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On Interval-Valued Intuitionistic Fuzzy Subalgebras of Sheffer Stroke Hilbert Algebras

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Abstract. This paper explores the structure of interval-valued intuitionistic fuzzy (IVIF) subsets within the framework of Sheffer stroke Hilbert algebras (SSHAs). After establishing the foundational definitions of interval-valued intuitionistic fuzzy Sheffer stroke subalgebras (IVIFSS-subalgebras), we investigate their algebraic properties, closure under Sheffer stroke operations, and stability under set-theoretic intersections and unions. A key result characterizes IVIFSS-subalgebras through a pair of membership conditions, and it is further shown that the level subsets corresponding to IVIF-degrees form classical subalgebras in the crisp setting. These findings demonstrate that IVIF extensions preserve core algebraic behaviors while offering a robust model for uncertainty. The results contribute to the ongoing generalization of fuzzy algebraic systems and lay the groundwork for further developments involving fuzzy ideals and logical applications.

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Key Words and Phrases: Sheffer stroke Hilbert algebra, interval-valued intuitionistic fuzzy set, interval-valued intuitionistic fuzzy Sheffer stroke subalgebra

1. Introduction

Sheffer stroke Hilbert algebras (SSHAs) are essential algebraic structures that play a significant role in logical systems and Boolean algebras. They serve as a foundation

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for representing logical operations and have broad applications across fields like formal logic, artificial intelligence, and quantum computing. The Sheffer stroke operation, typically defined by the NAND operation, is central to these algebras and is fundamental in constructing more advanced logical systems.

The application of the Sheffer stroke in various algebraic structures has been thoroughly explored, making significant contributions to multiple fields of study. Research has focused on Sheffer stroke reducts within basic algebras [1, 2], Sheffer stroke MTL-algebras [3], and ortholattices [4]. Additionally, the role of fuzzy sets in Sheffer stroke BE-algebras [5] has been investigated, underscoring the far-reaching effects of the Sheffer stroke operation across diverse algebraic systems.

The concept of fuzzy sets, proposed by Zadeh [6], revolutionized the study of uncertainty and vagueness, providing a powerful tool for modeling imprecision. Fuzzy set theory has since been applied in a variety of real-world contexts, sparking considerable research into its expansion. Following the introduction of fuzzy sets, numerous studies focused on generalizing the theory. One key direction has been the integration of fuzzy sets with other uncertainty models, such as soft sets and rough sets, as explored in various works [7–9]. Another significant extension of fuzzy sets is the notion of intuitionistic fuzzy sets, introduced by Atanassov [10]. These sets enhance the applicability of fuzzy sets by incorporating both membership and non-membership degrees, making them suitable for situations involving incomplete information or uncertainty. Intuitionistic fuzzy sets have been widely applied in domains such as medical diagnostics, optimization, and multi-criteria decision-making [11–13].

Hilbert algebras were initially introduced in the 1950s by Henkin [14] as a means to investigate implications within intuitionistic and other non-classical logics. In the 1960s, scholars such as Horn and Diego further developed these algebras from an algebraic standpoint. Diego demonstrated that Hilbert algebras form a locally finite variety [15]. Researchers such as Busneag [16, 17] and Jun [18] also contributed to the study of Hilbert algebras, focusing on the role of their filters in forming deductive systems. In the context of fuzzy logic, Dudek [19] explored the fuzzification of subalgebras and deductive systems in Hilbert algebras, adding a new layer of complexity to these structures.

The algebraic framework of SSHAs has evolved considerably over the past few years, establishing itself as a fertile ground for generalizations in fuzzy and soft logic. The foundational connection between Sheffer stroke operations and Hilbert algebras was rigorously explored by Oner et al. [20], leading to a formal algebraic structure that integrates logical minimalism with algebraic expressiveness. This foundation has since been extended in several directions. For instance, stabilization properties via ideals have been analyzed in [21] to explore internal algebraic regularities. The incorporation of fuzzy logic began with fuzzy filters [22], followed by the introduction of fuzzy ideals [23], and then generalized fuzzy subalgebras [24]. Parallel developments include investigations into length-based and mean-fuzzy structures [25, 26], as well as soft set extensions using $\mathcal{N}$-structures [27]. Collectively, these contributions not only demonstrate the flexibility of the SSHA model but also set the stage for more refined fuzzy generalizations, such as those based on intuitionistic and interval-valued constructs.

Despite the broad utilization of SSHAs, their connection with fuzzy logic and intuitionistic fuzzy set theory remains largely unexplored. The introduction of interval-valued intuitionistic fuzzy sets (IVIFSs) offers a novel approach to representing uncertainty and partial membership in algebraic structures. By replacing fixed values with intervals to denote membership and non-membership degrees, IVIFS provide a more flexible approach to dealing with vagueness and uncertainty.

The paper introduces the concept of fuzzy and intuitionistic fuzzy sets in algebraic structures, particularly within the context of Sheffer-styled Hilbert algebras. It emphasizes the importance of fuzzy subsets in capturing uncertainty and partial membership in algebraic systems. The study then focuses explicitly on interval-valued intuitionistic fuzzy (IVIF) subsets, providing fundamental definitions and properties. Section 3 elaborates on these IVIF subsets, examining how they can be characterized as subalgebras and their behavior under various set operations. The section aims to establish a solid theoretical foundation for integrating fuzzy logic principles into algebraic structures, thereby paving the way for future research and applications.

The list of acronyms is given in Table 1.

Table 1: List of acronyms

Acronyms	Representation
SSHA	Sheffer stroke Hilbert algebra
IFS	intuitionistic fuzzy set
IVIFS	interval-valued intuitionistic fuzzy set
IVIFSS-subalgebra	interval-valued intuitionistic fuzzy Sheffer stroke subalgebra

2. Preliminaries

In this section, to deepen the understanding of the foundational structures introduced earlier, we will elaborate on the key properties and relationships of the relevant algebraic and fuzzy constructs. Specifically, we will define certain operations and structures, and examine their characteristics and interconnections. This groundwork will ensure clarity and provide a solid theoretical basis for the subsequent sections. Our aim is to present these fundamental concepts and structures in a clear and precise manner, establishing the essential building blocks for the developments that follow.

Definition 1. *[28] Let $\mathscr{A} := (A, |)$ be a groupoid. Then the operation $|$ is said to be Sheffer stroke or Sheffer operation if it satisfies:*

(s1) $(\forall \mathfrak{d}, \mathfrak{g} \in A)\ (\mathfrak{d} \mid \mathfrak{g} = \mathfrak{g} \mid \mathfrak{d})$,

(s2) $(\forall \mathfrak{d}, \mathfrak{g} \in A)\ ((\mathfrak{d} \mid \mathfrak{d}) \mid (\mathfrak{d} \mid \mathfrak{g}) = \mathfrak{d})$,

(s3) $(\forall \mathfrak{d}, \mathfrak{g}, \mathfrak{w} \in A)\ (\mathfrak{d} \mid ((\mathfrak{g} \mid \mathfrak{w}) \mid (\mathfrak{g} \mid \mathfrak{w})) = ((\mathfrak{d} \mid \mathfrak{g}) \mid (\mathfrak{d} \mid \mathfrak{g})) \mid \mathfrak{w})$, (s4) $(\forall \mathfrak{d}, \mathfrak{g}, \mathfrak{w} \in A)$ $((\mathfrak{d} \mid ((\mathfrak{d} \mid \mathfrak{d}) \mid (\mathfrak{g} \mid \mathfrak{g}))) \mid (\mathfrak{d} \mid ((\mathfrak{d} \mid \mathfrak{d}) \mid (\mathfrak{g} \mid \mathfrak{g}))) = \mathfrak{d})$.

To improve the clarity of this manuscript, we introduce the following notation, which will be used consistently throughout the text:

$$\mathfrak{w} \mid (\mathfrak{g} \mid \mathfrak{g}) := \mathfrak{w}^{\mathfrak{g}}.$$

Definition 2. *[20] A Sheffer stroke Hilbert algebra (SSHA) is a groupoid $S_H := (S_H, |, 0)$ equipped with a Sheffer stroke operation that satisfies specific conditions:*

(sH1) $(\mathfrak{d} \mid (\mathfrak{g}^{\mathfrak{w}} \mid \mathfrak{g}^{\mathfrak{w}})) \mid ((\mathfrak{d}^{\mathfrak{g}} \mid (\mathfrak{d}^{\mathfrak{w}} \mid \mathfrak{d}^{\mathfrak{w}})) \mid (\mathfrak{d}^{\mathfrak{g}} \mid (\mathfrak{d}^{\mathfrak{w}} \mid \mathfrak{d}^{\mathfrak{w}}))) = \mathfrak{d}^{\mathfrak{d}}$,

(sH2) $\mathfrak{d}^{\mathfrak{g}} = \mathfrak{g}^{\mathfrak{d}} = \mathfrak{d}^{\mathfrak{d}} \ \Rightarrow \ \mathfrak{d} = \mathfrak{g}$

for all $\mathfrak{d}, \mathfrak{g}, \mathfrak{w} \in S_H$.

Proposition 1. *[20] Let $S_H := (S_H, |, 0)$ be an SSHA. Then the binary relation*

$$\mathfrak{d} \leq \mathfrak{g} \ \Leftrightarrow \ \mathfrak{d}^{\mathfrak{g}} = 0$$

is a partial order on S_H.

Definition 3. *[20] Let $S_H := (S_H, |, 0)$ be an SSHA. A nonempty subset A of S_H is called a subalgebra of S_H if $\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}} \in A$ for all $\mathfrak{d}, \mathfrak{g} \in A$.*

Definition 4. *[10] Let X be a nonempty set. The intuitionistic fuzzy set (IFS) on X is defined as a structure*

$$A := \{\langle \mathfrak{d}, \mu_A(\mathfrak{d}), \gamma_A(\mathfrak{d})\rangle \mid \mathfrak{d} \in X\}, \tag{1}$$

where $\mu_A : X \to [0,1]$ represents the degree of membership of $\mathfrak{d}$ in A, and $\gamma_A : X \to [0,1]$ represents the degree of non-membership of $\mathfrak{d}$ in A such that

$$0 \leq \mu_A(\mathfrak{d}) + \gamma_A(\mathfrak{d}) \leq 1.$$

The IFS in (1) is simply denoted by $A = (\mu_A, \gamma_A)$.

Let $\mathscr{D}[0,1]$ be the set of all closed subintervals of the interval $[0,1]$. Consider $I_1, I_2 \in \mathscr{D}[0,1]$. If $I_1 = [\mathfrak{m}_1, \mathfrak{n}_1]$ and $I_2 = [\mathfrak{m}_2, \mathfrak{n}_2]$, then

$$\mathrm{rmin}\{I_1, I_2\} = [\min\{\mathfrak{m}_1, \mathfrak{m}_2\}, \min\{\mathfrak{n}_1, \mathfrak{n}_2\}]$$

and

$$\mathrm{rmax}\{I_1, I_2\} = [\max\{\mathfrak{m}_1, \mathfrak{m}_2\}, \max\{\mathfrak{n}_1, \mathfrak{n}_2\}].$$

Thus, if $I_{\mathfrak{i}} = [\mathfrak{m}_{\mathfrak{i}}, \mathfrak{n}_{\mathfrak{i}}] \in \mathscr{D}[0,1]$ for $\mathfrak{i} = 1, 2, \ldots$, then we define

$$\mathrm{rsup}_{\mathfrak{i}}\{I_{\mathfrak{i}}\} = [\sup_{\mathfrak{i}}\{\mathfrak{m}_{\mathfrak{i}}\}, \sup_{\mathfrak{i}}\{\mathfrak{n}_{\mathfrak{i}}\}]$$

and

$$\mathrm{rinf}_{\mathfrak{i}}\{I_{\mathfrak{i}}\} = [\inf_{\mathfrak{i}}\{\mathfrak{m}_{\mathfrak{i}}\}, \inf_{\mathfrak{i}}\{\mathfrak{n}_{\mathfrak{i}}\}].$$

Now, we call $I_1 \geq I_2$ if and only if $\mathfrak{m}_1 \geq \mathfrak{m}_2$ and $\mathfrak{n}_1 \leq \mathfrak{n}_2$. Similarly, the relations $I_1 \leq I_2$ and $I_1 = I_2$ are defined.

Definition 5. *An interval-valued intuitionistic fuzzy set (IVIFS) A over a universe X is defined as an object of the form*

$$A = \{\langle \mathfrak{d}, \mu_A(\mathfrak{d}), \gamma_A(\mathfrak{d})\rangle \mid \mathfrak{d} \in X\},$$

where $\mu_A(\mathfrak{d}) : X \to \mathscr{D}[0,1]$ *and* $\gamma_A(\mathfrak{d}) : X \to \mathscr{D}[0,1]$. *The functions* $\mu_A(\mathfrak{d})$ *and* $\gamma_A(\mathfrak{d})$ *represent the intervals of the degree of membership and non-membership of the element* $\mathfrak{d}$ *in A, respectively, where* $\mu_A(\mathfrak{d}) = [\mu^{\mathfrak{l}}_A(\mathfrak{d}), \mu^{\mathfrak{u}}_A(\mathfrak{d})]$ *and* $\gamma_A(\mathfrak{d}) = [\gamma^{\mathfrak{l}}_A(\mathfrak{d}), \gamma^{\mathfrak{u}}_A(\mathfrak{d})]$ *for all* $\mathfrak{d} \in X$, *subject to condition* $0 \leq \mu^{\mathfrak{l}}_A(\mathfrak{d}) + \gamma^{\mathfrak{u}}_A(\mathfrak{d}) \leq 1$. *For simplicity, we denote the IVIFS A as* $A = (\mu_A, \gamma_A)$, *where* $A = \{\langle \mathfrak{d}, \mu_A(\mathfrak{d}), \gamma_A(\mathfrak{d})\rangle \mid \mathfrak{d} \in X\}$. *Additionally, the complements of* μ_A *and* γ_A *are given by* $\overline{\mu_A}(\mathfrak{d}) = [1 - \mu^{\mathfrak{u}}_A(\mathfrak{d}), 1 - \mu^{\mathfrak{l}}_A(\mathfrak{d})]$ *and* $\overline{\gamma_A}(\mathfrak{d}) = [1 - \gamma^{\mathfrak{u}}_A(\mathfrak{d}), 1 - \gamma^{\mathfrak{l}}_A(\mathfrak{d})]$, *where* $[\overline{\mu_A}(\mathfrak{d}), \overline{\gamma_A}(\mathfrak{d})]$, *represents the complement of* $\mathfrak{d}$ *in A.*

3. Interval-valued intuitionistic fuzzy Sheffer stroke subalgebras

The integration of IVIFSs with SSHAs provides a fertile ground for generalizing classical algebraic structures under uncertainty. While intuitionistic fuzzy frameworks have been extensively applied to various algebraic systems, their interval-valued extensions remain relatively underexplored within the context of SSHA. This motivates a deeper investigation into the algebraic behavior of such fuzzy structures.

In this section, we introduce the concept of interval-valued intuitionistic fuzzy Sheffer stroke subalgebras (IVIFSS-subalgebras) and examine their fundamental properties. The aim is to establish criteria under which IVIF subsets of an SSHA preserve subalgebraic structure with respect to the Sheffer operation. Special attention is given to their closure properties, interaction under set-theoretic operations, and logical coherence. These results not only enrich the theory of fuzzy algebraic systems but also offer tools for modeling imprecise reasoning in logic and computational intelligence.

Definition 6. *Let* $S_H := (S_H, |, 0)$ *be an SSHA. An IVIFS* $A = (\mu_A, \gamma_A)$ *in* S_H *is called an interval-valued intuitionistic fuzzy Sheffer stroke subalgebra (IVIFSS-subalgebra) of* S_H *if*

$$(\forall \mathfrak{d}, \mathfrak{g} \in S_H) \left(\begin{array}{l} \mu_A(\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}}) \geq \text{rmin}\{\mu_A(\mathfrak{d}), \mu_A(\mathfrak{g})\} \\ \gamma_A(\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}}) \leq \text{rmax}\{\gamma_A(\mathfrak{d}), \gamma_A(\mathfrak{g})\} \end{array} \right). \tag{2}$$

Proposition 2. *Let* $S_H := (S_H, |, 0)$ *be an SSHA. Every IVIFSS-subalgebra* $A = (\mu_A, \gamma_A)$ *of* S_H *satisfies*

$$(\forall \mathfrak{d} \in S_H) \left(\begin{array}{l} \mu_A(0) \geq \mu_A(\mathfrak{d}) \\ \gamma_A(0) \leq \gamma_A(\mathfrak{d}) \end{array} \right). \tag{3}$$

Proof. For any $\mathfrak{d} \in S_H$, we have

$$\begin{array}{rcl} \mu_A(0) & = & \mu_A(\mathfrak{d}^{\mathfrak{d}} \mid \mathfrak{d}^{\mathfrak{d}}) \\ & \geq & \text{rmin}\{\mu_A(\mathfrak{d}), \mu_A(\mathfrak{d})\} \\ & = & \text{rmin}\{[\mu^{\mathfrak{l}}_A(\mathfrak{d}), \mu^{\mathfrak{u}}_A(\mathfrak{d})], [\mu^{\mathfrak{l}}_A(\mathfrak{d}), \mu^{\mathfrak{u}}_A(\mathfrak{d})]\} \\ & = & [\mu^{l}_A(\mathfrak{d}), \mu^{\mathfrak{u}}_A(\mathfrak{d})] \\ & = & \mu_A(\mathfrak{d}) \end{array}$$

and

$$\begin{array}{rcl}\gamma_A(0) & = & \gamma_A(\mathfrak{d}^{\mathfrak{d}} \mid \mathfrak{d}^{\mathfrak{d}}) \\ & \leq & \mathrm{rmax}\{\gamma_A(\mathfrak{d}), \gamma_A(\mathfrak{d})\} \\ & = & \mathrm{rmax}\{[\gamma_A^{\mathfrak{l}}(\mathfrak{d}), \gamma_A^{\mathfrak{u}}(\mathfrak{d})], [\gamma_A^{l}(\mathfrak{d}), \gamma_A^{\mathfrak{u}}(\mathfrak{d})]\} \\ & = & [\gamma_A^{\mathfrak{l}}(\mathfrak{d}), \gamma_A^{\mathfrak{u}}(\mathfrak{d})] \\ & = & \gamma_A(\mathfrak{d}).\end{array}$$

Proposition 3. *Let $S_H := (S_H, |, 0)$ be an SSHA. Every IVIFSS-subalgebra $A = (\mu_A, \gamma_A)$ of S_H satisfies the following:*

$$(\forall \mathfrak{d}, \mathfrak{g} \in S_H) \left(\begin{array}{c} \mu_A(\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}}) \geq \mu_A(\mathfrak{g}) \\ \gamma_A(\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}}) \leq \gamma_A(\mathfrak{g}) \end{array} \right) \tag{4}$$

if and only if $\mu_A(0) = \mu_A(\mathfrak{d})$ and $\gamma_A(0) = \gamma_A(\mathfrak{d})$ for all $\mathfrak{d} \in S_H$.

Proof. Let $\mathfrak{d} \in S_H$. Then

$$\begin{array}{rcl}\mu_A(\mathfrak{d}) & = & \mu_A((\mathfrak{d}|0)|(\mathfrak{d}|0)) \\ & = & \mu_A(\mathfrak{d}^0 \mid \mathfrak{d}^0) \\ & \geq & \mu_A(0)\end{array}$$

and

$$\begin{array}{rcl}\gamma_A(\mathfrak{d}) & = & \gamma_A((\mathfrak{d}|0)|(\mathfrak{d}|0)) \\ & = & \gamma_A(\mathfrak{d}^0 \mid \mathfrak{d}^0) \\ & \leq & \gamma_A(0).\end{array}$$

Then by Proposition 2, $\mu_A(0) = \mu_A(\mathfrak{d})$ and $\gamma_A(0) = \gamma_A(\mathfrak{d})$. The converse is clear.

Theorem 1. *Let $S_H := (S_H, |, 0)$ be an SSHA. An IVIFS $A = ([\mu_A^{\mathfrak{l}}, \mu_A^{\mathfrak{u}}], [\gamma_A^{\mathfrak{l}}, \gamma_A^{\mathfrak{u}}])$ in S_H is an IVIFSS-subalgebra of S_H if and only if $\mu_A^{\mathfrak{l}}, \mu_A^{\mathfrak{u}}, \gamma_A^{\mathfrak{l}}$ and $\gamma_A^{\mathfrak{u}}$ are fuzzy subalgebras of S_H.*

Proof. Let $\mu_A^{\mathfrak{l}}$ and $\mu_A^{\mathfrak{u}}$ be fuzzy subalgebras of S_H and $\mathfrak{d}, \mathfrak{g} \in S_H$. Then

$$\mu_A^{l}(\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}}) \geq \min\{\mu_A^{\mathfrak{l}}(\mathfrak{d}), \mu_A^{\mathfrak{l}}(\mathfrak{g})\}$$

and

$$\mu_A^{\mathfrak{u}}(\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}}) \leq \min\{\mu_A^{\mathfrak{u}}(\mathfrak{d}), \mu_A^{\mathfrak{u}}(\mathfrak{g})\}.$$

Now,

$$\begin{array}{rcl}\mu_A(\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}}) & = & [\mu_A^{\mathfrak{l}}(\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}}), \mu_A^{\mathfrak{u}}(\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}})] \\ & \geq & [\min\{\mu_A^{\mathfrak{l}}(\mathfrak{d}), \mu_A^{\mathfrak{l}}(\mathfrak{g})\}, \min\{\mu_A^{\mathfrak{u}}(\mathfrak{d}), \mu_A^{\mathfrak{u}}(\mathfrak{g})\}] \\ & = & \mathrm{rmin}\{[\mu_A^{\mathfrak{l}}(\mathfrak{d}), \mu_A^{\mathfrak{u}}(\mathfrak{d})], [\mu_A^{\mathfrak{l}}(\mathfrak{g}), \mu_A^{\mathfrak{u}}(\mathfrak{g})]\} \\ & = & \mathrm{rmin}\{\mu_A(\mathfrak{d}), \mu_A(\mathfrak{g})\}.\end{array}$$

Let $\gamma_A^{\mathfrak{l}}$ and $\gamma_A^{\mathfrak{u}}$ be fuzzy subalgebras of S_H and $\mathfrak{d}, \mathfrak{g} \in S_H$. Then $\gamma_A^{\mathfrak{l}}(\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}}) \leq \max\{\gamma_A^{\mathfrak{l}}(\mathfrak{d}), \gamma_A^{\mathfrak{l}}(\mathfrak{g})\}$ and $\gamma_A^{\mathfrak{u}}(\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}}) \leq \max\{\gamma_A^{\mathfrak{u}}(\mathfrak{d}), \gamma_A^{\mathfrak{u}}(\mathfrak{g})\}$. Now,

$$\begin{aligned} \gamma_A(\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}}) &= [\gamma_A^{\mathfrak{l}}(\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}}), \gamma_A^{\mathfrak{u}}(\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}})] \\ &\leq [\max\{\gamma_A^{\mathfrak{l}}(\mathfrak{d}), \gamma_A^{\mathfrak{l}}(\mathfrak{g})\}, \max\{\gamma_A^{\mathfrak{u}}(\mathfrak{d}), \gamma_A^{\mathfrak{u}}(\mathfrak{g})\}] \\ &= \mathrm{rmax}\{[\gamma_A^{\mathfrak{l}}(\mathfrak{d}), \gamma_A^{\mathfrak{u}}(\mathfrak{d})], [\gamma_A^{\mathfrak{l}}(\mathfrak{g}), \gamma_A^{\mathfrak{u}}(\mathfrak{g})]\} \\ &= \mathrm{rmax}\{\gamma_A(\mathfrak{d}), \gamma_A(\mathfrak{g})\}. \end{aligned}$$

Hence, $A = ([\mu_A^{\mathfrak{l}}, \mu_A^{\mathfrak{u}}], [\gamma_A^{\mathfrak{l}}, \gamma_A^{\mathfrak{u}}])$ is an IVIFSS-subalgebra of S_H.

Conversely, assume that $A = ([\mu_A^{\mathfrak{l}}, \mu_A^{\mathfrak{u}}], [\gamma_A^{\mathfrak{l}}, \gamma_A^{\mathfrak{u}}])$ is an IVIFSS-subalgebra of S_H. For any $\mathfrak{d}, \mathfrak{g} \in S_H$,

$$\begin{aligned} [\mu_A^{\mathfrak{l}}(\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}}), \mu_A^{\mathfrak{u}}(\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}})] &= \mu_A(\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}}) \\ &\geq \mathrm{rmin}\{\mu_A(\mathfrak{d}), \mu_A(\mathfrak{g})\} \\ &= \mathrm{rmin}\{[\mu_A^{\mathfrak{l}}(\mathfrak{d}), \mu_A^{\mathfrak{u}}(\mathfrak{d})], [\mu_A^{l}(\mathfrak{g}), \mu_A^{\mathfrak{u}}(\mathfrak{g})]\} \\ &= [\min\{\mu_A^{\mathfrak{l}}(\mathfrak{d}), \mu_A^{l}(\mathfrak{g})\}, \min\{\mu_A^{\mathfrak{u}}(\mathfrak{d}), \mu_A^{\mathfrak{u}}(\mathfrak{g})\}], \end{aligned}$$

and

$$\begin{aligned} [\gamma_A^{\mathfrak{l}}(\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}}), \gamma_A^{\mathfrak{u}}(\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}})] &= \gamma_A(\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}}) \\ &\leq \mathrm{rmax}\{\gamma_A(\mathfrak{d}), \gamma_A(\mathfrak{g})\} \\ &= \mathrm{rmax}\{[\gamma_A^{\mathfrak{l}}(\mathfrak{d}), \gamma_A^{\mathfrak{u}}(\mathfrak{d})], [\gamma_A^{\mathfrak{l}}(\mathfrak{g}), \gamma_A^{\mathfrak{u}}(\mathfrak{g})]\} \\ &= [\max\{\gamma_A^{\mathfrak{l}}(\mathfrak{d}), \gamma_A^{\mathfrak{l}}(\mathfrak{g})\}, \max\{\gamma_A^{\mathfrak{u}}(\mathfrak{d}), \gamma_A^{\mathfrak{u}}(\mathfrak{g})\}]. \end{aligned}$$

Thus, $\mu_A^{\mathfrak{l}}(\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}}) \geq \min\{\mu_A^{\mathfrak{l}}(\mathfrak{d}), \mu_A^{\mathfrak{l}}(\mathfrak{g})\}, \mu_A^{\mathfrak{u}}(\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}}) \geq \min\{\mu_A^{\mathfrak{u}}(\mathfrak{d}), \mu_A^{\mathfrak{u}}(\mathfrak{g})\}, \gamma_A^{\mathfrak{l}}(\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}}) \leq \max\{\gamma_A^{\mathfrak{l}}(\mathfrak{d}), \gamma_A^{\mathfrak{l}}(\mathfrak{g})\}$ and $\gamma_A^{\mathfrak{u}}(\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}}) \leq \max\{\gamma_A^{\mathfrak{u}}(\mathfrak{d}), \gamma_A^{\mathfrak{u}}(\mathfrak{g})\}$. Therefore, $\mu_A^{\mathfrak{l}}, \mu_A^{\mathfrak{u}}, \gamma_A^{l}$ and $\gamma_A^{\mathfrak{u}}$ are fuzzy subalgebras of S_H.

Theorem 2. *Let $S_H := (S_H, |, 0)$ be an SSHA. If $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ are IVIFSS-subalgebras of S_H, then $A \cap B = (\mu_{A\cap B}, \gamma_{A\cup B})$ is an IVIFSS-subalgebra of S_H.*

Proof. Let $\mathfrak{d}, \mathfrak{g} \in A \cap B$. Since $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ are IVIFSS-subalgebras of S_H,

$$\begin{aligned} \mu_{A\cap B}(\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}}) &= [\mu_{A\cap B}^{\mathfrak{l}}(\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}}), \mu_{A\cap B}^{\mathfrak{u}}(\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}})] \\ &= [\min\{\mu_A^{\mathfrak{l}}(\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}}), \mu_B^{\mathfrak{l}}(\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}})\}, \min\{\mu_A^{\mathfrak{u}}(\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}}), \mu_B^{\mathfrak{u}}(\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}})\}] \\ &\geq [\min\{\mu_{A\cap B}^{\mathfrak{l}}(\mathfrak{d}), \mu_{A\cap B}^{\mathfrak{l}}(\mathfrak{g}))\}, \min\{\mu_{A\cap B}^{\mathfrak{u}}(\mathfrak{d}), \mu_{A\cap B}^{\mathfrak{u}}(\mathfrak{g}))\}] \\ &= \mathrm{rmin}\{\mu_{A\cap B}(\mathfrak{d}), \mu_{A\cap B}(\mathfrak{g})\} \end{aligned}$$

and

$$\begin{aligned} \gamma_{A\cup B}(\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}}) &= [\gamma_{A\cup B}^{\mathfrak{l}}(\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}}), \gamma_{A\cup B}^{\mathfrak{u}}(\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}})] \\ &= [\max\{\gamma_A^{\mathfrak{l}}(\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}}), \gamma_B^{\mathfrak{l}}(\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}})\}, \max\{\gamma_A^{\mathfrak{u}}(\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}}), \gamma_B^{\mathfrak{u}}(\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}})\}] \\ &\leq [\max\{\gamma_{A\cup B}^{\mathfrak{l}}(\mathfrak{d}), \gamma_{A\cup B}^{\mathfrak{l}}(\mathfrak{g})\}, \max\{\gamma_{A\cup B}^{\mathfrak{u}}(\mathfrak{d}), \gamma_{A\cup B}^{\mathfrak{u}}(\mathfrak{g})\}] \\ &= \mathrm{rmax}\{\gamma_{A\cup B}(\mathfrak{d}), \gamma_{A\cup B}(\mathfrak{g})\}. \end{aligned}$$

Hence, $A \cap B = (\mu_{A\cap B}, \gamma_{A\cup B})$ is an IVIFSS-subalgebra of S_H.

Definition 7. *Let $S_H := (S_H, |, 0)$ be an SSHA. Let $A = (\mu_A, \gamma_A)$ be an IVIFS defined on S_H. The operators $\oplus A$ and $\otimes A$ are defined as*

$$\oplus A = \{\langle \mathfrak{d}, \mu_A(\mathfrak{d}), \overline{\mu_A}(\mathfrak{d})\rangle \mid \mathfrak{d} \in S_H\}$$

and

$$\otimes A = \{\langle \mathfrak{d}, \overline{\gamma_A}(\mathfrak{d}), \gamma_A(\mathfrak{d})\rangle \mid \mathfrak{d} \in S_H\}.$$

Theorem 3. *Let $S_H := (S_H, |, 0)$ be an SSHA. If $A = (\mu_A, \gamma_A)$ is an IVIFSS-subalgebra of S_H, then $\oplus A$ and $\otimes A$ are both IVIFSS-subalgebras.*

Proof. Let $\mathfrak{d}, \mathfrak{g} \in S_H$. Then

$$\begin{array}{rcl} \mu_A(\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}}) & = & [1,1] - \mu_A(\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}}) \\ & \leq & [1,1] - \text{rmin}\{\mu_A(\mathfrak{d}), \mu_A(\mathfrak{g})\} \\ & = & \text{rmax}\{1 - \mu_A(\mathfrak{d}), 1 - \mu_A(\mathfrak{g})\} \\ & = & \text{rmax}\{\mu_A(\mathfrak{d}), \mu_A(\mathfrak{g})\}. \end{array}$$

Hence, $\oplus A$ is an IVIFSS-subalgebra of S_H. Let $\mathfrak{d}, \mathfrak{g} \in S_H$. Then

$$\begin{array}{rcl} \gamma_A(\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}}) & = & [1,1] - \gamma_A(\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}}) \\ & \geq & [1,1] - \text{rmax}\{\gamma_A(\mathfrak{d}), \gamma_A(\mathfrak{g})\} \\ & = & \text{rmin}\{1 - \gamma_A(\mathfrak{d}), 1 - \gamma_A(\mathfrak{g})\} \\ & = & \text{rmin}\{\gamma_A(\mathfrak{d}), \gamma_A(\mathfrak{g})\}. \end{array}$$

Hence, $\otimes A$ is an IVIFSS-subalgebra of S_H.

The sets $\{\mathfrak{d} \in S_H \mid \mu_A(\mathfrak{d}) = \mu_A(0)\}$ and $\{\mathfrak{d} \in S_H \mid \gamma_A(\mathfrak{d}) = \gamma_A(0)\}$ are denoted by μ_A^* and γ_A^*, respectively.

Theorem 4. *Let $S_H := (S_H, |, 0)$ be an SSHA. Let $A = (\mu_A, \gamma_A)$ be an IVIFSS-subalgebra of S_H, then the sets μ_A^* and γ_A^* are subalgebras of S_H.*

Proof. Let $\mathfrak{d}, \mathfrak{g} \in \mu_A^*$. Then $\mu_A(\mathfrak{d}) = \mu_A(0) = \mu_A(\mathfrak{g})$ and so $\mu_A(\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}}) \leq \text{rmin}\{\mu_A(\mathfrak{d}), \mu_A(\mathfrak{g})\} = \mu_A(0)$. By using Proposition 2, we have $\mu_A(\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}}) = \mu_A(0)$ and hence $\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}} \in \mu_A^*$. Again, let $\mathfrak{d}, \mathfrak{g} \in \gamma_A^*$. Then $\gamma_A(\mathfrak{d}) = \gamma_A(0) = \gamma_A(\mathfrak{g})$ and so $\gamma_A(\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}}) \leq \text{rmax}\{\gamma_A(\mathfrak{d}), \gamma_A(\mathfrak{g})\} = \gamma_A(0)$. Again, by Proposition 2, we hece $\gamma_A(x^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}}) = \gamma_A(0)$; hence $\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}} \in \gamma_A^*$. Therefore, the sets μ_A^* and γ_A^* are subalgebras of S_H.

Theorem 5. *Let $S_H := (S_H, |, 0)$ be an SSHA. Let B be a nonempty subset of S_H and $A = (\mu_A, \gamma_A)$ be an IVIFS in S_H defined by*

$$\mu_A(\mathfrak{d}) = \begin{cases} [\vartheta_1, \vartheta_2] & \text{if } \mathfrak{d} \in B \\ [\eta_1, \eta_2] & \text{otherwise} \end{cases}$$

and

$$\gamma_A(\mathfrak{d}) = \begin{cases} [\varpi_1, \varpi_2] & \text{if } \mathfrak{d} \in B \\ [\varsigma_1, \varsigma_2] & \text{otherwise} \end{cases}$$

for all $[\vartheta_1, \vartheta_2], [\eta_1, \eta_2], [\varpi_1, \varpi_2], [\delta_1, \delta_2] \in \mathscr{D}[0,1]$ with $[\vartheta_1, \vartheta_2] \geq [\eta_1, \eta_2]$ and $[\varpi_1, \varpi_2] \leq [\varsigma_1, \varsigma_2]$ and $\vartheta_2 + \varpi_2 \leq 1$ and $\eta_2 + \delta_2 \leq 1$. Then $A = (\mu_A, \gamma_A)$ is an IVIFSS-subalgebra of S_H if and only if B is a subalgebra of S_H. Moreover, $\mu_A^* = B = \gamma_A^*$.

Proof. Let $A = (\mu_A, \gamma_A)$ be an IVIFSS-subalgebra of S_H. Let $\mathfrak{d}, \mathfrak{g} \in S_H$ be such that $\mathfrak{d}, \mathfrak{g} \in B$. Then $\mu_A(\mathfrak{d}^\mathfrak{g} \mid \mathfrak{d}^\mathfrak{g}) \geq \text{rmin}\{\mu_A(\mathfrak{d}), \mu_A(\mathfrak{g})\} = \text{rmin}\{[\vartheta_1, \vartheta_2], [\vartheta_1, \vartheta_2]\} = [\vartheta_1, \vartheta_2]$ and $\gamma_A(\mathfrak{d}^\mathfrak{g} \mid \mathfrak{d}^\mathfrak{g}) \leq \text{rmax}\{\gamma_A(\mathfrak{d}), \gamma_A(\mathfrak{g})\} = \text{rmax}\{[\vartheta_1, \vartheta_2], [\vartheta_1, \vartheta_2]\} = [\vartheta_1, \vartheta_2]$. So, $\mathfrak{d}^\mathfrak{g} \mid \mathfrak{d}^\mathfrak{g} \in B$. Hence, B is a subalgebra of S_H.

Conversely, suppose that B is a subalgebra of S_H. Let $\mathfrak{d}, \mathfrak{g} \in S_H$. Consider two cases:

Case (i): If $\mathfrak{d}, \mathfrak{g} \in B$, then $\mathfrak{d}^\mathfrak{g} \mid \mathfrak{d}^\mathfrak{g} \in B$. Thus,

$$\mu_A(\mathfrak{d}^\mathfrak{g} \mid \mathfrak{d}^\mathfrak{g}) = [\varpi_1, \varpi_2] = \text{rmin}\{\mu_A(\mathfrak{d}), \mu_A(\mathfrak{g})\}$$

and

$$\gamma_A(\mathfrak{d}^\mathfrak{g} \mid \mathfrak{d}^\mathfrak{g}) = [\theta_1, \theta_2] = \text{rmax}\{\gamma_A(\mathfrak{d}), \gamma_A(\mathfrak{g})\}.$$

Case (ii): If $\mathfrak{d} \notin B$ or $\mathfrak{g} \notin B$, then

$$\mu_A(\mathfrak{d}^\mathfrak{g} \mid \mathfrak{d}^\mathfrak{g}) \geq [\eta_1, \eta_2] = \text{rmin}\{\mu_A(\mathfrak{d}), \mu_A(\mathfrak{g})\}$$

and

$$\gamma_A(\mathfrak{d}^\mathfrak{g} \mid \mathfrak{d}^\mathfrak{g}) \leq [\theta_1, \theta_2] = \text{rmax}\{\gamma_A(\mathfrak{d}), \gamma_A(\mathfrak{g})\}.$$

Hence, $A = (\mu_A, \gamma_A)$ is an IVIFSS-subalgebra of S_H. Now, $\mu_A^* = \{\mathfrak{d} \in S_H \mid \mu_A(\mathfrak{d}) = \mu_A(0)\} = \{\mathfrak{d} \in S_H \mid \mu_A(\mathfrak{d}) = [\vartheta_1, \vartheta_2]\} = B$ and $\gamma_A^* = \{\mathfrak{d} \in S_H \mid \gamma_A(\mathfrak{d}) = \gamma_A(0)\} = \{\mathfrak{d} \in S_H \mid \gamma_A(\mathfrak{d}) = [\varpi_1, \varpi_2]\} = B$.

To confirm that an IVIFSS-subalgebra aligns with the underlying crisp structure, we examine its level subsets. Specifically, we define α- and β-level sets and verify that these subsets form classical subalgebras of the SSHA. This ensures that the fuzzy extension preserves the essential algebraic properties.

Definition 8. *Let $S_H := (S_H, |, 0)$ be an SSHA. Let $A = (\mu_A, \gamma_A)$ is an IVIFSS-subalgebra of S_H. For $[\mathfrak{p}_1, \mathfrak{p}_2], [\mathfrak{q}_1, \mathfrak{q}_2] \in \mathscr{D}[0,1]$, the set $\mathscr{U}(\mu_A : [\mathfrak{p}_1, \mathfrak{p}_2]) = \{\mathfrak{d} \in S_H \mid \mu_A(\mathfrak{d}) \geq [\mathfrak{p}_1, \mathfrak{p}_2]\}$ is called an upper $[\mathfrak{p}_1, \mathfrak{p}_2]$-level of A and $\mathscr{L}(\gamma_A : [\mathfrak{q}_1, \mathfrak{q}_2]) = \{\mathfrak{d} \in S_H \mid \gamma_A(\mathfrak{d}) \leq [\mathfrak{q}_1, \mathfrak{q}_2]\}$ is called a lower $[\mathfrak{q}_1, \mathfrak{q}_2]$-level of A.*

Theorem 6. *Let $S_H := (S_H, |, 0)$ be an SSHA. If $A = (\mu_A, \gamma_A)$ is an IVIFSS-subalgebra of S_H, then the upper $[\mathfrak{p}_1, \mathfrak{p}_2]$-level and lower $[\mathfrak{q}_1, \mathfrak{q}_2]$-level of A are subalgebras of S_H.*

Proof. Let $\mathfrak{d}, \mathfrak{g} \in \mathscr{U}(\mu_A : [s_1, s_2])$. Then $\mu_A(\mathfrak{d}) \leq [\mathfrak{p}_1, \mathfrak{p}_2]$ and $\mu_A(\mathfrak{g}) \leq [\mathfrak{p}_1, \mathfrak{p}_2]$. It follows that $\mu_A(\mathfrak{d}^\mathfrak{g} \mid \mathfrak{d}^\mathfrak{g}) \leq \text{rmin}\{\mu_A(\mathfrak{d}), \mu_A(\mathfrak{g})\} \leq [\mathfrak{p}_1, \mathfrak{p}_2]$ so that $\mathfrak{d}^\mathfrak{g} \mid \mathfrak{d}^\mathfrak{g} \in \mathscr{U}(\mu_A : [\mathfrak{p}_1, \mathfrak{p}_2])$. Hence, $\mathscr{U}(\mu_A : [\mathfrak{p}_1, \mathfrak{p}_2])$ is a subalgebra of S_H. Let $\mathfrak{d}, \mathfrak{g} \in \mathscr{L}(\gamma_A : [\mathfrak{q}_1, \mathfrak{q}_2])$. Then $\gamma_A(\mathfrak{d}) \leq [\mathfrak{q}_1, \mathfrak{q}_2]$ and $\gamma_A(\mathfrak{g}) \leq [\mathfrak{q}_1, \mathfrak{q}_2]$. It follows that $\gamma_A(\mathfrak{d}^\mathfrak{g} \mid \mathfrak{d}^\mathfrak{g}) \leq \text{rmax}\{\gamma_A(\mathfrak{d}), \gamma_A(\mathfrak{g})\} \leq [\mathfrak{q}_1, \mathfrak{q}_2]$ so that $\mathfrak{d}^\mathfrak{g} \mid \mathfrak{d}^\mathfrak{g} \in \mathscr{L}(\gamma_A : [\mathfrak{q}_1, \mathfrak{q}_2])$. Hence, $\mathscr{L}(\gamma_A : [\mathfrak{q}_1, \mathfrak{q}_2])$ is a subalgebra of S_H.

Theorem 7. *Let $S_H := (S_H, |, 0)$ be an SSHA. Let $A = (\mu_A, \gamma_A)$ be an IVIFS in S_H such that the sets $\mathscr{U}(\mu_A : [\mathfrak{p}_1, \mathfrak{p}_2])$ and $\mathscr{L}(\gamma_A : [\mathfrak{q}_1, \mathfrak{q}_2])$ are subalgebras of S_H for every $[\mathfrak{p}_1, \mathfrak{p}_2], [\mathfrak{q}_1, \mathfrak{q}_2] \in \mathscr{D}[0,1]$. Then $A = (\mu_A, \gamma_A)$ is an IVIFSS-subalgebra of S_H.*

Proof. Let for every $[\mathfrak{p}_1, \mathfrak{p}_2], [\mathfrak{q}_1, \mathfrak{q}_2] \in \mathscr{D}[0,1]$, $\mathscr{U}(\mu_A : [\mathfrak{p}_1, \mathfrak{p}_2])$ and $\mathscr{L}(\gamma_A : [\mathfrak{q}_1, \mathfrak{q}_2])$ are subalgebras of S_H. In contrary, let $\mathfrak{d}_0, \mathfrak{g}_0 \in S_H$ be such that $\mu_A(\mathfrak{d}_0^{\mathfrak{g}_0} \mid \mathfrak{d}_0^{\mathfrak{g}_0}) < \text{rmin}\{\mu_A(\mathfrak{d}_0), \mu_A(\mathfrak{g}_0)\}$. Let $\mu_A(\mathfrak{d}_0) = [\theta_1, \theta_2], \mu_A(\mathfrak{g}_0) = [\theta_3, \theta_4]$ and $\mu_A(\mathfrak{d}_0^{\mathfrak{g}_0} \mid \mathfrak{d}_0^{\mathfrak{g}_0}) = [\mathfrak{p}_1, \mathfrak{p}_2]$. Then $[\mathfrak{p}_1, \mathfrak{p}_2] < \text{rmin}\{[\theta_1, \theta_2], [\theta_3, \theta_4]\} = [\min\{\theta_1, \theta_3\}, \min\{\theta_2, \theta_4\}]$. So, $\mathfrak{p}_1 < \min\{\theta_1, \theta_3\}$ and $\mathfrak{p}_2 < \min\{\theta_2, \theta_4\}$. Consider,

$$\begin{aligned} [\rho_1, \rho_2] &= \tfrac{1}{2}[\mu_A(\mathfrak{d}_0^{\mathfrak{g}_0} \mid \mathfrak{d}_0^{\mathfrak{g}_0}) + \text{rmin}\{\mu_A(\mathfrak{d}_0), \mu_A(\mathfrak{g}_0)\}] \\ &= \tfrac{1}{2}[[\mathfrak{p}_1, \mathfrak{p}_2] + [\min\{\theta_1, \theta_3\}, \min\{\theta_2, \theta_4\}]] \\ &= [\tfrac{1}{2}(\mathfrak{p}_1 + \min\{\theta_1, \theta_3\}), \tfrac{1}{2}(\mathfrak{p}_2 + \min\{\theta_2, \theta_4\})]. \end{aligned}$$

Therefore, $\min\{\theta_1, \theta_3\} > \rho_1 = \frac{1}{2}(\mathfrak{p}_1 + \min\{\theta_1, \theta_3\}) > \mathfrak{p}_1$ and $\min\{\theta_2, \theta_4\} > \rho_2 = \frac{1}{2}(\mathfrak{p}_2 + \min\{\theta_2, \theta_4\}) > \mathfrak{p}_2$. Hence, $[\min\{\theta_1, \theta_3\}, \min\{\theta_2, \theta_4\}] > [\rho_1, \rho_2] > [\mathfrak{p}_1, \mathfrak{p}_2]$, so that $\mathfrak{d}_0^{\mathfrak{g}_0} \mid \mathfrak{d}_0^{\mathfrak{g}_0} \notin \mathscr{U}(\mu_A : [\mathfrak{p}_1, \mathfrak{p}_2])$, a contradiction, since $\mu_A(\mathfrak{d}_0) = [\theta_1, \theta_2] \geq [\min\{\theta_1, \theta_3\}, \min\{\theta_2, \theta_4\}] > [\rho_1, \rho_2]$ and $\mu_A(\mathfrak{g}_0) = [\theta_3, \theta_4] \geq [\min\{\theta_1, \theta_3\}, \min\{\theta_2, \theta_4\}] > [\rho_1, \rho_2]$. This implies $\mathfrak{d}_0^{\mathfrak{g}_0} \mid \mathfrak{d}_0^{\mathfrak{g}_0} \in \mathscr{U}(\mu_A : [\mathfrak{p}_1, \mathfrak{p}_2])$. Thus $\mu_A(\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}}) \leq \text{rmin}\{\mu_A(\mathfrak{d}), \mu_A(\mathfrak{g})\}$ for all $\mathfrak{d}, \mathfrak{g} \in S_H$. Again, in contrary, let $\mathfrak{d}_0, \mathfrak{g}_0 \in S_H$ be such that $\gamma_A(\mathfrak{d}_0^{\mathfrak{g}_0} \mid \mathfrak{d}_0^{\mathfrak{g}_0}) > \text{rmax}\{\gamma_A(\mathfrak{d}_0), \gamma_A(\mathfrak{g}_0)\}$. Let $\gamma_A(\mathfrak{d}_0) = [\eta_1, \eta_2], \gamma_A(\mathfrak{g}_0) = [\eta_3, \eta_4]$ and $\gamma_A(\mathfrak{d}_0^{\mathfrak{g}_0} \mid \mathfrak{d}_0^{\mathfrak{g}_0}) = [\mathfrak{q}_1, \mathfrak{q}_2]$. Then $[\mathfrak{q}_1, \mathfrak{q}_2] > \text{rmax}\{[\eta_1, \eta_2], [\eta_3, \eta_4]\} = [\max\{\eta_1, \eta_3\}, \max\{\eta_2, \eta_4\}]$. So, $\mathfrak{q}_1 > \max\{\eta_1, \eta_3\}$ and $\mathfrak{q}_2 > \max\{\eta_2, \eta_4\}$. Let us consider,

$$\begin{aligned} [\lambda_1, \lambda_2] &= \tfrac{1}{2}[\gamma_A(\mathfrak{d}_0^{\mathfrak{g}_0} \mid \mathfrak{d}_0^{\mathfrak{g}_0}) + \text{rmax}\{\gamma_A(\mathfrak{d}_0), \gamma_A(\mathfrak{g}_0)\}] \\ &= \tfrac{1}{2}[[\mathfrak{q}_1, \mathfrak{q}_2] + [\max\{\eta_1, \eta_3\}, \max\{\eta_2, \eta_4\}] \\ &= \tfrac{1}{2}(\mathfrak{q}_1 + \max\{\eta_1, \eta_3\}), \tfrac{1}{2}(\mathfrak{q}_2 + \max\{\eta_2, \eta_4\}). \end{aligned}$$

Therefore, $\max\{\eta_1, \eta_3\} < \lambda_1 = \frac{1}{2}[(\mathfrak{q}_1 + \max\{\eta_1, \eta_3\})] < \mathfrak{q}_1$ and $\max\{\eta_2, \eta_4\} < \lambda_2 = \frac{1}{2}[(\mathfrak{q}_2 + \max\{\eta_2, \eta_4\})] < \mathfrak{q}_2$. Hence, $[\max\{\eta_1, \eta_3\}, \max\{\eta_2, \eta_4\}] < [\lambda_1, \lambda_2] < [\mathfrak{q}_1, \mathfrak{q}_2]$ so that $\mathfrak{d}_0^{\mathfrak{g}_0} \mid \mathfrak{d}_0^{\mathfrak{g}_0} \notin \mathscr{L}(\gamma_A : [\mathfrak{q}_1, \mathfrak{q}_2])$, a contradiction, since $\gamma_A(x_0) = [\eta_1, \eta_2] \leq [\max\{\eta_1, \eta_3\}, \max\{\eta_2, \eta_4\}] > [\beta_1, \beta_2]$ and $\gamma_A(\mathfrak{g}_0) = [\eta_3, \eta_4] \geq [\max\{\eta_1, \eta_3\}, \max\{\eta_2, \eta_4\}] > [\beta_1, \beta_2]$. Hence, $\mathfrak{d}_0^{\mathfrak{g}_0} \mid \mathfrak{d}_0^{\mathfrak{g}_0} \in \mathscr{L}(\gamma_A : [\mathfrak{q}_1, \mathfrak{q}_2])$. Thus, $\gamma_A(\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}}) \geq \text{rmax}\{\gamma_A(\mathfrak{d}), \gamma_A(\mathfrak{g})\}$ for all $\mathfrak{d}, \mathfrak{g} \in S_H$.

Theorem 8. *Let $S_H := (S_H, |, 0)$ be an SSHA. Any subalgebra of S_H can be realized as both the upper $[\mathfrak{p}_1, \mathfrak{p}_2]$-level and lower $[\mathfrak{q}_1, \mathfrak{q}_2]$-level of some IVIFSS-subalgebra of S_H.*

Proof. Let B be a subalgebra of S_H, and $A = (\mu_A, \gamma_A)$ be an IVIFS on S_H defined by

$$\mu_A(\mathfrak{d}) = \begin{cases} [\varrho_1, \varrho_2] & \text{if } \mathfrak{d} \in B \\ [0, 0] & \text{otherwise} \end{cases}$$

and

$$\gamma_A(\mathfrak{d}) = \begin{cases} [\vartheta_1, \vartheta_2] & \text{if } \mathfrak{d} \in B \\ [1, 1] & \text{otherwise} \end{cases}$$

for all $[\varrho_1, \varrho_2], [\vartheta_1, \vartheta_2] \in \mathscr{D}[0,1]$ and $\varrho_2 + \vartheta_2 \leq 1$. We consider the following cases:

Case (i): If $\mathfrak{d}, \mathfrak{g} \in B$, then $\mu_A(\mathfrak{d}) = [\varrho_1, \varrho_2]$, $\gamma_A(\mathfrak{d}) = [\vartheta_1, \vartheta_2]$ and $\mu_A(\mathfrak{g}) = [\varrho_1, \varrho_2]$, $\gamma_A(\mathfrak{g}) = [\beta_1, \beta_2]$. Thus,

$$\mu_A(\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}}) = [\varrho_1, \varrho_2] = \text{rmin}\{[\varrho_1, \varrho_2], [\alpha_1, \alpha_2]\} = \text{rmin}\{\mu_A(\mathfrak{d}), \mu_A(\mathfrak{g})\}$$

and

$$\gamma_A(\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}}) = [\vartheta_1, \vartheta_2] = \text{rmax}\{[\vartheta_1, \vartheta_2], [\vartheta_1, \vartheta_2]\} = \text{rmax}\{\gamma_A(\mathfrak{d}), \gamma_A(\mathfrak{g})\}.$$

Case (ii): If $\mathfrak{d} \in B$ and $\mathfrak{g} \notin B$, then $\mu_A(\mathfrak{d}) = [\varrho_1, \varrho_2]$, $\gamma_A(\mathfrak{d}) = [\vartheta_1, \vartheta_2]$ and $\mu_A(\mathfrak{g}) = [0, 0]$, $\gamma_A(\mathfrak{g}) = [1, 1]$. Thus,

$$\mu_A(\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}}) \geq [0, 0] = \text{rmin}\{[\varrho_1, \varrho_2], [0, 0]\} = \text{rmin}\{\mu_A(\mathfrak{d}), \mu_A(\mathfrak{g})\}$$

and

$$\gamma_A(\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}}) \geq [1, 1] = \text{rmax}\{[\vartheta_1, \vartheta_2], [1, 1]\} = \text{rmax}\{\gamma_A(\mathfrak{d}), \gamma_A(\mathfrak{g})\}.$$

Case (iii): If $\mathfrak{d} \notin B$ and $\mathfrak{g} \in B$, then $\mu_A(\mathfrak{d}) = [0, 0]$, $\gamma_A(\mathfrak{d}) = [1, 1]$, $\mu_A(\mathfrak{g}) = [\alpha_1, \alpha_2]$, $\gamma_A(\mathfrak{g}) = [\vartheta_1, \vartheta_2]$. Thus,

$$\mu_A(\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}}) \geq [0, 0] = \text{rmin}\{[0, 0], [\varrho_1, \varrho_2]\} = \text{rmin}\{\mu_A(\mathfrak{d}), \mu_A(\mathfrak{g})\}$$

and

$$\gamma_A(\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}}) \leq [1, 1] = \text{rmax}\{[1, 1], [\beta_1, \beta_2]\} = \text{rmax}\{\gamma_A(\mathfrak{d}), \gamma_A(\mathfrak{g})\}.$$

Case (iv): If $\mathfrak{d} \notin B$ and $\mathfrak{g} \notin B$, then $\mu_A(\mathfrak{d}) = [0, 0]$, $\gamma_A(\mathfrak{d}) = [1, 1]$ and $\mu_A(\mathfrak{g}) = [0, 0]$, $\gamma_A(\mathfrak{g}) = [1, 1]$. Now,

$$\mu_A(\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}}) \leq [0, 0] = \text{rmin}\{[0, 0], [0, 0]\} = \text{rmin}\{\mu_A(\mathfrak{d}), \mu_A(\mathfrak{g})\}$$

and

$$\gamma_A(\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}}) \geq [1, 1] = \text{rmax}\{[1, 1], [1, 1]\} = \text{rmax}\{\gamma_A(\mathfrak{d}), \gamma_A(\mathfrak{g})\}.$$

Therefore, $A = (\mu_A, \gamma_A)$ is an IVIFSS-subalgebra of S_H.

Theorem 9. *Let $S_H := (S_H, |, 0)$ be an SSHA. Let B be a subset of S_H and $A = (\mu_A, \gamma_A)$ be an IVIFS on S_H defined by*

$$\mu_A(\mathfrak{d}) = \begin{cases} [\varrho_1, \varrho_2] & \text{if } \mathfrak{d} \in B \\ [0, 0] & \text{otherwise} \end{cases}$$

and

$$\gamma_A(\mathfrak{d}) = \begin{cases} [\vartheta_1, \vartheta_2] & \text{if } \mathfrak{d} \in B \\ [1, 1] & \text{otherwise} \end{cases}$$

for all $[\varrho_1, \varrho_2], [\vartheta_1, \vartheta_2] \in \mathscr{D}[0, 1]$ and $\varrho_2 + \vartheta_2 \leq 1$. If $A = (\mu_A, \gamma_A)$ is realized as a lower-level subalgebra and an upper-level subalgebra of some IVIFSS-subalgebra of S_H, then B is a subalgebra of S_H.

Proof. Let $A = (\mu_A, \gamma_A)$ be an IVIFSS-subalgebra of S_H, and $\mathfrak{d}, \mathfrak{g} \in B$. Then $\mu_A(\mathfrak{d}) = [\varrho_1, \varrho_2] = \mu_A(\mathfrak{g})$ and $\gamma_A(\mathfrak{d}) = [\beta_1, \beta_2] = \gamma_A(\mathfrak{g})$. Thus,

$$\mu_A(\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}}) \leq \text{rmin}\{\mu_A(\mathfrak{d}), \mu_A(\mathfrak{g})\} = \text{rmin}\{[\varrho_1, \varrho_2], [\varrho_1, \varrho_2]\} = [\varrho_1, \varrho_2]$$

and

$$\gamma_A(\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}}) \geq \text{rmax}\{\gamma_A(\mathfrak{d}), \gamma_A(\mathfrak{g})\} = \text{rmax}\{[\vartheta_1, \vartheta_2], [\vartheta_1, \vartheta_2]\} = [\vartheta_1, \beta_2],$$

which imply that $\mathfrak{d}^{\mathfrak{g}} \mid \mathfrak{d}^{\mathfrak{g}} \in B$.

4. Conclusion

This study explored the fundamental properties of interval-valued intuitionistic fuzzy (IVIF) subsets and subalgebras within Sheffer stroke Hilbert algebras, highlighting their algebraic behavior and relationships under various set operations. The results demonstrated that these fuzzy structures could serve as effective extensions of classical algebraic systems, allowing for the flexible capture of uncertainty and partial membership. The theoretical insights provided by this work lay a foundation for integrating fuzzy logic principles into algebraic frameworks, with potential applications in logical reasoning, decision-making, and information processing.

Future research could focus on extending the concept of IVIF subsets and ideals to other algebraic systems such as Boolean algebras, orthomodular lattices, and residuated structures. Developing computational algorithms for the automatic identification and manipulation of IVIF subalgebras will facilitate practical applications in areas such as artificial intelligence, fuzzy decision-making, and data analysis. Additionally, investigating the application of these fuzzy algebraic structures in real-world scenarios—such as medical diagnosis, control systems, and information security—may provide valuable insights and enhance their practical utility. Further exploration of their interactions with probabilistic and other uncertainty-based frameworks can also open new avenues for theoretical advancement.

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