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Super Hop Roman Domination in Graphs

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Abstract. Let $G = (V(G), E(G))$ be a simple undirected graph. A function $f : V(G) \to \{0, 1, 2\}$ is a super hop Roman dominating function (SHRDF) on G if for every $v \in V(G)$ with $f(v) = 0$, there exist $w, u \in V(G)$ with $f(w) = 2$ and $f(u) \neq 0$ such that $d_G(v, w) = 2$, and $N^2_G(u) \cap \{x \in V(G) : f(x) = 0\} = \{v\}$. The *weight* of SHRDF f, denoted $\omega^{shR}_G(f)$, is given by $\omega^{shR}_G(f) = \sum_{y\in V(G)} f(y)$. The *super hop Roman domination number* of a graph G, denoted $\gamma_{shR}(G)$, is the minimum weight of an SHRDF on G, that is, $\gamma_{shR}(G) = min\{\omega^{shR}_G(f) : f \textit{ is an } \text{SHRDF} \textit{ on } G\}$. In this paper, we make an initial investigation of this newly defined variation of hop Roman domination in graphs. Some bounds and exact values of the parameter are obtained and some characterizations on some classes of graphs are given.

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Key Words and Phrases: Super domination, hop Roman domination, super hop Roman domination

1. Introduction

Domination is one of the major concepts in graph theory that is rigorously studied by several discrete mathematicians due to its interesting theoretic structures [1], [2], [3], [4], [5], [6], [7], [8], [9]. Roman dominating function is one of the topics in the theory of domination that remains intriguing and have been a center of mathematics research. Roman domination was pioneered by Cockayne et al. [5] in 2004 which is based on the defence strategy of Roman Emperor Constantine the great around the fourth century A.D. Currently, there are now several variations of Roman domination that has been published in the literature of graph theory and can be found in [10], [11], [12]. In the year 2017, Shabani et al. [13] formally introduced the hop Roman domination in graphs which is extensively studied recently. In addition, super dominating sets in graphs initiated by

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Lemańska et al. [14] also captures the attention of many graph theorists. Motivated by hop Roman domination and super domination, the author introduced a new parameter called super hop Roman domination and investigated its mathematical properties.

Let $G = (V(G), E(G))$ be a simple, undirected and finite graph where $V(G)$ is the vertex set and $E(G)$ is the edge set of G. The cardinality of $V(G)$ denoted by $|V(G)|$ is called the order of G and the cardinality of $E(G)$ denoted by $|E(G)|$ is called the size of G. The *complement* of a graph G denoted by $\overline{G}$ is the graph that satisfies the following conditions: (i) $V(\overline{G}) = V(G)$; and (ii) $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$. All needed basic concepts and terminologies used in this study which are not define are found in [15], [16], [6]. Let $x \in V(G)$. Then the *open neighborhood* of x in G is the set $N_G(x) = \{y \in V(G) : xy \in E(G)\}$ and the *closed neighborhood* of a vertex $x \in V(G)$ is the set $N_G[x] = N_G(x) \cup \{x\}$. Let $O \subseteq V(G)$. Then, the set $N_G(O) = N(O) = \bigcup\limits_{v \in O} N_G(v)$ is called the *open neighborhood* of O and the set $N_G[O] = N[O] = N(X) \cup X$ is called the *closed neighborhood* of O. Let u and v be two distinct vertices in graph G. Then, the distance between u and v denoted by $d_G(u, v)$ is the length of the shortest walk between u and v in G. If there is no such walk between u and v in G, then we define the distance as $d_G(u, v) = \infty$. Now, let $v \in V(G)$. Then, the set $N^2_G(v) = \{u \in V(G) : deg_G(u, v) = 2\}$ is called the *open hop-neighborhood* and each element of $N^2_G(v)$ is called hop-neigbor of vertex v. Moreover, for $H \subseteq V(G)$, $N^2_G(H) = \bigcup\limits_{v \in H} N^2_G(v)$ and $N^2_G[H] = N^2_G(H) \cup H$.

A subset D of vertex set $V(G)$ is a *dominating set* of G if for every $v \in V(G) \setminus D$, there exists $u \in D$ such that uv is an edge of G [6]. In that case, $N[D] = V(G)$. The *domination number* denoted by $\gamma(G)$ is the minimum cardinality of a dominating set D in G. If D is a dominating set with $|D| = \gamma(G)$, then we call D a *minimum dominating set* of G or a γ-set in G. A dominating set $S \subseteq V(G)$ is called a *super dominating set* of G if for every vertex $u \in V(G) \setminus S$, there exists $v \in S$ such that $N(v) \cap (V(G) \setminus S) = \{u\}$ [8]. In that case, v is a private neighbor of u with respect to $V(G) \setminus S$. The smallest cardinality of a super dominating set of G is called the *super domination number* denoted by $\gamma_{sp}(G)$. A super dominating set of cardinality $\gamma_{sp}(G)$ is called γ_{sp}-set in G.

A set $S \subseteq V(G)$ is called a *hop dominating set* of G if for every vertex in $v \in V(G) \setminus S$, there exists $u \in S$ such that $d_G(u, v) = 2$ [17]. The smallest cardinality of a hop dominating set in G, denoted $\gamma_h(G)$, is called the *hop domination number* of G. A hop dominating set of cardinality $\gamma_h(G)$ is called a γ_h-set in G. A hop dominating set $S \subseteq V(G)$ is called *super hop dominating* if for every vertex $v \in V(G) \setminus S$, there exists $u \in D$ such that $N^2_G(u) \cap (V(G) \setminus S) = \{v\}$ [18]. The smallest cardinality of a super hop dominating set in G, denoted $\gamma_{sh}(G)$, is called the *super hop domination number* of G. A super hop dominating set of cardinality $\gamma_{sh}(G)$ is called a γ_{sh}-set in G. Hop domination and some of its variants have beed studied previously in [18], [19], [20], [21], [22], and [23].

Let $f : V(G) \to \{0, 1, 2\}$ be a function on G. Let the sets V_0, V_1, V_2 be given as follows:

$$\begin{aligned} V_0 &= \{v \in V(G) : f(v) = 0\}; \\ V_1 &= \{v \in V(G) : f(v) = 1\}; and \end{aligned}$$

$$V_2 = \{v \in V(G) : f(v) = 2\}.$$

In this case, we may denote f by $f = (V_0, V_1, V_2)$. A function $f = (V_0, V_1, V_2)$ is a *hop Roman dominating function* (HRDF) on G if for every $v \in V_0$, there exists $u \in V_2$ such that $d_G(u, v) = 2$. The *weight* of f is given by $\omega_G^{hR}(f) = \sum_{v \in V(G)} f(v)$. The *hop Roman domination number* of G, denoted $\gamma_{hR}(G)$, is the minimum weight of an HRDF on G, that is, $\gamma_{hR}(G) = min\{\omega_G^{hR}(f) : f \textit{ is an } \text{HRDF} \textit{ on } G\}$. Any HRDF f on G with $\omega_G^{hR}(f) = \gamma_{hR}(G)$ is called a γ_{hR}-function on G.

A function $f = (V_0, V_1, V_2)$ is a *super hop Roman dominating function* (SHRDF) on G if it satisfies the following conditions:

(SHR1) f is a hop Roman dominating function on G; and

(SHR2) for each $v \in V_0$, there exists $w \in V_1 \cup V_2$ such that $N_G^2(w) \cap V_0 = \{v\}$.

The *weight* $\omega_G^{shR}(f)$ of an SHRDF f is given by $\omega_G^{shR}(f) = \sum_{u \in V(G)} f(u)$, that is, $\omega_G^{shR}(f) = |V_1| + 2|V_2|$. The *super hop Roman domination number* of G, denoted $\gamma_{shR}(G)$, is the minimum weight of an SHRDF on G, that is, $\gamma_{shR}(G) = min\{\omega_G^{shR}(f) : f \textit{ is an } \text{SHRDF} \textit{ on } G\}$. Any SHRDF f on G with $\omega_G^{shR}(f) = \gamma_{shR}(G)$ is called a γ_{shR}-function on G.

Consider the graph G with $|V(G)| = 10$ in Figure 1 below. Let $f = (V_0, V_1, V_2)$ be a function on G such that $V_0 = \{v_5, v_6, v_8\}$, $V_1 = \{v_1, v_2, v_4, v_7, v_9, v_{10}\}$, and $V_2 = \{v_3\}$. Observe that f is a γ_{shR}-function on G. Hence, $\gamma_{shR}(G) = 8$.

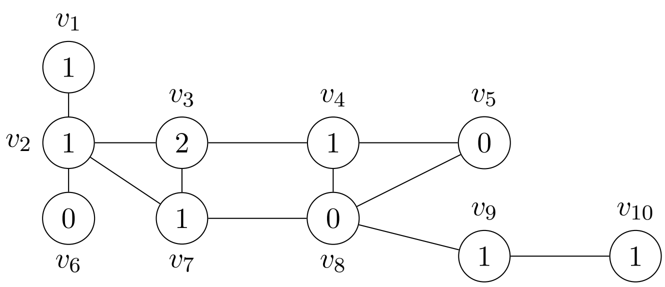

Figure 1: A graph G with $\gamma_{shR}(G) = 8$.

In this paper, we do an initial investigation of super hop Roman domination in graphs.

2. Known Results

We shall need the results obtained by Canoy et al. in [18].

Theorem 1. *Let G be a graph of order $n \geq 1$. Then $\lceil \frac{n}{2} \rceil \leq \gamma_{sh}(G)$. In particular, $n \leq 2\gamma_{sh}(G)$.*

Theorem 2. *Let G be a graph of order $n \geq 1$. Then $\gamma_{sh}(G) = n$ if and only if each component G' of G is a complete graph.*

Corollary 1. *Let G be a connected graph of order n. Then $\gamma_{sh}(G) = n$ if and only if $G = K_n$.*

3. Results

This section explores the properties of the super hop Roman dominating function in graphs.

Proposition 1. *Let G be a graph of order $n \geq 1$ and $f = (V_0, V_1, V_2)$ be an SHRDF on G. Then $V_1 \cup V_2$ is a super hop dominating set on G. Moreover, if f is a γ_{shR}-function on G, then the following statements hold:*

(i) *$V_0 = \varnothing$ if and only if $V_2 = \varnothing$. In this case, $\gamma_{shR}(G) = |V(G)| = n$; and*

(ii) *If $V_1 = \varnothing$, then V_2 is a γ_{sh}-set and $\gamma_{shR}(G) = 2\gamma_{sh}(G) = n$.*

Proof. Assume that $f = (V_0, V_1, V_2)$ is an SHRDF on G and let $v \in V(G) \setminus (V_1 \cup V_2)$. Then $v \in V_0$. Since f satisfies (SHR2), there exists $w \in V_1 \cup V_2$ such that $N_G^2(w) \cap V_0 = \{v\}$. This implies that $V_1 \cup V_2$ is a super hop dominating set on G.

Now, suppose f is a γ_{shR}-function on G. Let $V_0 = \varnothing$. Assume for a moment that $V_2 \neq \varnothing$, say $w \in V_2$. Let $W_0 = V_0$, $W_1 = V_1 \cup \{w\}$, and $W_2 = V_2 \setminus \{w\}$. Then $g = (W_0, W_1, W_2)$ is a super hop Roman dominating function on G. It follows that

$$\begin{aligned}
\omega_G^{shR}(g) &= |W_1| + 2|W_2| \\
&= (|V_1| + 1) + 2(|V_2| - 1) \\
&= |V_1| + 2|V_2| - 1 \\
&< \omega_G^{shR}(f) \\
&= \gamma_{shR}(G),
\end{aligned}$$

a contradiction to the assumption that f is a γ_{shR}-function on G. Thus, $V_2 = \varnothing$.

Conversely, suppose that $|V_2| = 0$. Since f satisfies (SHR1), the assumption that $|V_2| = 0$ forces $|V_0| = 0$. This, in turn, implies that $|V_1| = n$. Thus, $\gamma_{shR}(G) = |V_1| + 2|V_2| = |V_1| = |V(G)| = n$, showing that (i) holds.

Next, suppose $V_1 = \varnothing$. Then $V_1 \cup V_2 = V_2$ is a super hop dominating set. Suppose V_2 is not a γ_{sh}-set in G. Let V_2' be a γ_{sh}-set in G. Then, we obtain $|V_2'| < |V_2|$. Define a function $g = (W_0, W_1, W_2)$ on G where $W_0 = V(G) \setminus V_2'$, $W_1 = \varnothing$ and $W_2 = V_2'$. Then g is an SHRDF on G and $\omega_G^{shR}(g) = 2|W_2| < 2|V_2| = \gamma_{shR}(G)$, a contradiction. Therefore V_2 is a γ_{sh}-set on G and $\gamma_{shR}(G) = 2|V_2| = 2\gamma_{sh}(G)$. By Theorem 1, $\gamma_{shR}(G) = n$. This shows that (ii) holds. □

Lemma 1. *Let G be a graph of order n and let $f = (V_0, V_1, V_2)$ be a γ_{shR}-function on G. Then $|V_2| \leq |V_0|$ and $|V_0| \leq |V_1| + |V_2|$. Moreover, each of the following statements hold:*

- (i) *If $|V_0| = |V_2|$, then $\gamma_{shR}(G) = n$.*
- (ii) *If $\gamma_{shR}(G) < n$, then $1 \leq |V_1| < n$.*

Proof. If $|V_0| = 0$, then $|V_2| = 0$ by Proposition 1. Hence, $|V_0| = |V_2|$. So suppose $|V_0| \neq 0$. Since f satisfies (SHR1), it follows that for each $v \in V_0$, there exists $z_v \in V_2 \cap N_G^2(v)$. Hence, the assignment $v \to z_v$ defines a function ψ from V_0 into V_2. Since f is a γ_{shR}-function on G, ψ must be onto. Thus, $|V_2| \leq |V_0|$.

Now, by Proposition 1(i), $V_1 \cup V_2$ is a super hop dominating set. Hence, for every $v \in V_0$, there exists $w_v \in V_1 \cup V_2$ such that $N_G^2(w_v) \cap V_0 = \{v\}$. Define the function $h : V_0 \to \{w_v : v \in V_0\}$ by $h(v) = w_v$ for each $v \in V_0$. Then h is a one-one and onto function. Thus, $|V_0| = |\{w_v : v \in V_0\}|$. Since $\{w_v : v \in V_0\} \subseteq V_1 \cup V_2$, it follows that $|V_0| \leq |V_1 \cup V_2|$.

Next, if $|V_0| = |V_2|$, then we have $\gamma_{shR}(G) = |V_1| + 2|V_2| = |V_1| + |V_2| + |V_0| = |V(G)| = n$. This shows that (i) holds.

Finally, suppose that $\gamma_{shR}(G) < n$. Then $|V_0| \neq |V_2|$ by (the contrapostive of) (i). Assume that $|V_1| = 0$. Then V_2 is a γ_{sh}-set on G and $\gamma_{shR}(G) = 2\gamma_{sh}(G) = n$ by Proposition 1. This contradicts the assumption that $\gamma_{shR}(G) < n$. Therefore $|V_1| \geq 1$, showing that (ii) holds. This proves the assertion. □

Proposition 2. *Let G be a graph of order n and let $f = (V_0, V_1, V_2)$ be a γ_{shR}-function on G. Then each of the following statements holds:*

- (i) *$\gamma_{shR}(G) < n$ if and only if $1 \leq |V_2| < |V_0|$.*
- (ii) *$\gamma_{shR}(G) = n$ if and only if $|V_0| = |V_2|$.*

Proof. (i) Suppose $\gamma_{shR}(G) < n$. By Lemma 1, and property (SHR1), we have $1 \leq |V_2| < |V_0|$.

For the converse, suppose that $1 \leq |V_2| < |V_0|$. Then

$$\begin{aligned} \gamma_{shR}(G) = \omega_G^{shR}(f) &= |V_1| + 2|V_2| \\ &< |V_1| + |V_2| + |V_0| \\ &= n. \end{aligned}$$

(ii) Suppose $\gamma_{shR}(G) = n$. Assume for a moment that $|V_0| \neq |V_2|$. By Proposition 1(i), $|V_0| \neq 0$ and $|V_2| \neq 0$. Lemma 1 would now imply that $1 \leq |V_2| < |V_0|$. This implies that $\gamma_{shR}(G) < n$ by (i), a contradiction to our assumption. Therefore, $|V_0| = |V_2|$.

The converse follows from Lemma 1(i). □

Theorem 3. *Let G be a graph of order n and let $f = (V_0, V_1, V_2)$ be an SHRDF on G. Then, $V_1 \cup V_2$ is a minimal super hop dominating set of G if and only if each $u \in V_2$, there exists a vertex $v \in V_0$ such that $N_G^2(v) \cap V_2 = \{u\}$ or $d_G(v, w) \neq 2$ for all $w \in (V_1 \cup V_2) \setminus \{u\}$.*

Proof. Let $f = (V_0, V_1, V_2)$ be an SHRDF on G of order n. Then by Proposition 1, $V_1 \cup V_2$ is a super hop dominating set on G.
($\Rightarrow$) Assume that $V_1 \cup V_2$ is a minimal super hop dominating set on G. Then for every $u \in V_1 \cup V_2$, $(V_1 \cup V_2) \setminus \{u\}$ is not a super hop dominating set of G. This means that there exists $v \in V(G) \setminus ((V_1 \cup V_2) \setminus \{u\})$ such that $d_G(v, w) \neq 2$ for all $w \in (V_1 \cup V_2) \setminus \{u\}$. Suppose that $v \neq u$. It is worth noting that $V_1 \cup V_2$ is a super hop dominating set, hence, v must be super hop dominated by $V_1 \cup V_2$. So, it follows that $d_G(u, v) = 2$ which implies that $N_G^2(v) \cap V_2 = \{u\}$. Now, suppose that $v = u$. Then it simply follows that $d_G(v, w) \neq 2$ for every $w \in (V_1 \cup V_2) \setminus \{u\}$.
($\Leftarrow$) As for the converse, we assume that for every $u \in V_2$, there exists $v \in V_0$ such that $N_G^2(v) \cap V_2 = \{u\}$. Then it follows that for every $v \in V_0$ is not hop dominated by the set $(V_1 \cup V_2) \setminus \{u\}$. On the other hand, assume that for every $u \in V_2$, we have $d_G(u, w) \neq 2$ for all $w \in (V_1 \cup V_2) \setminus \{u\}$. This implies that u can not be super hop dominated by any vertex $x \in (V_1 \cup V_2) \setminus \{u\}$ and hence, $(V_1 \cup V_2) \setminus \{u\}$ is not a super hop dominating set of G. Therefore, it is concluded that $V_1 \cup V_2$ is a minimal super hop dominating set of G. □

The next result gives some bounds on the super hop Roman domination number of a graph.

Theorem 4. *Let G be a connected graph of order $n \geq 1$. Then,*

$$max\{\gamma_{sh}(G), \gamma_R(G)\} \leq \gamma_{shR}(G) \leq n.$$

Proof. Since every super hop Roman dominating function is Roman dominating, it follows that $\gamma_R(G) \leq \gamma_{shR}(G)$. Let $f = (V_0, V_1, V_2)$ be a γ_{shR}-function on G. By Proposition 1, $V_1 \cup V_2$ is a super hop dominating set. This implies that $\gamma_{sh}(G) \leq |V_1| + |V_2| \leq |V_1| + 2|V_2| = \gamma_{shR}(G)$. Therefore, $max\{\gamma_{sh}(G), \gamma_R(G)\} \leq \gamma_{shR}(G)$. Since $h = (\varnothing, V(G), \varnothing)$ is an SHRDF on G, we have $\gamma_{shR}(G) \leq \omega_G^{shR}(h) = |V(G)| = n$. □

We note that for any graph G, it is easy to show that $\gamma_{shR}(G) \leq 2\gamma_{sh}(G)$. Indeed, if S be a γ_{sh}-set on G, then $g = (V_0', V_1', V_2')$, where $V_0' = V(G) \setminus S$, $V_1' = \varnothing$, and $V_2' = S$, is a SHRDF on G. Thus, we obtain $\gamma_{shR}(G) \leq \omega_G^{shR}(g) = |V_1'| + 2|V_2'| = 2|S| = 2\gamma_{sh}(G)$. However, $2\gamma_{sh}(G)$ is not always the best upper bound because $n \leq 2\gamma_{sh}(G)$ by Theorem 1.

The next result simply says that the super hop Roman domination number of a graph is the sum of the super Roman domination numbers of its components. For completeness, we show its proof here.

Theorem 5. *Let $G_1, G_2, \ldots, G_k$ be the components of graph G of order n. Then f is an SHRDF on G if and only if the restriction function $f|_{G_j}$ is an SHRDF on G_j for each $j \in \{1, 2, ...k\}$. Moreover, $\gamma_{shR}(G) = \sum_{j=1}^{k} \gamma_{shR}(G_j)$.*

Proof. Suppose $f = (V_0, V_1, V_2)$ is an SHRDF on G. For each $j \in \{1, 2, ..., k\}$, let $V_0^j = V_0 \cap V(G_j)$, $V_1^j = V_1 \cap V(G_j)$ and $V_2^j = V_2 \cap V(G_j)$. Then $f|_{G_j} = (V_0^j, V_1^j, V_2^j)$ for all $j \in \{1, 2, ..., k\}$. Let $j \in \{1, 2, ..., k\}$ and let $v \in V_0^j$. Then $v \in V_0$. Since f is a hop Roman dominating function on G, it follows that there exists $w \in V_2$ such that $d_G(w, v) = 2$. This implies that $w \in V_2^j$ and $d_{G_j}(w, v) = 2$, showing that $f|_{G_j}$ is a hop Roman dominating function on G_j. Moreover, since f satisfies (SHR2), there exists $z \in V_1 \cup V_2$ such that $N_G^2(z) \cap V_0 = \{v\}$. This means that $z \in V_1^j \cup V_2^j$ and $N_{G_j}^2(z) \cap V_0 = \{v\}$. Therefore, $f|_{G_j}$ is an SHRDF on G_j. If, in particular, f is a γ_{shR}-function on G, then

$$\begin{aligned}
\gamma_{shR}(G) = \omega_G^{shR}(f) &= |V_1| + 2|V_2| \\
&= \sum_{j=1}^{k} |V_1^j| + 2\sum_{j=1}^{k} |V_2^j| \\
&= \sum_{j=1}^{k} \left(|V_1^j| + 2|V_2^j|\right) \\
&\geq \sum_{j=1}^{k} \gamma_{shR}(G_j).
\end{aligned}$$

Next, suppose that $f|_{G_j} = (W_0^j, W_1^j, W_2^j)$ is an SHRDF on G_j for each $j \in \{1, 2, ...k\}$. Then $V_0 = \bigcup_{j=1}^{k} W_0^j$, $V_1 = \bigcup_{j=1}^{k} W_1^j$ and $V_2 = \bigcup_{j=1}^{k} W_2^j$. Let $\overline{v} \in V_0$. Then $\overline{v} \in V_0^j$ for some $j \in \{1, 2, ...k\}$. Since $f|_{G_j}$ is a hop Roman dominating function on G, it follows that there exists $\overline{w} \in V_2^j$ such that $d_{G_j}(\overline{w}, \overline{v}) = 2$. It follows that $\overline{w} \in V_2$. Also, since $f|_{G_j}$ satisfies (SHR2), there exists $\overline{z} \in V_1^j \cup V_2^j$ such that $N_G^2(\overline{z}) \cap V_0^j = \{\overline{v}\}$. This implies that $\overline{z} \in V_1 \cup V_2$ and $N_G^2(\overline{z}) \cap V_0 = \{\overline{v}\}$. Accordingly, f is an SHRDF on G. If, in particular, $f|_{G_j}$ is a γ_{shR}-function on G_j for all $j \in \{1, 2, ...k\}$, then

$$\begin{aligned}
\sum_{j=1}^{k} \gamma_{shR}(G_j) = \sum_{j=1}^{k} \omega_G^{shR}(f|_{G_j}) &= \sum_{j=1}^{k} \left(|W_1^j| + 2|W_2^j|\right) \\
&= \sum_{j=1}^{k} |W_1^j| + 2\sum_{i=1}^{k} |W_2^j| \\
&= |V_1| + 2|V_2| \\
&\geq \gamma_{shR}(G).
\end{aligned}$$

This proves the assertion. □

Corollary 2. *Let $G_1, G_2, \ldots, G_k$ be the components of a graph G of order n. If G_j is complete for every $j \in \{1, 2, \cdots, k\}$, then $\gamma_{shR}(G) = n$. In particular, $\gamma_{shR}(K_n) = \gamma_{shR}(\overline{K}_n) = n$.*

Proof. This follows from Theorem 2, Theorem 4, and Theorem 5. □

Proposition 3. *Let G be a graph of order $n \geq 1$. Then*

(i) *$\gamma_{shR}(G) = 1$ if and only if $G = K_1$;*

(ii) *$\gamma_{shR}(G) = 2$ if and only if $G \in \{K_2, \overline{K_2}\}$; and*

(iii) *$\gamma_{shR}(G) = 3$ if and only if $G \in \{P_3, K_3, \overline{K_3}, K_2 \cup K_1\}$.*

Proof. Let $f = (V_0, V_1, V_2)$ be a γ_{shR}-function on G.

(i) Suppose $\gamma_{shR}(G) = 1$. Then $|V_1| + 2|V_2| = 1$. This implies that $|V_2| = 0$. By Proposition 1(i), $|V_0| = 0$. Hence, $|V_1| = |V(G)| = 1$, i.e., $G = K_1$.

The converse is clear.

(ii) Suppose $\gamma_{shR}(G) = |V_1| + 2|V_2| = 2$. Then $|V_2| \leq 1$. Suppose $|V_2| = 1$. Then $|V_1| = 0$ and $|V_0| \neq 0$. Let $V_2 = \{v\}$ and let $w \in V_0$. Then $d_G(v, w) = 2$. Let $x \in N_G(w) \cap N_G(v)$. Since $|V_1| = 0$, this implies that V_2 is not a hop dominating set in G, a contradiction. Thus, $|V_2| = 0$. This implies that $|V_0| = 0$ and $\gamma_{shR}(G) = |V_1| = |V(G)| = 2$. Therefore, $G \in \{K_2, \overline{K_2}\}$.

Conversely, suppose that $G \in \{K_2, \overline{K_2}\}$. Then clearly, $\gamma_{shR}(G) = 2$.

(iii) Suppose $\gamma_{shR}(G) = |V_1| + 2|V_2| = 3$. Then $|V_2| \leq 1$. If $|V_2| = 0$, then $|V_0| = 0$ and $|V_1| = |V(G)| = 3$. Hence, $G \in \{P_3, K_3, \overline{K_3}, P_2 \cup K_1\}$. Next, suppose that $|V_2| = 1$. Then $|V_1| = 1$ and $|V_0| \geq 1$. Let $v \in V_2$ and $u \in V_0$. Then $d_G(u, v) = 2$. Now, let $x \in N_G(v) \cap N_G(u)$. Since f satisfies (SHR1) and $x \in N_G(v)$, it follows that $x \in V_1$. Suppose now that $|V(G)| \geq 4$. Let $w \in V(G) \setminus \{u, v, x\}$. Since $|V_1| = |V_2| = 1$, it follows that $w \in V_0$ and $d_G(w, v) = 2$. Let $y \in N_G(v) \cap N_G(w)$. Again, this will imply that $y \in V_1$. Hence, $x = y$. However, since $u \neq w$, it follows that f does not satisfy (SHR2), a contradiction. Thus, $|V(G)| = 3$ and $G = P_3$.

The converse is clear. □

Theorem 6. *Let G be a graph. Then $\gamma_{shR}(G) = 4$ if and only if it satisfies one of the following:*

(i) *$|V(G)| = 4$; or*

(ii) *$V(G) = \{x, y, p, q, v\}$ such that $N_G(v) = \{p, q\}$, $N_G^2(v) = \{x, y\}$, $x \in N_G^2(p) \setminus N_G^2(q)$ and $y \in N_G^2(q) \setminus N_G^2(p)$.*

Proof. Let $f = (V_0, V_1, V_2)$ be a γ_{shR}-function on G. Assume that $\gamma_{shR}(G) = 4$. Then $|V_1| + 2|V_2| = 4$ and so, $|V_2| \leq 2$. First, suppose that $|V_2| = 0$. Then $|V_0| = 0$ and $|V_1| = |V(G)| = 4$. Next, suppose $|V_2| = 1$. Then $|V_1| = 1$ and $|V_0| \geq 1$. Suppose $|V_0| \geq 3$. Let $a, b, c \in V_0$. Since f satisfies (SHR1), we have $N^2_G(v) \cap V_0 = \{a, b, c\}$. This forces $|N_G(p) \cap V_0| = 1$ for every $p \in V_1$ because f satisfies (SHR2). However, this is not possible because $|V_1| = 2$ and $|V_0| \geq 3$. Therefore, $|V_0| \leq 2$. Consequently, $4 \leq |V(G)| \leq 5$. If $|V_0| = 1$, then $|V(G)| = 4$. Suppose $|V_0| = 2$. Then $|V(G)| = 5$. Let $V_0 = \{x, y\}$, $V_1 = \{p, q\}$ and $V_2 = \{v\}$. Then $N_G(v) \cap V_0 = V_0 = \{x, y\}$. Since f satisfies (SHR2), we may assume that $N^2_G(p) \cap V_0 = \{x\}$ and $N^2_G(q) \cap V_0 = \{y\}$. Let $z \in N_G(x) \cap N_G(v)$. Since $p \in N^2_G(x)$ and $y \in N^2_G(v)$, it follows that $z \notin \{p, y\}$. This forces $z = q$. Hence, $q \in N_G(v)$. Similarly, $p \in N_G(v)$. This shows that (ii) holds. Finally, suppose $|V_2| = 2$. Then $|V_1| = 0$ and, by Lemma 1, $|V_0| = 2$. It follows that $|V(G)| = 4$.

For the converse, suppose first that $|V(G)| = 4$. By Theorem 4 and Proposition 3, we have $\gamma_{shR}(G) = 4$. Next, suppose that (ii) holds. Let $V_0 = \{x, y\}$, $V_1 = \{p, q\}$, and $V_2 = \{v\}$. By assumption, $h = (V_0, V_1, V_2)$ is an SHRDF on G. Hence, Proposition 3 would imply that $\gamma_{shR}(G) = \omega^{shR}_G(h) = 4$. □

Proposition 4. *Let $G = P_n$ with $n \geq 1$. Then*

$$\gamma_{shR}(G) = \begin{cases} n, & \text{if } n \leq 8, \\ n - k, & \text{if } n \geq 9, \end{cases}$$

where $k = \lfloor \frac{n+1}{10} \rfloor$.

Proof. Assume that $G = P_n = [v_1, v_2, ..., v_n]$ with $n \geq 1$. Let $f = (V_0, V_1, V_2)$ be a γ_{shR}-function on G and let $n \leq 8$. If $V_1 = V(G)$, then we are done. Assume for a moment that $\gamma_{shR}(G) < n$. Then $V_1 \neq V(G)$ and by Proposition 2, it follows that $|V_0| > |V_2|$. This implies that there exists $u \in V_2$ such that $|N^2_G(u) \cap V_0| = 2$. Let $x, y \in N^2_G(u) \cap V_0$. Then by definition of super dominating set, there exists $a, b \in V_1 \cup V_2$ such that $N^2_G(a) \cap V_0 = \{x\}$ and $N^2_G(b) \cap V_0 = \{y\}$. Clearly, $u \notin \{a, b\}$. Hence, $n \geq 9$, a contradiction since $n \leq 8$. Thus, $\gamma_{shR}(G) = n$ whenever $n \leq 8$. Now, let $n \geq 9$. Then, consider the following cases:
Case 1: $n \equiv 0 \pmod{10}$

Let $n = 10k$ where $k \in \mathbb{N}$. Then $k = \lfloor \frac{n+1}{10} \rfloor = \frac{n}{10}$. Now, let $S_i = \{v_{10i-9}, v_{10i-1}, v_{10i}\}$ and $D_i = \{v_{10i-6}, v_{10i-5}, v_{10i-4}\}$ for each $i \in \{1, 2, ..., k\}$. Set $W_1 = \bigcup_{i=1}^{k} S_i$, $W_2 = \bigcup_{i=1}^{k} D_i$ and $W_0 = V(G) \setminus (W_1 \cup W_2)$. Then, $g = (W_0, W_1, W_2)$ is an HRDF on G. Then, it is easy to see that for every $v = v_j \in W_0$, there exists $u = v_t \in W_2$ such that $d_G(u, v) = 2$ and there exists $w = v_l \in (W_1 \cup W_2)$ such that $N^2_G(w) \cap W_0 = \{v\}$. By construction, it follows that g is a γ_{shR}-function on G. Thus, we get

$$\begin{aligned} \gamma_{shR}(G) = \omega^{shR}_G(g) &= |W_1| + 2|W_2| \\ &= \sum_{i=1}^{k} |S_i| + 2 \sum_{i=1}^{k} |D_i| \end{aligned}$$

$$
\begin{aligned}
&= 3k + 2(3k)\\
&= 9k\\
&= 10k - k\\
&= n - k
\end{aligned}
$$

where $k = \frac{n}{10}$.

Case 2: $n \equiv r \pmod{10}$ where $1 \leq r \leq 8$

Let $n = 10k + r$ where $k \in \mathbb{N}$ and $1 \leq r \leq 8$. Then $k = \lfloor \frac{n+1}{10} \rfloor = \frac{n-r}{10}$ where $r \in \{1, 2, ..., 8\}$. In view of *Case 1*, we let $S_i' = \{v_{10i-9}, v_{10i-1}, v_{10i}\}$ and $D_i' = \{v_{10i-6}, v_{10i-5}, v_{10i-4}\}$ for each $i \in \{1, 2, ..., k\}$. Again, set $W_1' = \big(\bigcup_{i=1}^{k} S_i'\big) \cup \{v_{n-r+1}, ..., v_n\}$ where $1 \leq r \leq 8$, $W_2' = \bigcup_{i=1}^{k} D_i'$ and $W_0' = V(G) \setminus (W_1' \cup W_2')$. So, $g' = (W_0', W_1', W_2')$ is an HRDF on G. Note that for every $v' = v_j \in W_0'$, there exists $u' = v_t \in W_2'$ such that $d_G(u', v') = 2$ and there exists $w' = v_l \in (W_1' \cup W_2')$ such that $N_G^2(w') \cap W_0' = \{v'\}$. By construction, it implies that g' is a γ_{shR}-function on G. Hence, we have

$$
\begin{aligned}
\omega_{shR}(g') &= |W_1'| + 2|W_2'|\\
&= \big(\sum_{i=1}^{k} |S_i| + r\big) + 2\sum_{i=1}^{k} |D_i|\\
&= 3k + r + 2(3k)\\
&= 9k + r\\
&= (10k + r) - k\\
&= n - k
\end{aligned}
$$

where $k = \frac{n-r}{10}$ for all $r \in \{1, 2, ..., 8\}$.

Case 3: $n \equiv 9 \pmod{10}$

Let $n = 10p + 9$ where $p \in \mathbb{N} \cup \{0\}$. Then $k = \lfloor \frac{n+1}{10} \rfloor = p + 1 = \frac{n+1}{10}$ where $p \in \mathbb{N} \cup \{0\}$. Again, by *Case 1*, we let $S_i'' = \{v_{10i-9}, v_{10i-1}, v_{10i}\}$ and $D_i'' = \{v_{10i-6}, v_{10i-5}, v_{10i-4}\}$ for each $i \in \{1, 2, ..., k\}$. Now, set $W_1'' = \big(\bigcup_{i=1}^{k} S_i'\big) \cup \{v_{n-8}, v_n\}$, $W_2'' = \big(\bigcup_{i=1}^{k} D_i''\big) \cup \{v_{n-5}, v_{n-4}, v_{n-3}\}$ and $W_0'' = V(G) \setminus (W_1'' \cup W_2'')$. Thus, $g'' = (W_0'', W_1'', W_2'')$ is an HRDF on G. Now, for every $v'' = v_j \in W_0''$, there exists $u'' = v_t \in W_2''$ such that $d_G(u'', v'') = 2$ and there exists $w'' = v_l \in (W_1'' \cup W_2'')$ such that $N_G^2(w'') \cap W_0'' = \{v''\}$. By construction, it follows that g'' is a γ_{shR}-function on G. Hence, we have

$$
\begin{aligned}
\gamma_{shR}(G) = \omega_G^{shR}(g'') &= |W_1''| + 2|W_2''|\\
&= \big(\sum_{i=1}^{p} |S_i''| + 2\big) + 2\big(\sum_{i=1}^{p} |D_i''| + 3\big)
\end{aligned}
$$

$$
\begin{aligned}
&= 3p + 2 + 2(3p + 3)\\
&= 9p + 8\\
&= (10p + 9) - (p + 1)\\
&= n - k
\end{aligned}
$$

where $k = \frac{n+1}{10}$.
This proves the assertion. □

The proof of the next result is similar to that of Proposition 4.

Proposition 5. *Let $G = C_n$ with $n \geq 3$. Then,*

$$
\gamma_{shR}(G) = \begin{cases} n, & \text{if } n \in \{3, 4, 6, 7, 8, 9\} \\ 4, & \text{if } n = 5 \\ n - k, & \text{if } n \geq 10, \end{cases}
$$

where $k = \lfloor \frac{n}{10} \rfloor$.

Theorem 7. *Let G be a graph of order n. Then $\gamma_{sh}(G) = \gamma_{shR}(G)$ if and only if each component of G is complete. In this case, $\gamma_{sh}(G) = \gamma_{shR}(G) = n$.*

Proof. Suppose $\gamma_{sh}(G) = \gamma_{shR}(G)$ and let $f = (V_0, V_1, V_2)$ be a γ_{shR}-function on G. By Proposition 1, $V_1 \cup V_2$ is a super hop dominating set on G. Hence, $\gamma_{sh}(G) \leq |V_1| + |V_2| \leq |V_1| + 2|V_2| = \gamma_{shR}(G)$. Since $\gamma_{sh}(G) = \gamma_{shR}(G)$, it follows that $|V_2| = 0$. By Proposition 1(i), we have $|V_0| = 0$. This implies that $|V_1| = n = |V(G)|$. Thus, $\gamma_{sh}(G) = \gamma_{shR}(G) = n$. By Theorem 2, we find that each component of G is complete.

For the converse, suppose that every component of G is complete. By Theorem 2 and Corollary 2, we have $\gamma_{sh}(G) = \gamma_{shR}(G) = n$. □

Theorem 8. *Let G be a graph of order n such that $\gamma_{sh}(G) < \gamma_{shR}(G)$. Then $\gamma_{shR}(G) = \gamma_{sh}(G) + 1$ if and only if there exist a set $S \subseteq V(G)$ and vertex v such that $S \cup \{v\}$ is a γ_{sh}-set of G and $V(G) \setminus (S \cup \{v\}) \subseteq N_G^2(v)$.*

Proof. Suppose $\gamma_{shR}(G) = \gamma_{sh}(G) + 1$ and let $f = (V_0, V_1, V_2)$ be a γ_{shR}-function on G. If $\gamma_{sh}(G) = |V_1| + |V_2|$, then the assumption implies that $|V_1| + |V_2| + 1 = |V_1| + 2|V_2|$. Hence, $|V_2| = 1$ and $|V_1| = \gamma_{sh}(G) - 1$. This implies that $V_1 \cup V_2$ is a γ_{sh}-set in G. Let $V_2 = \{v\}$ and $S = V_1$. Then $V_0 = V(G) \setminus (S \cup \{v\})$. Since f satisfies (SHR1), we have $V(G) \setminus (S \cup \{v\}) \subseteq N_G^2(v)$. Next, suppose that $\gamma_{sh}(G) < |V_1| + |V_2|$. Then $\gamma_{sh}(G) + 1 \leq |V_1| + |V_2|$. Since $\gamma_{shR}(G) = \gamma_{sh}(G) + 1$, we have $|V_1| + 2|V_2| \leq |V_1| + |V_2|$. Hence, it implies that $|V_2| = 0$ and so, $|V_0| = 0$ and $|V_1| = n$. Consequently, $\gamma_{sh}(G) = n-1$. Let $Q = V(G) \setminus \{x\}$ be a γ_{sh}-set on G. Then there exists $v \in Q$ such that $x \in N_G^2(v)$. Let $S = Q \setminus \{v\}$. Then $S \cup \{v\} = Q$ is γ_{sh}-set in G and $V(G) \setminus (S \cup \{v\}) = \{x\} \subseteq N_G^2(v)$.

For the converse, suppose there exist a set $S \subseteq V(G)$ and vertex v such that $S \cup \{v\}$ is a γ_{sh}-set of G and $V(G) \setminus (S \cup \{v\}) \subseteq N_G^2(v)$. Let $V_0 = V(G) \setminus (S \cup \{v\})$, $V_1 = S$, and

$V_2 = \{v\}$. Then $g = (V_0, V_1, V_2)$ is an SHRDF on G. Hence,

$$\gamma_{shR}(G) \leq \omega_G^{shR}(g) = |V_1| + 2|V_2| = |S| + 2 = (\gamma_{sh}(G) - 1) + 2 = \gamma_{sh}(G) + 1.$$

Since $\gamma_{sh}(G) < \gamma_{shR}(G)$, it follows that $\gamma_{shR}(G) = \gamma_{sh}(G) + 1$. □

The *join* of graphs G and H is the graph $G+H$ with vertex set $V(G+H) = V(G) \cup V(H)$ and edge set $E(G + H) = E(G) \cup E(H) \in \{uv : u \in V(G) \text{ and } v \in V(H)\}$.

Theorem 9. *Let H be a non-complete graph. Then $f = (V_0, V_1, V_2)$ is an SHRDF on a graph $G = K_n + H$ if and only if the following conditions are satisfied:*

(i) $V(K_n) \subseteq V_1 \cup V_2$*; and*

(ii) $f|_H$ *is an SHRDF on* H.

Proof. Suppose that $f = (V_0, V_1, V_2)$ is an SHRDF on G. Let $x \in V(K_n)$. Since $xy \in E(G)$ for all $y \in V(G) \setminus \{x\}$ and f is an HRDF on G, it follows that $x \notin V_0$. Hence, $x \in V_1 \cup V_2$. This shows that (i) holds. This implies that $V_0 \subseteq V(H)$. Note that $f|_H = (V_0^H, V_1^H, V_2^H)$, where $V_0^H = V_0$, $V_1^H = V_1 \cap V(H)$, and $V_2^H = V_2 \cap V(H)$. Let $v \in V_0^H$. Since f is an HRDF on G, there exists $w \in V_2 \cap N_G^2(v)$. Hence, $w \in V_2^H$, showing that $f|_H$ is an HRDF on H. Since $V_1 \cup V_2$ is a super dominating set in G, there exists $z \in V_1 \cup V_2$ such that $N_G^2(z) \cap V_0 = \{v\}$. This implies that $z \in V_1^H \cup V_2^H$. Thus, $f|_H$ is a SHRDF on H. This shows that (ii) also holds.

For the converse, suppose (i) and (ii) hold. Let $V_1^H = V_1 \cap V(H)$, $V_2^H = V_2 \cap V(H)$, $V_1^n = V_1 \cap V(K_n)$, and $V_2^n = V_2 \cap V(K_n)$. From (i), it follows that $V_0 = V_0^H \subseteq V(H)$. If $V_0 = \varnothing$, then $f = (\varnothing, V_1^H \cup V_1^n, V_2^H \cup V_2^n)$ is an SHRDF on G. Suppose $V_0 \neq \varnothing$. Let $x \in V_0$. Since $f|_H = (V_0, V_1^H, V_2^H)$ is an HRDF on H, there exists $y \in V_2^H \cap N_H^2(x)$. Hence, $y \in V_2 \cap N_G^2(x)$, showing that $f = (V_0, V_1, V_2)$ is an HRDF on G. Also, since $f|_H$ satisfies (SHR2) on H, it follows that there exists $z \in V_1^H \cup V_2^H$ such that $N_H^2(z) \cap V_0 = \{x\}$. Since $V_1^H \cup V_2^H \subseteq V_1 \cup V_2$, we find that $z \in V_1 \cup V_2$. Consequently, $f = (V_0, V_1, V_2)$ is an SHRDF on a graph G. □

The following corollaries below are direct consequence of Theorem 9.

Corollary 3. *Let G be any graph. Then $\gamma_{shR}(K_n + G) = n + \gamma_{shR}(G)$.*

Proof. Let $D = V(K_n)$ and let $g = (V_0, V_1, V_2)$ be a γ_{shR}-function on G. Let $V_0' = V_0$, $V_1' = D \cup V_1$, and $V_2' = V_2$. Then $f = (V_0', V_1', V_2')$ is an SHRDF on $K_n + G$ by Theorem 9. Hence, we have

$$\begin{aligned}
\gamma_{shR}(K_n + G) \leq \omega_{K_n+G}^{shR}(f) &= |V_1'| + 2|V_2'| \\
&= (|D \cup V_1|) + 2|V_2| \\
&= (|D| + |V_1|) + 2|V_2| \\
&= |D| + (|V_1| + 2|V_2|) \\
&= n + \gamma_{shR}(G).
\end{aligned}$$

On the other hand, let $h = (W_0, W_1, W_2)$ be a γ_{shR}-function on $K_n + G$. By Theorem 9, we have $V(K_n) \subseteq W_1 \cup W_2$ and $h|_G$ is an SHRDF on G. Since h is a γ_{shR}-function, we have $V(K_n) \subseteq W_1$, i.e., $W_1 = V(K_n) \cup (W_1 \cap V(G))$ and $h|G = (W_0, W_1 \cap V(G), W_2)$ Thus,

$$\begin{aligned}\gamma_{shR}(K_n + G) = \omega^{shR}_{K_n+G}(h) &= |W_1| + 2|W_2| \\ &= (|V(K_n)| + |W_1 \cap V(G)|) + 2|W_2| \\ &= n + |W_1 \cap V(G)| + 2|W_2| \\ &= n + \omega^{shR}_G(h|_G) \\ &\geq n + \gamma_{shR}(G).\end{aligned}$$

Therefore, $\gamma_{shR}(K_n + G) = n + \gamma_{shR}(G)$. This establishes the desired equality. □

Corollary 4. *Let n be a positive integer greater than or equal to 3. Then each the following holds:*

(i) $\gamma_{shR}(S_n) = \gamma_{shR}(K_{1,n}) = n + 1$;

(ii) $\gamma_{shR}(F_n) = \gamma_{shR}(K_1 + P_n) = \begin{cases} n+1, & \text{if } n \leq 8 \\ n-k+1, & \text{if } n \geq 9, \end{cases}$
where $k = \lfloor \frac{n+1}{10} \rfloor$.

(iii) $\gamma_{shR}(W_n) = \gamma_{shR}(K_1 + C_n) = \begin{cases} n+1, & \text{if } 3 \leq n \leq 9 \\ n-k+1, & \text{if } n \geq 10, \end{cases}$
where $k = \lfloor \frac{n}{10} \rfloor$.

Theorem 10. *Let G and H be any non-complete graphs. Then, $f = (V_0, V_1, V_2)$ is a SHRDF on $G + H$ if and only if $f|_G$ and $f|_H$ are SHRDF on G and H, respectively.*

Proof. Let $f = (V_0, V_1, V_2)$ be an RDF on $G + H$, and let $V_i^G = V_i \cap V(G)$ and $V_i^H = V_i \cap V(H)$ for $i \in \{0, 1, 2\}$. Then $f|_G = (V_0^G, V_1^G, V_2^G)$ and $f|_H = (V_0^H, V_1^H, V_2^H)$.

Suppose that f is an SHRDF on $G + H$. Let $v \in V_0^G$. Since f satisfies (SHR1) and (SHR2) on $G+H$, there exists $u \in V_2$ such that $d_{G+H}(u, v) = 2$ and there exists $w \in V_1 \cup V_2$ such that $N^2_{G+H}(w) \cap V_0 = \{v\}$. Since $vx \in E(G + H)$ for all $x \in V(H)$, it follows that $u \in V_2^G$, $w \in V_1^G \cup V_2^G$, $d_G(u, v) = 2$, and $N^2_G(w) \cap V_0^G = \{v\}$. Thus, $f|_G$ is an SHRDF on G. Using similar argument, $f|_H$ is also an SHRDF on H.

Conversely, suppose that $f|_G$ and $f|_H$ are SHRDF on G and H, respectively. Let $v' \in V_0$. Then, either $v' \in V_0^G$ or $v' \in V_0^H$. Without loss of generality, suppose that $v' \in V_0^G$. Since $f|_G$ is an SHRDF on G, there exists $u' \in V_2^G$ such that $d_G(u', v') = 2$ and there exists $w' \in V_1^G \cup V_2^G$ such that $N^2_G(w') \cap V_0^G = \{v'\}$. Since $V_2^G \subset V_2$ and $V_1^G \cup V_2^G \subseteq V_1 \cup V_2$, it implies that $u' \in V_2$ and $w' \in V_1 \cup V_2$ for which $d_{G+H}(u', v') = 2$ and $N^2_{G+H}(w') \cap V_0^G = \{v'\}$. Thus, f is an SHRDF on $G + H$. □

The next result follows from Theorem 10.

Corollary 5. *Let G and H be any two graphs. Then, $\gamma_{shR}(G+H) = \gamma_{shR}(G) + \gamma_{shR}(H)$.*

Proof. Let $g = (V_0^G, V_1^G, V_2^G)$ and $h = (V_0^H, V_1^H, V_2^H)$ be γ_{shR}-functions on G and H, respectively. Let $V_i = V_i^G \cup V_i^H$ for each $i \in \{0,1,2\}$. Then $g = f|_G$ and $g = f|_H$ where $f = (V_0, V_1, V_2)$. By Theorem 10, f is an SHRDF on $G + H$. Hence, we have

$$\begin{aligned}\gamma_{shR}(G+H) \leq \omega_{G+H}^{shR}(f) &= |V_1| + 2|V_2| \\ &= |V_1^G \cup V_1^H| + 2|V_2^G \cup V_2^H| \\ &= (|V_1^G| + 2|V_2^G|) + (|V_1^H| + 2|V_2^H|) \\ &= \gamma_{shR}(G) + \gamma_{shR}(H).\end{aligned}$$

Now, let $f' = (V_0', V_1', V_2')$ be a γ_{shR}-function on $G + H$. Then it follows that $f'|_G = (V_0' \cap V(G), V_1' \cap V(G), V_2' \cap V(G))$ and $f'|_H = (V_0' \cap V(H), V_1' \cap V(H), V_2' \cap V(H))$ are SHRDF on G and H, respectively, by Theorem 10. Thus, we get

$$\begin{aligned}\gamma_{shR}(G+H) = \omega_{G+H}^{shR}(f') &= |V_1'| + 2|V_2'| \\ &= |(V_1' \cap V(G)) \cup (V_1' \cap V(H))| + 2|(V_2' \cap V(G)) \cup (V_2' \cap V(H))| \\ &= (|V_1' \cap V(G)| + 2|V_2' \cap V(G)|) + (|V_1' \cap V(G)| + 2|V_2' \cap V(G)|) \\ &= \omega_G^{shR}(f'|_G) + \omega_H^{shR}(f'|_H) \\ &\geq \gamma_{shR}(G) + \gamma_{shR}(H).\end{aligned}$$

Therefore, $\gamma_{shR}(G+H) = \gamma_{shR}(G) + \gamma_{shR}(H)$. This establishes the desired equality. □

The next result is a direct consequence of Corollary 2 and Corollary 5.

Corollary 6. *If $G = K_{m,n}$ where $m, n \geq 1$, then $\gamma_{shR}(G) = m + n$.*

4. Conclusion

This study introduced a new variation of hop Roman domination called super hop Roman domination. Some bounds and exact values of the super hop Roman domination number of some classes of graphs were determined. Necessary and sufficient conditions for functions to be super hop Roman dominating in the join of some graphs were obtained. The newly defined parameter can be studied for other classes of graphs and sharp and tight bounds on the parameter may be obtained.

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