EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2026, Vol. 19, Issue 1, Article Number 7089
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Pythagorean Fuzzy Soft Structures on Boolean Rings

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Abstract. This paper introduces the concept of Pythagorean fuzzy soft (PFS) structures within the framework of Boolean rings (BRs), combining the expressive power of soft set theory with that of Pythagorean fuzzy sets in algebraic systems. We begin by defining fundamental operations on PFSSs—such as intersection, union, AND, and OR—and then specialize these structures to form Pythagorean fuzzy soft Boolean rings (PFSBRs). We further define Pythagorean fuzzy soft ideals (PFSIs) as a refined subclass of PFSBRs that exhibit ideal-like properties under the operations of the ring. Several theorems are established to demonstrate closure properties under these operations, with examples provided to illustrate the applicability and consistency of the proposed framework. This approach enhances the modeling of uncertainty in algebraic contexts and offers potential for future applications in decision science and soft computing.

2020 Mathematics Subject Classifications: 06D72, 03E72, 08A72, 16Y99

Key Words and Phrases: Fuzzy Set (FS), Pythagorean Fuzzy (PF), Soft Set (SS), Fuzzy Soft Set (FSS), Pythagorean Fuzzy Soft (PFS), Boolean Ring (BR), Pythagorean Fuzzy Soft Set (PFSS), Pythagorean Fuzzy Soft Boolean Ring (PFSBR), Pythagorean Fuzzy Soft Ideal (PFSI), Fuzzy Soft Boolean Ring (FSBR).

1. Introduction

Fuzzy sets (FSs) were initially proposed by Zadeh [1] in 1965, introducing the idea of partial membership and enabling formal reasoning under uncertainty. Building on this foundation, Abou-Zaid [2] investigated fuzzy subnear-rings and ideals, laying early algebraic groundwork for incorporating fuzziness into ring-theoretic structures. Molodtsov [3] later initiated the theory of soft sets, providing a parameterized framework for representing uncertainty that cannot be efficiently modeled using classical fuzzy or probabilistic

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DOI: https://doi.org/10.29020/nybg.ejpam.v19i1.7089

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approaches. Maji et al. [4] further formalized soft set theory and explored its algebraic and computational aspects.

Ahmat and Kharal [5] extended these ideas by introducing fuzzy soft sets (FSSs), which combine the uncertainty-handling ability of fuzzy sets with the parametric flexibility of soft sets. Acar et al. [6] subsequently studied soft rings and their algebraic properties, opening the way to integrating soft set theory with classical algebraic systems. Yager [7] then proposed the concept of Pythagorean membership grades, leading to the development of Pythagorean fuzzy sets (PF-sets), a generalization of intuitionistic fuzzy sets with the condition $\mu^2 + \nu^2 \leq 1$.

Because traditional crisp and even intuitionistic frameworks were inadequate for modeling complex real-world problems—such as those arising in economics, engineering, and environmental systems—fuzzy and soft paradigms gained increasing significance. In 2001, Maji et al. [8] introduced fuzzy soft sets (FSSs) as an extension of standard soft sets and demonstrated their utility in decision-making applications. Rehman et al. [9] later analyzed the algebraic behavior of FSSs under basic operations like union, intersection, AND, and OR, providing counterexamples to highlight their distinctive properties. Finally, the concept of Pythagorean fuzzy soft sets (PFSSs) was established by Peng et al. [10] in 2015, who defined essential operations such as complement, union, intersection, AND/OR, addition, and multiplication, thereby bridging Pythagorean fuzzy logic with soft set theory.

Some authors have examined the algebraic properties of fuzzy soft sets (FSSs). Initially, Maji et al. [8] defined FSSs and obtained several foundational results. FSSs were represented over Boolean rings (BRs) by Rao et al. [11]. The structure of soft Boolean near-rings (BNRs) was studied by Rao et al. [12], who applied soft set theory to classical near-ring concepts. The notion of soft intersection BNRs was later introduced by Rao et al. [13], emphasizing their structural properties and potential applications. Further, Rao et al. [14] extended this framework by developing $(\in, \in \vee q_k)$-fuzzy soft BNRs, thereby broadening the algebraic treatment of fuzzy soft systems. Rao et al. [15] also refined the algebraic foundation of soft computing through fuzzy soft BNRs and their idealistic variants, formalizing new fuzzy logic procedures and structural generalizations.

Building upon these developments, Rao et al. [16] introduced intuitionistic fuzzy soft Boolean rings (IFS-BRs), marking a transition toward intuitionistic environments. More recently, Rao et al. [17] proposed $(\in, \in \vee q_k)$-intuitionistic fuzzy soft Boolean near-rings, offering a unified framework that integrates intuitionistic membership and quasi-coincidence relations. Their work significantly strengthens the algebraic underpinnings of fuzzy soft systems, aligning closely with our aim to extend such intuitionistic and Pythagorean fuzzy logic structures over Boolean rings. In this study, we rigorously formalize the intuitionistic fuzzy framework introduced by Platil and Tanaka [18], underscoring the role of structured trade-off modeling in fuzzy decision-making BNRs and advancing soft algebraic frameworks through generalized membership notions. The algebraic aspects of bipolar fuzzy soft Boolean rings (BFSBRs) were subsequently presented by Rao et al. [19], highlighting further extensions toward dual and bipolar fuzzy environments.

In this study, we introduce and apply the PFSS idea to BRs. The operations on four different types of PFSSs—the intersection, union, AND, and OR—are then examined.

Similar multi-criteria evaluation frameworks have been explored in the context of intuitionistic fuzzy sets by Platil and Tanaka [18], reinforcing the relevance of structured trade-off modeling in fuzzy decision-making.

2. Preliminaries

First, the definitions of the main terms—BR and PFSS—that will be used in the next section are covered. The algebraic foundation for uncertainty modeling has also been generalized through structures such as fuzzy Γ-semimodules over Γ-semirings [20], which provide a broader algebraic environment for representing graded membership relations.

Definition 1. *Let S symbolise a universe and K signify a set of parameters. $Q(S)$ is a model of S's power set. In this case, $\varkappa : G \to P(S)$ is a set-valued function, and the couple $(\varkappa, G)$ over S.*

Definition 2. *Assume for the moment that $(\varkappa, G)$ is a non-null FSS. Then an FSBR over $\mathbb{R}$ is described as $(\varkappa, G)$ if for each $g^* \in G, \varkappa(g^*) = \varkappa_{g^*}$ is an F-sub-BR of $\mathbb{R}$, i.e.,*
(i) $\varkappa_{g^*}(\mathrm{u}_1^* - \tau_1^*) \geq \varkappa_{g^*}(\mathrm{u}_1^*) \wedge \varkappa_{g^*}(\tau_1^*)$
(ii) $\varkappa_{g^*}(\mathrm{u}_1^* \tau_1^*) \geq \varkappa_{g^*}(\mathrm{u}_1^*) \wedge \varkappa_{g^*}(\tau_1^*)$, $\forall \mathrm{u}_1^*, \tau_1^* \in \mathbb{R}$.

Definition 3. *Let S mean a universe, K mean a set of parameters, and let $PF(S)$ denote the collection of all PF-sets on S. If $G \subseteq K$ and $\varkappa : G \to PF(S)$ is a mapping, then the pair $(\varkappa, G)$ is called a PFSS over S.*

A PFSS over S is essentially a parameterized family of PF-sets of S. For any parameter $g \in G, \varkappa(g)$ is a PF-set associated with the parameter g, and can be expressed as:

$$\varkappa(g) = \left\{ \left\langle s, \alpha_{\varkappa(s)}, \beta_{\varkappa(s)} \right\rangle \mid s \in S \right\}$$

In practical decision-making processes, compared to intuitionistic fuzzy soft sets (IFSSs), PFSSs offer a larger membership space for description. They overcome the limitation of IFSSs, where the total of degrees of both membership and absence cannot effectively describe cases when their sum exceeds 1. As a result, PFSSs possess stronger practical applicability.

Definition 4. *Let $G, V \subseteq K$ and $(\varkappa, G)$ and (ϖ, V) be two PFSSs over S. If $(\varkappa, G)$ and (ϖ, V) fulfill the two requirements listed below:*
(i) $V \subseteq G$
(ii) $\forall v \in V, s \in S$, $\alpha_{\varpi(v)}(s) \leq \alpha_{\varkappa(v)}(s)$ and $\beta_{\varpi(v)}(s) \geq \beta_{\varkappa(v)}(s)$.
Then we call (ϖ, V) the PFS-subset of $(\varkappa, G)$, denoted by $(\varpi, V) \subseteq (\varkappa, G)$.

Definition 5. *Let $(\varkappa, G)$ and (ϖ, V) be two PFSSs over U. If $(\varpi, V) \subseteq (\varkappa, G)$ and $(\varkappa, G) \subseteq (\varpi, V)$, then we call $(\varkappa, G)$ equal (ϖ, V), are commonly referred to as PFSS equals and indicated by $(\varkappa, G) = (\varpi, V)$.*

Definition 6. *Let $(\varkappa, G)$ and (ϖ, V) be two PFSSs over U. The union of $(\varkappa, G)$ and (ϖ, V) is defined to be the PFSS $(\varkappa, G) \cup (\varpi, V) = (\varrho, R)$ meeting the axioms listed below:*
(i) $R = G \cup V$
(ii) $\forall r \in R$ *and* $s \in S$,

$$\alpha_\varrho(s) = \begin{cases} \alpha_\varkappa(s) \vee \alpha_\varpi(s) & \text{if } r \in G \cap V \\ \alpha_\varkappa(s) & \text{if } r \in G - V \\ \alpha_\varpi(s) & \text{if } r \in V - G \end{cases}$$

and

$$\beta_\varrho(s) = \begin{cases} \beta_\varkappa(s) \wedge \beta_\varpi(s) & \text{if } r \in G \cap V \\ \beta_\varkappa(s) & \text{if } r \in G - V \\ \beta_\varpi(s) & \text{if } r \in V - G \end{cases}$$

Definition 7. *Let $(\varkappa, G)$ and (ϖ, V) be two PFSSs over U. The intersection of $(\varkappa, G)$ and (ϖ, V) is defined to be the PFSS $(\varkappa, G) \cap (\varpi, V) = (\varrho, R)$ meeting the axioms listed below:*
(i) $R = G \cap V \neq \emptyset$
(ii) $\forall r \in R$ *and* $s \in S$,

$$\alpha_\varrho(s) = \alpha_\varkappa(s) \wedge \alpha_\varpi(s) \text{ and } \beta_\varrho(s) = \beta_\varkappa(s) \vee \beta_\varpi(s).$$

Definition 8. *Let $(\varkappa, G)$ be a PFSS over S. The complement, denoted by $(\varkappa, G)^c = (\varkappa^c, G)$, is defined as: $\forall s \in S$,*

$$\alpha_{\varkappa^c}(s) = \beta_\varkappa(s) \text{ and } \beta_{\varkappa^c}(s) = \alpha_\varkappa(s).$$

Definition 9. *Let $(\varkappa, G)$ and (ϖ, V) be two PFSSs over S. Then, the AND operation between $(\varkappa, G)$ and (ϖ, V) is a new PFSS denoted by*

$$(\varkappa, G) \wedge (\varpi, V) = (\varrho, G \times V)$$

where $\forall (g, v) \in G \times V$ and $s \in S$,

$$\varrho(g, v) = \left((\alpha_{g^*}(s) \wedge \alpha_v(s)) \,, \, (\beta_{g^*}(s) \vee \beta_v(s)) \right).$$

Definition 10. *Let $(\varkappa, G)$ and (ϖ, V) be two PFSSs over S. Then, the OR operation of $(\varkappa, G)$ and (ϖ, V) is a new PFSS denoted by*

$$(\varkappa, G) \vee (\varpi, V) = (\varrho, G \times V)$$

where $\forall (g, v) \in G \times V$ and $s \in S$,

$$\varrho(g, v) = \left((\alpha_{g^*}(s) \vee \alpha_v(s)) \,, \, (\beta_{g^*}(s) \wedge \beta_v(s)) \right).$$

3. Pythagorean Fuzzy Soft Boolean Rings

In this section, the foundational aspects of Pythagorean fuzzy soft Boolean ring (PFSBR) theory are introduced. Subsequently, we explore the induced algebraic structures in detail and provide rigorous proofs for several theorems that form the backbone of the proposed framework.

Definition 11. *Let $\mathbb{R}$ be a BR. A PFSS $\varkappa = (\alpha_\varkappa, \beta_\varkappa)$ over $\mathbb{R}$ is called a PFSBR if $\forall \mathfrak{u}_1^*, \tau_1^* \in \mathbb{R}$, the prerequisites listed below are met:*
(i) $[\alpha_\varkappa(\varphi_1^* + \tau_1^*)]^2 \geq [\alpha_\varkappa(\varphi_1^*)]^2 \wedge [\alpha_\varkappa(\tau_1^*)]^2$
(ii) $[\alpha_\varkappa(\varphi_1^* \tau_1^*)]^2 \geq [\alpha_\varkappa(\varphi_1^*)]^2 \wedge [\alpha_\varkappa(\tau_1^*)]^2$
(iii) $[\beta_\varkappa(\varphi_1^* + \tau_1^*)]^2 \leq [\beta_\varkappa(\varphi_1^*)]^2 \vee [\beta_\varkappa(\tau_1^*)]^2$
(iv) $[\beta_\varkappa(\varphi_1^* \tau_1^*)]^2 \leq [\beta_\varkappa(\varphi_1^*)]^2 \vee [\beta_\varkappa(\tau_1^*)]^2$.

Example 1. *Suppose that the nonempty set $\mathbb{R} = \{0, g^*, s^*, r^*\}$ is equipped with two binary operations, $+$ and $\cdot$, such that the algebraic structure $(\mathbb{R}, +, \cdot)$ forms a BR. The operations $+$ and $\cdot$ are defined in accordance with the axioms of Boolean rings, as described below.*

$+$	0	g^*	s^*	r^*
0	0	g^*	s^*	r^*
g^*	g^*	0	r^*	s^*
s^*	s^*	r^*	0	g^*
r^*	r^*	s^*	g^*	0

$\cdot$	0	g^*	s^*	r^*
0	0	0	0	0
g^*	0	g^*	r^*	s^*
s^*	0	r^*	s^*	g^*
r^*	0	s^*	g^*	r^*

Hence, $(\varkappa, G)$ is a PFSS over a BR $\mathbb{R}$. Let $G = \{k_1^1, k_2^1, k_3^1\}$ be the set of parameters defined by

$$\begin{aligned}
\varkappa(k_1^1) &= \{(0, 0.8, 0.12), (g^*, 0.46, 0.24), (s^*, 0.48, 0.48), (r^*, 0.46, 0.46)\},\\
\varkappa(k_2^1) &= \{(0, 0.74, 0.14), (g^*, 0.51, 0.37), (s^*, 0.31, 0.14), (r^*, 0.31, 0.37)\},\\
\varkappa(k_3^1) &= \{(0, 0.62, 0.28), (g^*, 0.26, 0.48), (s^*, 0.58, 0.14), (r^*, 0.26, 0.48)\}.
\end{aligned}$$

Therefore, $(\varkappa, G)$ is a PFSBR of $\mathbb{R}$.

Theorem 1. *A PFSBR of $\mathbb{R}$ is the result of the intersection of two PFSBRs of $\mathbb{R}$.*

Proof. Suppose that $\varkappa = (\alpha_\varkappa, \beta_\varkappa)$ and $\varpi = (\alpha_\varpi, \beta_\varpi)$ are two PFSBRs of $\mathbb{R}$. Then $\forall \varphi_1^*, \tau_1^* \in \mathbb{R}$, we have

$$\begin{aligned}
[\alpha_\varrho(\varphi_1^* + \tau_1^*)]^2 &= [\alpha_{\varkappa\cap\varpi}(\varphi_1^* + \tau_1^*)]^2\\
&= [\alpha_\varkappa(\varphi_1^* + \tau_1^*)]^2 \wedge [\alpha_\varpi(\varphi_1^* + \tau_1^*)]^2\\
&\geq ([\alpha_\varkappa(\varphi_1^*)]^2 \wedge [\alpha_\varkappa(\tau_1^*)]^2) \wedge ([\alpha_\varpi(\varphi_1^*)]^2 \wedge [\alpha_\varpi(\tau_1^*)]^2)\\
&= ([\alpha_\varkappa(\varphi_1^*)]^2 \wedge [\alpha_\varpi(\varphi_1^*)]^2) \wedge ([\alpha_\varkappa(\tau_1^*)]^2 \wedge [\alpha_\varpi(\tau_1^*)]^2)\\
&= [\alpha_{\varkappa\cap\varpi}(\varphi_1^*)]^2 \wedge [\alpha_{\varkappa\cap\varpi}(\tau_1^*)]^2\\
&= [\alpha_\varrho(\varphi_1^*)]^2 \wedge [\alpha_\varrho(\tau_1^*)]^2,
\end{aligned}$$

$$\begin{aligned}
[\alpha_{\varrho}(\varphi_1^*\tau_1^*)]^2 &= [\alpha_{\varkappa\cap\varpi}(\varphi_1^*\tau_1^*)]^2 \\
&= [\alpha_{\varkappa}(\varphi_1^*\tau_1^*)]^2 \wedge [\alpha_{\varpi}(\varphi_1^*\tau_1^*)]^2 \\
&\geq ([\alpha_{\varkappa}(\varphi_1^*)]^2 \wedge [\alpha_{\varkappa}(\tau_1^*)]^2) \wedge ([\alpha_{\varpi}(\varphi_1^*)]^2 \wedge [\alpha_{\varpi}(\tau_1^*)]^2) \\
&= ([\alpha_{\varkappa}(\varphi_1^*)]^2 \wedge [\alpha_{\varpi}(\varphi_1^*)]^2) \wedge ([\alpha_{\varkappa}(\tau_1^*)]^2 \wedge [\alpha_{\varpi}(\tau_1^*)]^2) \\
&= [\alpha_{\varkappa\cap\varpi}(\varphi_1^*)]^2 \wedge [\alpha_{\varkappa\cap\varpi}(\tau_1^*)]^2 \\
&= [\alpha_{\varrho}(\varphi_1^*)]^2 \wedge [\alpha_{\varrho}(\tau_1^*)]^2,
\end{aligned}$$

$$\begin{aligned}
[\beta_{\varrho}(\varphi_1^*+\tau_1^*)]^2 &= [\beta_{\varkappa\cap\varpi}(\varphi_1^*+\tau_1^*)]^2 \\
&= [\beta_{\varkappa}(\varphi_1^*+\tau_1^*)]^2 \vee [\beta_{\varpi}(\varphi_1^*+\tau_1^*)]^2 \\
&\leq ([\beta_{\varkappa}(\varphi_1^*)]^2 \vee [\beta_{\varkappa}(\tau_1^*)]^2) \vee ([\beta_{\varpi}(\varphi_1^*)]^2 \vee [\beta_{\varpi}(\tau_1^*)]^2) \\
&= ([\beta_{\varkappa}(\varphi_1^*)]^2 \vee [\beta_{\varpi}(\varphi_1^*)]^2) \vee ([\beta_{\varkappa}(\tau_1^*)]^2 \vee [\beta_{\varpi}(\tau_1^*)]^2) \\
&= [\beta_{\varkappa\cap\varpi}(\varphi_1^*)]^2 \vee [\beta_{\varkappa\cap\varpi}(\tau_1^*)]^2 \\
&= [\beta_{\varrho}(\varphi_1^*)]^2 \vee [\beta_{\varrho}(\tau_1^*)]^2,
\end{aligned}$$

$$\begin{aligned}
[\beta_{\varrho}(\varphi_1^*\tau_1^*)]^2 &= [\beta_{\varkappa\cap\varpi}(\varphi_1^*\tau_1^*)]^2 \\
&= [\beta_{\varkappa}(\varphi_1^*\tau_1^*)]^2 \vee [\beta_{\varpi}(\varphi_1^*\tau_1^*)]^2 \\
&\leq ([\beta_{\varkappa}(\varphi_1^*)]^2 \vee [\beta_{\varkappa}(\tau_1^*)]^2) \vee ([\beta_{\varpi}(\varphi_1^*)]^2 \vee [\beta_{\varpi}(\tau_1^*)]^2) \\
&= ([\beta_{\varkappa}(\varphi_1^*)]^2 \vee [\beta_{\varpi}(\varphi_1^*)]^2) \vee ([\beta_{\varkappa}(\tau_1^*)]^2 \vee [\beta_{\varpi}(\tau_1^*)]^2) \\
&= [\beta_{\varkappa\cap\varpi}(\varphi_1^*)]^2 \vee [\beta_{\varkappa\cap\varpi}(\tau_1^*)]^2 \\
&= [\beta_{\varrho}(\varphi_1^*)]^2 \vee [\beta_{\varrho}(\tau_1^*)]^2.
\end{aligned}$$

Hence, $(\varkappa, G) \cap (\varpi, V)$ is a PFSBR of $\mathbb{R}$.

Theorem 2. *A PFSBR of $\mathbb{R}$ is the result of the union of two PFSBRs of $\mathbb{R}$.*

Proof. Suppose that $\varkappa = (\alpha_{\varkappa}, \beta_{\varkappa})$ and $\varpi = (\alpha_{\varpi}, \beta_{\varpi})$ are two PFSBRs of $\mathbb{R}$. Then $\forall \varphi_1^*, \tau_1^* \in \mathbb{R}$, we consider 3 cases.

Case I: If $r \in G - V$, then

$$\begin{aligned}
[\alpha_{\varrho}(\varphi_1^*+\tau_1^*)]^2 &= [\alpha_{\varkappa}(\varphi_1^*+\tau_1^*)]^2 \\
&\geq [\alpha_{\varkappa}(\varphi_1^*)]^2 \wedge [\alpha_{\varkappa}(\tau_1^*)]^2 \\
&= [\alpha_{\varrho}(\varphi_1^*)]^2 \wedge [\alpha_{\varrho}(\tau_1^*)]^2,
\end{aligned}$$

$$\begin{aligned}
[\alpha_{\varrho}(\varphi_1^*\tau_1^*)]^2 &= [\alpha_{\varkappa}(\varphi_1^*\tau_1^*)]^2 \\
&\geq [\alpha_{\varkappa}(\varphi_1^*)]^2 \wedge [\alpha_{\varkappa}(\tau_1^*)]^2 \\
&= [\alpha_{\varrho}(\varphi_1^*)]^2 \wedge [\alpha_{\varrho}(\tau_1^*)]^2,
\end{aligned}$$

$$[\beta_{\varrho}(\varphi_1^*+\tau_1^*)]^2 = [\beta_{\varkappa}(\varphi_1^*+\tau_1^*)]^2$$

$$\begin{aligned} &\leq [\beta_{\varkappa}(\varphi_1^*)]^2 \vee [\beta_{\varkappa}(\tau_1^*)]^2 \\ &= [\beta_{\varrho}(\varphi_1^*)]^2 \vee [\beta_{\varrho}(\tau_1^*)]^2, \end{aligned}$$

$$\begin{aligned} [\beta_{\varrho}(\varphi_1^*\tau_1^*)]^2 &= [\beta_{\varkappa}(\varphi_1^*\tau_1^*)]^2 \\ &\leq [\beta_{\varkappa}(\varphi_1^*)]^2 \vee [\beta_{\varkappa}(\tau_1^*)]^2 \\ &= [\beta_{\varrho}(\varphi_1^*)]^2 \vee [\beta_{\varrho}(\tau_1^*)]^2. \end{aligned}$$

Case II: If $r \in V - G$, then

$$\begin{aligned} [\alpha_{\varrho}(\varphi_1^* + \tau_1^*)]^2 &= [\alpha_{\varpi}(\varphi_1^* + \tau_1^*)]^2 \\ &\geq [\alpha_{\varpi}(\varphi_1^*)]^2 \wedge [\alpha_{\varpi}(\tau_1^*)]^2 \\ &= [\alpha_{\varrho}(\varphi_1^*)]^2 \wedge [\alpha_{\varrho}(\tau_1^*)]^2, \end{aligned}$$

$$\begin{aligned} [\alpha_{\varrho}(\varphi_1^*\tau_1^*)]^2 &= [\alpha_{\varpi}(\varphi_1^*\tau_1^*)]^2 \\ &\geq [\alpha_{\varpi}(\varphi_1^*)]^2 \wedge [\alpha_{\varpi}(\tau_1^*)]^2 \\ &= [\alpha_{\varrho}(\varphi_1^*)]^2 \wedge [\alpha_{\varrho}(\tau_1^*)]^2, \end{aligned}$$

$$\begin{aligned} [\beta_{\varrho}(\varphi_1^* + \tau_1^*)]^2 &= [\beta_{\varpi}(\varphi_1^* + \tau_1^*)]^2 \\ &\leq [\beta_{\varpi}(\varphi_1^*)]^2 \vee [\beta_{\varpi}(\tau_1^*)]^2 \\ &= [\beta_{\varrho}(\varphi_1^*)]^2 \vee [\beta_{\varrho}(\tau_1^*)]^2, \end{aligned}$$

$$\begin{aligned} [\beta_{\varrho}(\varphi_1^*\tau_1^*)]^2 &= [\beta_{\varpi}(\varphi_1^*\tau_1^*)]^2 \\ &\leq [\beta_{\varpi}(\varphi_1^*)]^2 \vee [\beta_{\varpi}(\tau_1^*)]^2 \\ &= [\beta_{\varrho}(\varphi_1^*)]^2 \vee [\beta_{\varrho}(\tau_1^*)]^2. \end{aligned}$$

Case III: If $r \in G \cap V$, then

$$\begin{aligned} [\alpha_{\varrho}(\varphi_1^* + \tau_1^*)]^2 &= [\alpha_{\varkappa}(\varphi_1^* + \tau_1^*)]^2 \vee [\alpha_{\varpi}(\varphi_1^* + \tau_1^*)]^2 \\ &\geq ([\alpha_{\varkappa}(\varphi_1^*)]^2 \wedge [\alpha_{\varkappa}(\tau_1^*)]^2) \vee ([\alpha_{\varpi}(\varphi_1^*)]^2 \wedge [\alpha_{\varpi}(\tau_1^*)]^2) \\ &= ([\alpha_{\varkappa}(\varphi_1^*)]^2 \vee [\alpha_{\varpi}(\varphi_1^*)]^2) \wedge ([\alpha_{\varkappa}(\tau_1^*)]^2 \vee [\alpha_{\varpi}(\tau_1^*)]^2) \\ &= [\alpha_{\varkappa\vee\varpi}(\varphi_1^*)]^2 \wedge [\alpha_{\varkappa\vee\varpi}(\tau_1^*)]^2 \\ &= [\alpha_{\varrho}(\varphi_1^*)]^2 \wedge [\alpha_{\varrho}(\tau_1^*)]^2, \end{aligned}$$

$$\begin{aligned} [\alpha_{\varrho}(\varphi_1^*\tau_1^*)]^2 &= [\alpha_{\varkappa}(\varphi_1^*\tau_1^*)]^2 \vee [\alpha_{\varpi}(\varphi_1^*\tau_1^*)]^2 \\ &\geq ([\alpha_{\varkappa}(\varphi_1^*)]^2 \wedge [\alpha_{\varkappa}(\tau_1^*)]^2) \vee ([\alpha_{\varpi}(\varphi_1^*)]^2 \wedge [\alpha_{\varpi}(\tau_1^*)]^2) \\ &= ([\alpha_{\varkappa}(\varphi_1^*)]^2 \vee [\alpha_{\varpi}(\varphi_1^*)]^2) \wedge ([\alpha_{\varkappa}(\tau_1^*)]^2 \vee [\alpha_{\varpi}(\tau_1^*)]^2) \\ &= [\alpha_{\varkappa\vee\varpi}(\varphi_1^*)]^2 \wedge [\alpha_{\varkappa\vee\varpi}(\tau_1^*)]^2 \end{aligned}$$

$$= [\alpha_\varrho(\varphi_1^*)]^2 \wedge [\alpha_\varrho(\tau_1^*)]^2,$$

$$\begin{aligned}
[\beta_\varrho(\varphi_1^*+\tau_1^*)]^2 &= [\beta_\varkappa(\varphi_1^*+\tau_1^*)]^2 \wedge [\beta_\varpi(\varphi_1^*+\tau_1^*)]^2 \\
&\leq ([\beta_\varkappa(\varphi_1^*)]^2 \vee [\beta_\varkappa(\tau_1^*)]^2) \wedge ([\beta_\varpi(\varphi_1^*)]^2 \vee [\beta_\varpi(\tau_1^*)]^2) \\
&= ([\beta_\varkappa(\varphi_1^*)]^2 \wedge [\beta_\varpi(\varphi_1^*)]^2) \vee ([\beta_\varkappa(\tau_1^*)]^2 \wedge [\beta_\varpi(\tau_1^*)]^2) \\
&= [\beta_{\varkappa\wedge\varpi}(\varphi_1^*)]^2 \vee [\beta_{\varkappa\wedge\varpi}(\tau_1^*)]^2 \\
&= [\beta_\varrho(\varphi_1^*)]^2 \vee [\beta_\varrho(\tau_1^*)]^2,
\end{aligned}$$

$$\begin{aligned}
[\beta_\varrho(\varphi_1^*\tau_1^*)]^2 &= [\beta_\varkappa(\varphi_1^*\tau_1^*)]^2 \wedge [\beta_\varpi(\varphi_1^*\tau_1^*)]^2 \\
&\leq ([\beta_\varkappa(\varphi_1^*)]^2 \vee [\beta_\varkappa(\tau_1^*)]^2) \wedge ([\beta_\varpi(\varphi_1^*)]^2 \vee [\beta_\varpi(\tau_1^*)]^2) \\
&= ([\beta_\varkappa(\varphi_1^*)]^2 \wedge [\beta_\varpi(\varphi_1^*)]^2) \vee ([\beta_\varkappa(\tau_1^*)]^2 \wedge [\beta_\varpi(\tau_1^*)]^2) \\
&= [\beta_{\varkappa\wedge\varpi}(\varphi_1^*)]^2 \vee [\beta_{\varkappa\wedge\varpi}(\tau_1^*)]^2 \\
&= [\beta_\varrho(\varphi_1^*)]^2 \vee [\beta_\varrho(\tau_1^*)]^2.
\end{aligned}$$

Hence, $(\varkappa, G) \cup (\varpi, V)$ is a PFSBR of $\mathbb{R}$.

Theorem 3. *A PFSBR of $\mathbb{R}$ is the result of the AND operation of two PFSBRs of $\mathbb{R}$.*

Proof. For all $\varphi_1^*, \tau_1^* \in \mathbb{R}$, we have

$$\begin{aligned}
[\alpha_\varrho(\varphi_1^*+\tau_1^*)]^2 &= [\alpha_{\varkappa\wedge\varpi}(\varphi_1^*+\tau_1^*)]^2 \\
&= [\alpha_\varkappa(\varphi_1^*+\tau_1^*)]^2 \wedge [\alpha_\varpi(\varphi_1^*+\tau_1^*)]^2 \\
&\geq ([\alpha_\varkappa(\varphi_1^*)]^2 \wedge [\alpha_\varkappa(\tau_1^*)]^2) \wedge ([\alpha_\varpi(\varphi_1^*)]^2 \wedge [\alpha_\varpi(\tau_1^*)]^2) \\
&= ([\alpha_\varkappa(\varphi_1^*)]^2 \wedge [\alpha_\varpi(\varphi_1^*)]^2) \wedge ([\alpha_\varkappa(\tau_1^*)]^2 \wedge [\alpha_\varpi(\tau_1^*)]^2) \\
&= [\alpha_{\varkappa\wedge\varpi}(\varphi_1^*)]^2 \wedge [\alpha_{\varkappa\wedge\varpi}(\tau_1^*)]^2 \\
&= [\alpha_\varrho(\varphi_1^*)]^2 \wedge [\alpha_\varrho(\tau_1^*)]^2,
\end{aligned}$$

$$\begin{aligned}
[\alpha_\varrho(\varphi_1^*\tau_1^*)]^2 &= [\alpha_{\varkappa\wedge\varpi}(\varphi_1^*\tau_1^*)]^2 \\
&= [\alpha_\varkappa(\varphi_1^*\tau_1^*)]^2 \wedge [\alpha_\varpi(\varphi_1^*\tau_1^*)]^2 \\
&\geq ([\alpha_\varkappa(\varphi_1^*)]^2 \wedge [\alpha_\varkappa(\tau_1^*)]^2) \wedge ([\alpha_\varpi(\varphi_1^*)]^2 \wedge [\alpha_\varpi(\tau_1^*)]^2) \\
&= ([\alpha_\varkappa(\varphi_1^*)]^2 \wedge [\alpha_\varpi(\varphi_1^*)]^2) \wedge ([\alpha_\varkappa(\tau_1^*)]^2 \wedge [\alpha_\varpi(\tau_1^*)]^2) \\
&= [\alpha_{\varkappa\wedge\varpi}(\varphi_1^*)]^2 \wedge [\alpha_{\varkappa\wedge\varpi}(\tau_1^*)]^2 \\
&= [\alpha_\varrho(\varphi_1^*)]^2 \wedge [\alpha_\varrho(\tau_1^*)]^2,
\end{aligned}$$

$$\begin{aligned}
[\beta_\varrho(\varphi_1^*+\tau_1^*)]^2 &= [\beta_{\varkappa\vee\varpi}(\varphi_1^*+\tau_1^*)]^2 \\
&= [\beta_\varkappa(\varphi_1^*+\tau_1^*)]^2 \vee [\beta_\varpi(\varphi_1^*+\tau_1^*)]^2 \\
&\leq ([\beta_\varkappa(\varphi_1^*)]^2 \vee [\beta_\varkappa(\tau_1^*)]^2) \vee ([\beta_\varpi(\varphi_1^*)]^2 \vee [\beta_\varpi(\tau_1^*)]^2)
\end{aligned}$$

$$\begin{aligned}
&= ([\beta_\varkappa(\varphi_1^*)]^2 \vee [\beta_\varpi(\varphi_1^*)]^2) \vee ([\beta_\varkappa(\tau_1^*)]^2 \vee [\beta_\varpi(\tau_1^*)]^2)\\
&= [\beta_{\varkappa\vee\varpi}(\varphi_1^*)]^2 \vee [\beta_{\varkappa\vee\varpi}(\tau_1^*)]^2\\
&= [\beta_\varrho(\varphi_1^*)]^2 \vee [\beta_\varrho(\tau_1^*)]^2,
\end{aligned}$$

$$\begin{aligned}
[\beta_\varrho(\varphi_1^*\tau_1^*)]^2 &= [\beta_{\varkappa\vee\varpi}(\varphi_1^*\tau_1^*)]^2\\
&= [\beta_\varkappa(\varphi_1^*\tau_1^*)]^2 \vee [\beta_\varpi(\varphi_1^* + \tau_1^*)]^2\\
&\leq ([\beta_\varkappa(\varphi_1^*)]^2 \vee [\beta_\varkappa(\tau_1^*)]^2) \vee ([\beta_\varpi(\varphi_1^*)]^2 \vee [\beta_\varpi(\tau_1^*)]^2)\\
&= ([\beta_\varkappa(\varphi_1^*)]^2 \vee [\beta_\varpi(\varphi_1^*)]^2) \vee ([\beta_\varkappa(\tau_1^*)]^2 \vee [\beta_\varpi(\tau_1^*)]^2)\\
&= [\beta_{\varkappa\vee\varpi}(\varphi_1^*)]^2 \vee [\beta_{\varkappa\vee\varpi}(\tau_1^*)]^2\\
&= [\beta_\varrho(\varphi_1^*)]^2 \vee [\beta_\varrho(\tau_1^*)]^2.
\end{aligned}$$

Hence, $(\varkappa, G) \wedge (\varpi, V)$ is a PFSBR of $\mathbb{R}$.

Theorem 4. *A PFSBR of* $\mathbb{R}$ *is the result of the OR operation of two PFSBRs of* $\mathbb{R}$.

Proof. For all $\varphi_1^*, \tau_1^* \in \mathbb{R}$, we have

$$\begin{aligned}
[\alpha_\varrho(\varphi_1^* + \tau_1^*)]^2 &= [\alpha_{\varkappa\vee\varpi}(\varphi_1^* + \tau_1^*)]^2\\
&= \alpha_\varkappa(\varphi_1^* + \tau_1^*)]^2 \vee [\alpha_\varpi(\varphi_1^* + \tau_1^*)]^2\\
&\geq ([\alpha_\varkappa(\varphi_1^*)]^2 \wedge [\alpha_\varkappa(\tau_1^*)]^2) \vee ([\alpha_\varpi(\varphi_1^*)]^2 \wedge [\alpha_\varpi(\tau_1^*)]^2)\\
&= ([\alpha_\varkappa(\varphi_1^*)]^2 \vee [\alpha_\varpi(\varphi_1^*)]^2) \wedge ([\alpha_\varkappa(\tau_1^*)]^2 \vee [\alpha_\varpi(\tau_1^*)]^2)\\
&= [\alpha_{\varkappa\vee\varpi}(\varphi_1^*)]^2 \wedge [\alpha_{\varkappa\vee\varpi}(\tau_1^*)]^2\\
&= [\alpha_\varrho(\varphi_1^*)]^2 \wedge [\alpha_\varrho(\tau_1^*)]^2,
\end{aligned}$$

$$\begin{aligned}
[\alpha_\varrho(\varphi_1^*\tau_1^*)]^2 &= [\alpha_{\varkappa\vee\varpi}(\varphi_1^*\tau_1^*)]^2\\
&= \alpha_\varkappa(\varphi_1^*\tau_1^*)]^2 \vee [\alpha_\varpi(\varphi_1^*\tau_1^*)]^2\\
&\geq ([\alpha_\varkappa(\varphi_1^*)]^2 \wedge [\alpha_\varkappa(\tau_1^*)]^2) \vee ([\alpha_\varpi(\varphi_1^*)]^2 \wedge [\alpha_\varpi(\tau_1^*)]^2)\\
&= ([\alpha_\varkappa(\varphi_1^*)]^2 \vee [\alpha_\varpi(\varphi_1^*)]^2) \wedge ([\alpha_\varkappa(\tau_1^*)]^2 \vee [\alpha_\varpi(\tau_1^*)]^2)\\
&= [\alpha_{\varkappa\vee\varpi}(\varphi_1^*)]^2 \wedge [\alpha_{\varkappa\vee\varpi}(\tau_1^*)]^2\\
&= [\alpha_\varrho(\varphi_1^*)]^2 \wedge [\alpha_\varrho(\tau_1^*)]^2,
\end{aligned}$$

$$\begin{aligned}
[\beta_\varrho(\varphi_1^* + \tau_1^*)]^2 &= [\beta_{\varkappa\wedge\varpi}(\varphi_1^* + \tau_1^*)]^2\\
&= [\beta_\varkappa(\varphi_1^* + \tau_1^*)]^2 \wedge [\beta_\varpi(\varphi_1^* + \tau_1^*)]^2\\
&\leq ([\beta_\varkappa(\varphi_1^*)]^2 \vee [\beta_\varkappa(\tau_1^*)]^2) \wedge ([\beta_\varpi(\varphi_1^*)]^2 \vee [\beta_\varpi(\tau_1^*)]^2)\\
&= ([\beta_\varkappa(\varphi_1^*)]^2 \wedge [\beta_\varpi(\varphi_1^*)]^2) \vee ([\beta_\varkappa(\tau_1^*)]^2 \wedge [\beta_\varpi(\tau_1^*)]^2)\\
&= [\beta_{\varkappa\wedge\varpi}(\varphi_1^*)]^2 \vee [\beta_{\varkappa\wedge\varpi}(\tau_1^*)]^2\\
&= [\beta_\varrho(\varphi_1^*)]^2 \wedge [\beta_\varrho(\tau_1^*)]^2,
\end{aligned}$$

$$\begin{aligned}
[\beta_{\varrho}(\varphi_1^*\tau_1^*)]^2 &= [\beta_{\varkappa\wedge\varpi}(\varphi_1^*\tau_1^*)]^2 \\
&= [\beta_{\varkappa}(\varphi_1^*\tau_1^*)]^2 \wedge [\beta_{\varpi}(\varphi_1^*\tau_1^*)]^2 \\
&\leq ([\beta_{\varkappa}(\varphi_1^*)]^2 \vee [\beta_{\varkappa}(\tau_1^*)]^2) \wedge ([\beta_{\varpi}(\varphi_1^*)]^2 \vee [\beta_{\varpi}(\tau_1^*)]^2) \\
&= ([\beta_{\varkappa}(\varphi_1^*)]^2 \wedge [\beta_{\varpi}(\varphi_1^*)]^2) \vee ([\beta_{\varkappa}(\tau_1^*)]^2 \wedge [\beta_{\varpi}(\tau_1^*)]^2) \\
&= [\beta_{\varkappa\wedge\varpi}(\varphi_1^*)]^2 \vee [\beta_{\varkappa\wedge\varpi}(\tau_1^*)]^2 \\
&= [\beta_{\varrho}(\varphi_1^*)]^2 \wedge [\beta_{\varrho}(\tau_1^*)]^2.
\end{aligned}$$

Hence, $(\varkappa, G) \vee (\varpi, V)$ is a PFSBR of $\mathbb{R}$.

4. Pythagorean Fuzzy Soft Ideals

This section introduces the foundational concepts of Pythagorean fuzzy soft ideals (PFSIs) within the framework of BR theory. It further explores the underlying algebraic structures of PFSIs over BRs, culminating in the formulation and proof of several related theorems that establish their key properties.

While PFSBRs capture the structural properties of PFSSs over BRs, the notion of PFSIs serves to refine this structure further by incorporating an ideal-theoretic perspective. Intuitively, a PFSI can be viewed as a PFSBR that satisfies additional absorption-like conditions, mirroring the role of ideals in classical ring theory. This additional constraint makes PFSIs suitable for modeling situations where certain subsets exhibit more restrictive algebraic behavior, such as closure under multiplication by ring elements. The definitions and results presented in this section aim to formalize and distinguish these stronger structural features.

Definition 12. *A PFSS $(\varkappa, G)$ over $\mathbb{R}$ is called a PFSI over $\mathbb{R}$ if $\forall \varphi_1^*, \tau_1^* \in \mathbb{R}$, the prerequisites listed below are met:*
(i) $[\alpha_{\varkappa}(\varphi_1^* + \tau_1^*)]^2 \geq [\alpha_{\varkappa}(\varphi_1^*)]^2 \wedge [\alpha_{\varkappa}(\tau_1^*)]^2$
(ii) $[\alpha_{\varkappa}(\varphi_1^*\tau_1^*)]^2 \geq [\alpha_{\varkappa}(\tau_1^*)]^2$
(iii) $[\beta_{\varkappa}(\varphi_1^* + \tau_1^*)]^2 \leq [\beta_{\varkappa}(\varphi_1^*)]^2 \vee [\beta_{\varkappa}(\tau_1^*)]^2$
(iv) $[\beta_{\varkappa}(\varphi_1^*\tau_1^*)]^2 \leq [\beta_{\varkappa}(\tau_1^*)]^2$.

Example 2. *We shall use the BR $\mathbb{R}$ defined above in Example 1. Using the parameters $G = \{k_1^1, k_2^1, k_3^1\}$, define a PFSS $(\varkappa, G)$ over $\mathbb{R}$ by*

$$\begin{aligned}
\varkappa(k_1^1) &= \{(0, 0.80, 0.12), (g^*, 0.46, 0.24), (s^*, 0.48, 0.48), (r^*, 0.46, 0.46)\}, \\
\varkappa(k_2^1) &= \{(0, 0.74, 0.14), (g^*, 0.51, 0.37), (s^*, 0.31, 0.14), (r^*, 0.31, 0.37)\}, \\
\varkappa(k_3^1) &= \{(0, 0.62, 0.28), (g^*, 0.26, 0.48), (s^*, 0.58, 0.14), (r^*, 0.26, 0.48)\}.
\end{aligned}$$

Verifying that $(\varkappa, G)$ is a PFSI of $\mathbb{R}$.

Theorem 5. *A PFSI of $\mathbb{R}$ is the result of the intersection of two PFSIs of $\mathbb{R}$.*

Proof. Suppose that $\varkappa = (\alpha_{\varkappa}, \beta_{\varkappa})$ and $\varpi = (\alpha_{\varpi}, \beta_{\varpi})$ are two PFSIs of $\mathbb{R}$. Then $\forall \varphi_1^*, \tau_1^* \in \mathbb{R}$, we have

$$[\alpha_{\varrho}(\varphi_1^* + \tau_1^*)]^2 = [\alpha_{\varkappa\wedge\varpi}(\varphi_1^* + \tau_1^*)]^2$$

$$\begin{aligned}
&= [\alpha_{\varkappa}(\varphi_1^* + \tau_1^*)]^2 \wedge [\alpha_{\varpi}(\varphi_1^* + \tau_1^*)]^2 \\
&\geq ([\alpha_{\varkappa}(\varphi_1^*)]^2 \wedge [\alpha_{\varkappa}(\tau_1^*)]^2) \wedge ([\alpha_{\varpi}(\varphi_1^*)]^2 \wedge [\alpha_{\varpi}(\tau_1^*)]^2) \\
&= ([\alpha_{\varkappa}(\varphi_1^*)]^2 \wedge [\alpha_{\varpi}(\varphi_1^*)]^2) \wedge ([\alpha_{\varkappa}(\tau_1^*)]^2 \wedge [\alpha_{\varpi}(\tau_1^*)]^2) \\
&= [\alpha_{\varkappa\wedge\varpi}(\varphi_1^*)]^2 \wedge [\alpha_{\varkappa\wedge\varpi}(\tau_1^*)]^2 \\
&= [\alpha_{\varrho}(\varphi_1^*)]^2 \wedge [\alpha_{\varrho}(\tau_1^*)]^2,
\end{aligned}$$

$$\begin{aligned}
[\alpha_{\varrho}(\varphi_1^*\tau_1^*)]^2 &= [\alpha_{\varkappa\wedge\varpi}(\varphi_1^*\tau_1^*)]^2 \\
&= [\alpha_{\varkappa}(\varphi_1^*\tau_1^*)]^2 \wedge [\alpha_{\varpi}(\varphi_1^*\tau_1^*)]^2 \\
&\geq [\alpha_{\varkappa}(\tau_1^*)]^2 \wedge [\alpha_{\varpi}(\tau_1^*)]^2 \\
&= [\alpha_{\varkappa\wedge\varpi}(\tau_1^*)]^2 \\
&= [\alpha_{\varrho}(\tau_1^*)]^2,
\end{aligned}$$

$$\begin{aligned}
[\beta_{\varrho}(\varphi_1^* + \tau_1^*)]^2 &= [\beta_{\varkappa\vee\varpi}(\varphi_1^* + \tau_1^*)]^2 \\
&= [\beta_{\varkappa}(\varphi_1^* + \tau_1^*)]^2 \vee [\beta_{\varpi}(\varphi_1^* + \tau_1^*)]^2 \\
&\leq ([\beta_{\varkappa}(\varphi_1^*)]^2 \vee [\beta_{\varkappa}(\tau_1^*)]^2) \vee ([\beta_{\varpi}(\varphi_1^*)]^2 \vee [\beta_{\varpi}(\tau_1^*)]^2) \\
&= ([\beta_{\varkappa}(\varphi_1^*)]^2 \vee [\beta_{\varpi}(\varphi_1^*)]^2) \vee ([\beta_{\varkappa}(\tau_1^*)]^2 \vee [\beta_{\varpi}(\tau_1^*)]^2) \\
&= [\beta_{\varkappa\vee\varpi}(\varphi_1^*)]^2 \vee [\beta_{\varkappa\vee\varpi}(\tau_1^*)]^2 \\
&= [\beta_{\varrho}(\varphi_1^*)]^2 \vee [\beta_{\varrho}(\tau_1^*)]^2,
\end{aligned}$$

$$\begin{aligned}
[\beta_{\varrho}(\varphi_1^*\tau_1^*)]^2 &= [\beta_{\varkappa\vee\varpi}(\varphi_1^*\tau_1^*)]^2 \\
&= [\beta_{\varkappa}(\varphi_1^*\tau_1^*)]^2 \vee [\beta_{\varpi}(\varphi_1^*\tau_1^*)]^2 \\
&\leq [\beta_{\varkappa}(\tau_1^*)]^2) \vee [\beta_{\varpi}(\tau_1^*)]^2 \\
&= [\beta_{\varkappa\vee\varpi}(\tau_1^*)]^2 \\
&= [\beta_{\varrho}(\tau_1^*)]^2.
\end{aligned}$$

Thus, $(\varkappa, G) \cap (\varpi, V)$ is a PFSI of $\mathbb{R}$.

Theorem 6. *A PFSI of* $\mathbb{R}$ *is the result of the union of two PFSIs of* $\mathbb{R}$*.*

Proof. Suppose that $\varkappa = (\alpha_{\varkappa}, \beta_{\varkappa})$ and $\varpi = (\alpha_{\varpi}, \beta_{\varpi})$ are two PFSIs of $\mathbb{R}$. Then $\forall \varphi_1^*, \tau_1^* \in \mathbb{R}$, we consider 3 cases.

Case I: If $r \in G - V$, then

$$\begin{aligned}
[\alpha_{\varrho}(\varphi_1^* + \tau_1^*)]^2 &= [\alpha_{\varkappa}(\varphi_1^* + \tau_1^*)]^2 \\
&\geq ([\alpha_{\varkappa}(\varphi_1^*)]^2 \wedge [\alpha_{\varkappa}(\tau_1^*)]^2) \\
&= [\alpha_{\varrho}(\varphi_1^*)]^2 \wedge [\alpha_{\varrho}(\tau_1^*)]^2,
\end{aligned}$$

$$[\alpha_{\varrho}(\varphi_1^*\tau_1^*)]^2 = [\alpha_{\varkappa}(\varphi_1^*\tau_1^*)]^2$$

$$
\begin{aligned}
&\geq [\alpha_{\varkappa}(\tau_1^*)]^2 \\
&= [\alpha_{\varrho}(\tau_1^*)]^2,
\end{aligned}
$$

$$
\begin{aligned}
[\beta_{\varrho}(\varphi_1^* + \tau_1^*)]^2 &= [\beta_{\varkappa}(\varphi_1^* + \tau_1^*)]^2 \\
&\leq ([\beta_{\varkappa}(\varphi_1^*)]^2 \vee [\beta_{\varkappa}(\tau_1^*)]^2) \\
&= [\beta_{\varrho}(\varphi_1^*)]^2 \vee [\beta_{\varrho}(\tau_1^*)]^2,
\end{aligned}
$$

$$
\begin{aligned}
[\beta_{\varrho}(\varphi_1^*\tau_1^*)]^2 &= [\beta_{\varkappa}(\varphi_1^*\tau_1^*)]^2 \\
&\leq [\beta_{\varkappa}(\tau_1^*)]^2 \\
&= [\beta_{\varrho}(\tau_1^*)]^2.
\end{aligned}
$$

Case II: If $r \in V - G$, then

$$
\begin{aligned}
[\alpha_{\varrho}(\varphi_1^* + \tau_1^*)]^2 &= [\alpha_{\varpi}(\varphi_1^* + \tau_1^*)]^2 \\
&\geq ([\alpha_{\varpi}(\varphi_1^*)]^2 \wedge [\alpha_{\varpi}(\tau_1^*)]^2) \\
&= [\alpha_{\varrho}(\varphi_1^*)]^2 \wedge [\alpha_{\varrho}(\tau_1^*)]^2,
\end{aligned}
$$

$$
\begin{aligned}
[\alpha_{\varrho}(\varphi_1^*\tau_1^*)]^2 &= [\alpha_{\varpi}(\varphi_1^* + \tau_1^*)]^2 \\
&\geq [\alpha_{\varpi}(\tau_1^*)]^2 \\
&= [\alpha_{\varrho}(\tau_1^*)]^2,
\end{aligned}
$$

$$
\begin{aligned}
[\beta_{\varrho}(\varphi_1^* + \tau_1^*)]^2 &= [\beta_{\varpi}(\varphi_1^* + \tau_1^*)]^2 \\
&\leq ([\beta_{\varpi}(\varphi_1^*)]^2 \vee [\beta_{\varpi}(\tau_1^*)]^2) \\
&= [\beta_{\varrho}(\varphi_1^*)]^2 \vee [\beta_{\varrho}(\tau_1^*)]^2,
\end{aligned}
$$

$$
\begin{aligned}
[\beta_{\varrho}(\varphi_1^*\tau_1^*)]^2 &= [\beta_{\varpi}(\varphi_1^*\tau_1^*)]^2 \\
&\leq [\beta_{\varpi}(\tau_1^*)]^2 \\
&= [\beta_{\varrho}(\tau_1^*)]^2.
\end{aligned}
$$

Case III: If $r \in G \cap V$, then

$$
\begin{aligned}
[\alpha_{\varrho}(\varphi_1^* + \tau_1^*)]^2 &= [\alpha_{\varkappa}(\varphi_1^* + \tau_1^*)]^2 \vee [\alpha_{\varpi}(\varphi_1^* + \tau_1^*)]^2 \\
&\geq ([\alpha_{\varkappa}(\varphi_1^*)]^2 \wedge [\alpha_{\varkappa}(\tau_1^*)]^2) \vee ([\alpha_{\varpi}(\varphi_1^*)]^2 \wedge [\alpha_{\varpi}(\tau_1^*)]^2) \\
&= ([\alpha_{\varkappa}(\varphi_1^*)]^2 \vee [\alpha_{\varpi}(\varphi_1^*)]^2) \wedge ([\alpha_{\varkappa}(\tau_1^*)]^2 \vee [\alpha_{\varpi}(\tau_1^*)]^2) \\
&= [\alpha_{\varkappa\vee\varpi}(\varphi_1^*)]^2 \wedge [\alpha_{\varkappa\vee\varpi}(\tau_1^*)]^2 \\
&= [\alpha_{\varrho}(\varphi_1^*)]^2 \wedge [\alpha_{\varrho}(\tau_1^*)]^2,
\end{aligned}
$$

$$
[\alpha_{\varrho}(\varphi_1^*\tau_1^*)]^2 = [\alpha_{\varkappa}(\varphi_1^*\tau_1^*)]^2 \vee [\alpha_{\varpi}(\varphi_1^*\tau_1^*)]^2
$$

$$\begin{aligned} &\geq [\alpha_{\varkappa}(\tau_1^*)]^2 \vee [\alpha_{\varpi}(\tau_1^*)]^2 \\ &= [\alpha_{\varkappa\vee\varpi}(\tau_1^*)]^2 \\ &= [\alpha_{\varrho}(\tau_1^*)]^2, \end{aligned}$$

$$\begin{aligned} [\beta_{\varrho}(\varphi_1^* + \tau_1^*)]^2 &= [\beta_{\varkappa}(\varphi_1^* + \tau_1^*)]^2 \wedge [\beta_{\varpi}(\varphi_1^* + \tau_1^*)]^2 \\ &\leq ([\beta_{\varkappa}(\varphi_1^*)]^2 \vee [\beta_{\varkappa}(\tau_1^*)]^2) \wedge ([\beta_{\varpi}(\varphi_1^*)]^2 \vee [\beta_{\varpi}(\tau_1^*)]^2) \\ &= ([\beta_{\varkappa}(\varphi_1^*)]^2 \wedge [\beta_{\varpi}(\varphi_1^*)]^2) \vee ([\beta_{\varkappa}(\tau_1^*)]^2 \wedge [\beta_{\varpi}(\tau_1^*)]^2) \\ &= [\beta_{\varkappa\wedge\varpi}(\varphi_1^*)]^2 \vee [\beta_{\varkappa\wedge\varpi}(\tau_1^*)]^2 \\ &= [\beta_{\varrho}(\varphi_1^*)]^2 \vee [\beta_{\varrho}(\tau_1^*)]^2, \end{aligned}$$

$$\begin{aligned} [\beta_{\varrho}(\varphi_1^*\tau_1^*)]^2 &= [\beta_{\varkappa}(\varphi_1^*\tau_1^*)]^2 \wedge [\beta_{\varpi}(\varphi_1^*\tau_1^*)]^2 \\ &\leq [\beta_{\varkappa}(\tau_1^*)]^2 \wedge [\beta_{\varpi}(\tau_1^*)]^2 \\ &= [\beta_{\varkappa\wedge\varpi}(\tau_1^*)]^2 \\ &= [\beta_{\varrho}(\tau_1^*)]^2. \end{aligned}$$

Thus, $(\varkappa, G) \cup (\varpi, V)$ is a PFSI of $\mathbb{R}$.

Theorem 7. *A PFSI of* $\mathbb{R}$ *is the result of the AND operation of two PFSIs of* $\mathbb{R}$.

Proof. For all $\varphi_1^*, \tau_1^* \in \mathbb{R}$, we have

$$\begin{aligned} [\alpha_{\varrho}(\varphi_1^* + \tau_1^*)]^2 &= [\alpha_{\varkappa}(\varphi_1^* + \tau_1^*)]^2 \wedge [\alpha_{\varpi}(\varphi_1^* + \tau_1^*)]^2 \\ &\geq ([\alpha_{\varkappa}(\varphi_1^*)]^2 \wedge [\alpha_{\varkappa}(\tau_1^*)]^2) \wedge ([\alpha_{\varpi}(\varphi_1^*)]^2 \wedge [\alpha_{\varpi}(\tau_1^*)]^2) \\ &= ([\alpha_{\varkappa}(\varphi_1^*)]^2 \wedge [\alpha_{\varpi}(\varphi_1^*)]^2) \wedge ([\alpha_{\varkappa}(\tau_1^*)]^2 \wedge [\alpha_{\varpi}(\tau_1^*)]^2) \\ &= [\alpha_{\varkappa\wedge\varpi}(\varphi_1^*)]^2 \wedge [\alpha_{\varkappa\wedge\varpi}(\tau_1^*)]^2 \\ &= [\alpha_{\varrho}(\varphi_1^*)]^2 \wedge [\alpha_{\varrho}(\tau_1^*)]^2, \end{aligned}$$

$$\begin{aligned} [\alpha_{\varrho}(\varphi_1^*\tau_1^*)]^2 &= [\alpha_{\varkappa\wedge\varpi}(\varphi_1^*\tau_1^*)]^2 \\ &= [\alpha_{\varkappa}(\varphi_1^*\tau_1^*)]^2 \wedge [\alpha_{\varpi}(\varphi_1^*\tau_1^*)]^2 \\ &\geq [\alpha_{\varkappa}(\tau_1^*)]^2 \wedge [\alpha_{\varpi}(\tau_1^*)]^2 \\ &= [\alpha_{\varkappa\wedge\varpi}(\tau_1^*)]^2 \\ &= [\alpha_{\varrho}(\tau_1^*)]^2, \end{aligned}$$

$$\begin{aligned} [\beta_{\varrho}(\varphi_1^* + \tau_1^*)]^2 &= [\beta_{\varkappa\vee\varpi}(\varphi_1^* + \tau_1^*)]^2 \\ &= [\beta_{\varkappa}(\varphi_1^* + \tau_1^*)]^2 \vee [\beta_{\varpi}(\varphi_1^* + \tau_1^*)]^2 \\ &\leq ([\beta_{\varkappa}(\varphi_1^*)]^2 \vee [\beta_{\varkappa}(\tau_1^*)]^2) \vee ([\beta_{\varpi}(\varphi_1^*)]^2 \vee [\beta_{\varpi}(\tau_1^*)]^2) \\ &= ([\beta_{\varkappa}(\varphi_1^*)]^2 \vee [\beta_{\varpi}(\varphi_1^*)]^2) \vee ([\beta_{\varkappa}(\tau_1^*)]^2 \vee [\beta_{\varpi}(\tau_1^*)]^2) \end{aligned}$$

$$\begin{aligned} &= [\beta_{\varkappa\vee\varpi}(\varphi_1^*)]^2 \vee [\beta_{\varkappa\vee\varpi}(\tau_1^*)]^2 \\ &= [\beta_\varrho(\varphi_1^*)]^2 \vee [\beta_\varrho(\tau_1^*)]^2, \end{aligned}$$

$$\begin{aligned} [\beta_\varrho(\varphi_1^*\tau_1^*)]^2 &= [\beta_{\varkappa\vee\varpi}(\varphi_1^*\tau_1^*)]^2 \\ &= [\beta_\varkappa(\varphi_1^*+\tau_1^*)]^2 \vee [\beta_\varpi(\varphi_1^*+\tau_1^*)]^2 \\ &\leq [\beta_\varkappa(\tau_1^*)]^2 \vee [\beta_\varpi(\tau_1^*)]^2 \\ &= [\beta_{\varkappa\vee\varpi}(\tau_1^*)]^2 \\ &= [\beta_\varrho(\tau_1^*)]^2. \end{aligned}$$

Thus, $(\varkappa, G) \wedge (\varpi, V)$ is a PFSI of $\mathbb{R}$.

Theorem 8. *A PFSI of $\mathbb{R}$ is the result of the OR operation of two PFSIs of $\mathbb{R}$.*

Proof. For all $\varphi_1^*, \tau_1^* \in \mathbb{R}$, we have

$$\begin{aligned} [\alpha_\varrho(\varphi_1^*+\tau_1^*)]^2 &= [\alpha_\varkappa(\varphi_1^*+\tau_1^*)]^2 \vee [\alpha_\varpi(\varphi_1^*+\tau_1^*)]^2 \\ &\geq ([\alpha_\varkappa(\varphi_1^*)]^2 \wedge [\alpha_\varkappa(\tau_1^*)]^2) \vee ([\alpha_\varpi(\varphi_1^*)]^2 \wedge [\alpha_\varpi(\tau_1^*)]^2) \\ &= ([\alpha_\varkappa(\varphi_1^*)]^2 \vee [\alpha_\varpi(\varphi_1^*)]^2) \wedge ([\alpha_\varkappa(\tau_1^*)]^2 \vee [\alpha_\varpi(\tau_1^*)]^2) \\ &= [\alpha_{\varkappa\vee\varpi}(\varphi_1^*)]^2 \wedge [\alpha_{\varkappa\vee\varpi}(\tau_1^*)]^2 \\ &= [\alpha_\varrho(\varphi_1^*)]^2 \wedge [\alpha_\varrho(\tau_1^*)]^2, \end{aligned}$$

$$\begin{aligned} [\alpha_\varrho(\varphi_1^*\tau_1^*)]^2 &= [\alpha_{\varkappa\vee\varpi}(\varphi_1^*\tau_1^*)]^2 \\ &= [\alpha_\varkappa(\varphi_1^*\tau_1^*)]^2 \vee [\alpha_\varpi(\varphi_1^*\tau_1^*)]^2 \\ &\geq [\alpha_\varkappa(\tau_1^*)]^2 \vee [\alpha_\varpi(\tau_1^*)]^2 \\ &= [\alpha_{\varkappa\vee\varpi}(\tau_1^*)]^2 \\ &= [\alpha_\varrho(\tau_1^*)]^2, \end{aligned}$$

$$\begin{aligned} [\beta_\varrho(\varphi_1^*+\tau_1^*)]^2 &= [\beta_{\varkappa\wedge\varpi}(\varphi_1^*+\tau_1^*)]^2 \\ &= [\beta_\varkappa(\varphi_1^*+\tau_1^*)]^2 \wedge [\beta_\varpi(\varphi_1^*+\tau_1^*)]^2 \\ &\leq ([\beta_\varkappa(\varphi_1^*)]^2 \vee [\beta_\varkappa(\tau_1^*)]^2) \wedge ([\beta_\varpi(\varphi_1^*)]^2 \vee [\beta_\varpi(\tau_1^*)]^2) \\ &= ([\beta_\varkappa(\varphi_1^*)]^2 \wedge [\beta_\varpi(\varphi_1^*)]^2) \vee ([\beta_\varkappa(\tau_1^*)]^2 \wedge [\beta_\varpi(\tau_1^*)]^2) \\ &= [\beta_{\varkappa\wedge\varpi}(\varphi_1^*)]^2 \vee [\beta_{\varkappa\wedge\varpi}(\tau_1^*)]^2 \\ &= [\beta_\varrho(\varphi_1^*)]^2 \vee [\beta_\varrho(\tau_1^*)]^2, \end{aligned}$$

$$\begin{aligned} [\beta_\varrho(\varphi_1^*\tau_1^*)]^2 &= [\beta_{\varkappa\wedge\varpi}(\varphi_1^*\tau_1^*)]^2 \\ &= [\beta_\varkappa(\varphi_1^*\tau_1^*)]^2 \wedge [\beta_\varpi(\varphi_1^*\tau_1^*)]^2 \\ &\leq [\beta_\varkappa(\tau_1^*)]^2 \wedge [\beta_\varpi(\tau_1^*)]^2 \end{aligned}$$

$$\begin{aligned} &= [\beta_{\varkappa \wedge \varpi}(\tau_1^*)]^2 \\ &= [\beta_{\varrho}(\tau_1^*)]^2. \end{aligned}$$

Thus, $(\varkappa, G) \vee (\varpi, V)$ is a PFSI of $\mathbb{R}$.

The consideration of inverse or edge-case fuzzy structures, as discussed in the anti-fuzzy framework of Platil and Vilela [21], may provide insight into dual or adversarial extensions of the present Pythagorean fuzzy soft ideals.

5. Conclusion

In this study, we have introduced and systematically investigated the notion of Pythagorean fuzzy soft sets (PFSSs) defined over Boolean rings (BRs). A rigorous algebraic framework has been developed to unify fuzzy membership modeling with Boolean ring operations. Within this framework, the notions of Pythagorean fuzzy soft subrings and ideals were formally established, and several fundamental results were derived to characterize their closure properties under the operations of union, intersection, AND, and OR.

The results obtained confirm that Pythagorean fuzzy soft rings and ideals retain the essential algebraic consistency inherent to Boolean structures while simultaneously accommodating the dual nature of membership and non-membership degrees under the Pythagorean fuzzy paradigm. This integration of fuzzy logic with algebraic systems not only enriches the theoretical understanding of soft algebraic frameworks but also provides a foundation for extending fuzzy algebraic analysis to diverse mathematical and decision-theoretic settings.

Beyond the immediate algebraic implications, the developed PFSS framework may serve as a bridge between abstract fuzzy algebra and applied fuzzy decision modeling. Recent works, such as the multi-criteria evaluation approach for intuitionistic fuzzy sets by Platil and Tanaka [18], demonstrate the relevance of structured trade-off modeling that aligns well with Pythagorean fuzzy representations. Similarly, generalized algebraic structures for uncertainty modeling—such as fuzzy Γ-semimodules over Γ-semirings proposed by Platil and Petalcorin [20]—offer valuable avenues for further generalization of the PFSS framework. Furthermore, the anti-fuzzy perspective introduced by Platil and Vilela [21] may inspire future studies on dual or inverse Pythagorean fuzzy soft ideals, potentially illuminating new classes of constraint-driven or adversarial algebraic systems.

Overall, this research establishes a mathematically consistent and conceptually versatile foundation for studying fuzzy algebraic systems under the Pythagorean setting. The proposed PFSS on Boolean Rings not only extends the existing fuzzy soft algebraic theory but also opens promising directions for applications in uncertainty quantification, optimization, and fuzzy multi-criteria analysis.

Acknowledgements

This research was supported by University of Phayao and Thailand Science Research and Innovation Fund (Fundamental Fund 2026, Grant No. 2252/2568).

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