \hbar_2)$.

Theorem 2. *Let* $\hbar_1, \hbar_2 \in [0, 1)$ *and* $n \in \mathbb{N}$. *If* $g \in \mathbf{G}^{\tau}(A_1, A_2)$, *then* $\mathcal{I}i_n(z)$ *is in the class* $\mathbf{F}(\hbar_1, \hbar_2)$ *if*

$$
(8\hbar_2-2\hbar_1\hbar_2-\hbar_1+3)\ln 2-2\hbar_2 \le \frac{n\,(1-\gamma)}{2(A_1-A_2)|\tau|}. \tag{21}
$$

Proof. In view of Lemma 1, it suffices to show that

$$
M(\hbar_1,\hbar_2):=\sum_{\rho=2}^{\infty}\rho(\rho-\hbar_1)\,(\hbar_2(\rho-1)+1)\,\frac{1}{((\rho-1)\,n+1)\,(\rho-1)!}\,|a_\rho| \le 1-\gamma.
$$

Since $g \in \mathbf{G}^{\tau}(A_1, A_2)$, then by Lemma 2, we have

$$
|a_\rho| \le \frac{(A_1-A_2)\,|\tau|}{\rho}. \tag{22}
$$

Thus, we have

$$
\begin{aligned}
M(\hbar_1,\hbar_2)&=\sum_{\rho=2}^{\infty}[\hbar_2\rho^3+(1-\hbar_2(\hbar_1+1))\rho^2\ +\hbar_1(\hbar_2-1))\rho]\frac{1}{((\rho-1)\,n+1)\,(\rho-1)!}\,|a_\rho|\\
&\le (A_1-A_2)|\tau|\sum_{\rho=2}^{\infty}[\hbar_2\rho^2+(1-\hbar_2(\hbar_1+1))\rho\ +\hbar_1(\hbar_2-1))]\frac{1}{((\rho-1)\,n+1)\,(\rho-1)!}.
\end{aligned}
$$

By (18) and (19), we get

$$
\begin{aligned}
M(\hbar_1,\hbar_2)&\le (A_1-A_2)|\tau|\sum_{\rho=2}^{\infty}[\hbar_2(\rho-1)(\rho-2)+(\hbar_2(2-\hbar_1)+1)(\rho-1)\ +1-\hbar_1]\\
&\quad\times\frac{1}{((\rho-1)\,n+1)\,(\rho-1)!}\\
&=(A_1-A_2)|\tau|\left(\sum_{\rho=3}^{\infty}\frac{\hbar_2}{((\rho-1)\,n+1)\,(\rho-3)!}+\sum_{\rho=2}^{\infty}\frac{\hbar_2(2-\hbar_1)+1}{((\rho-1)\,n+1)\,(\rho-2)!}\right.
\end{aligned}
$$

$$+\sum_{\rho=2}^{\infty}\frac{1-\hbar_1}{((\rho-1)\,n+1)\rho!}\Bigg).$$

By (15) and (16), we get

$$M(\hbar_1,\hbar_2)\leq\frac{(A_1-A_2)|\tau|}{n}\left(16\sum_{\rho=3}^{\infty}\frac{\hbar_2}{(\rho-1)\,2^{\rho}}+8\sum_{\rho=2}^{\infty}\frac{\hbar_2(2-\hbar_1)+1}{(\rho-1)\,2^{\rho}}+4\sum_{\rho=2}^{\infty}\frac{1-\hbar_1}{(\rho-1)\,2^{\rho}}\right).$$

By (11) and (12), we get

$$M(\hbar_1,\hbar_2)\leq\frac{(A_1-A_2)|\tau|}{n}\left[2\left(8\hbar_2-2\hbar_1\hbar_2-\hbar_1+3\right)\ln 2-4\hbar_2\right].$$

But this last expression is bounded by $1-\hbar_1$, if (21) holds.

4. An integral operator

Theorem 3. *Let* $\hbar_1,\hbar_2\in[0,1)$ *and* $n\in\mathbb{N}$. *The integral operator*

$$\mathcal{G}i_n(z):=\int_0^z\frac{\varpi i_n(t)}{t}dt,\ z\in\Delta, \tag{23}$$

is in the class $\mathbf{F}(\hbar_1,\hbar_2)$ *if and only if the inequality*

$$\left(8\hbar_2-2\hbar_1\hbar_2-\hbar_1+3\right)\ln 2-2\hbar_2\leq\frac{n\,(1-\gamma)}{2} \tag{24}$$

holds.

Proof. According to (9) it follows that

$$\mathcal{G}i_n(z)=z-\sum_{\rho=2}^{\infty}\frac{1}{((\rho-1)\,n+1)\,(\rho-1)!}\frac{z^{\rho}}{\rho},\ z\in\Delta.$$

Using Lemma 1, the function $\mathcal{G}i_n(z)$ belongs to $\mathbf{F}(\hbar_1,\hbar_2)$ if and only if

$$\sum_{\rho=2}^{\infty}[\rho(\rho-\hbar_1)\,(\hbar_2(\rho-1)+1)]\frac{1}{\rho((\rho-1)\,n+1)\,(\rho-1)!}\leq 1-\hbar_1.$$

By a similar proof of Theorem 2 we get that $\mathcal{G}i_n\in\mathbf{F}(\hbar_1,\hbar_2)$ if and only if (24) holds.

We obtain many corollaries by specializing the parameters $\hbar_1$ and $\hbar_2$ in our theorems, for example, if $\hbar_2=0$, we get the following corollary.

Corollary 1. *If* $n\in\mathbb{N}$ *and* $\hbar_1\in[0,1)$, *then*

(i) $\varpi i_n(z) \in \mathcal{K}(\hbar_1)$ if and only if

$$[22 - 6\hbar_1] \ln 2 - 4 \leq n(1 - \hbar_1).$$

(ii) For $g \in \mathbf{G}^{\tau}(A_1, A_2)$, $\mathcal{I}i_n(z) \in \mathcal{K}(\hbar_1)$ if and only if

$$(3 - \hbar_1) \ln 2 \leq \frac{n(1 - \hbar_1)}{2(A_1 - A_2)|\tau|}.$$

(iii) $\mathcal{G}i_n(z) \in \mathcal{K}(\hbar_1)$ if and only if

$$(3 - \hbar_1) \ln 2 \leq \frac{n(1 - \hbar_1)}{2}.$$

5. Conclusion

For the generalized normalized imaginary error function ϖi_n to belong to the class $\mathbf{F}(\hbar_1, \hbar_2)$ of analytic functions defined on the open unit disk Δ, we establish necessary and sufficient criteria. We also investigate the action of the function $\mathcal{I}i_n(z)$ on the class $\mathbf{F}(\hbar_1, \hbar_2)$. Furthermore, we derive a necessary and sufficient condition for the integral operator $\mathcal{G}i_n(z)$ to lie in the same class. This study may further motivate researchers to derive new criteria under which the generalized normalized imaginary error function ϖi_n belongs to other families of analytic functions defined on Δ.

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