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Bi-Univalent Function Families Involving q-Rabotnov Function and q-Analogues of Fibonacci Numbers

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Abstract. Motivated by the interplay between q-calculus and geometric function theory, this paper introduces and investigates a new subclass of bi-univalent functions associated with shell-like curves defined via the q-Rabotnov function and the q-analogue of Fibonacci numbers. By employing the subordination principle, we derive coefficient bounds for the initial Taylor–Maclaurin coefficients, specifically $|a_2|$ and $|a_3|$, and further establish sharp Fekete–Szegö type inequalities for the proposed function class. Our results not only extend and generalize several recent contributions in the theory of bi-univalent functions but also highlight novel connections between q-special functions, shell-like domains, and analytic inequalities. The findings presented herein contribute to a deeper understanding of the structural properties of bi-univalent functions and open avenues for future applications in operator theory and related analytic frameworks.

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1. Introduction

Let $\mathcal{A}$ denote the family of all analytic functions defined on the open unit disk $\mathscr{O}$, where $\mathscr{O}$ is the set of all complex numbers $z = a + ib$ (with $a, b \in \mathbb{R}$) satisfying $|z| < 1$. Geometrically, $\mathscr{O}$ represents the collection of all points in the complex plane that lie strictly inside the unit circle centered at the origin.

The functions $f \in \mathcal{A}$ are normalized to satisfy the following initial conditions:

$$f(0) = 0 \quad \text{and} \quad f'(0) = 1.$$

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These normalization conditions ensure that the functions are uniquely determined and facilitate the study of their properties within the unit disk. For every function $f \in \mathcal{A}$, the Taylor-Maclaurin series expansion can be expressed in the following form:

$$f(z) = z + \sum_{n=2}^{\infty} \alpha_n z^n, \quad (z \in \mathscr{O}). \tag{1}$$

An analytic function f that satisfies $|f(z)| < 1$ and $f(0) = 0$ within the domain $\mathscr{O}$ is called a Schwartz functions. When considering two functions f_1 and f_2 from $\mathcal{A}$, f_1 is referred to as subordinate to f_2, denoted by $f_1 \prec f_2$, if a Schwarz function g exists such that $f_1(z) = f_2(g(z))$ for all $z \in \mathscr{O}$. Additionally, examine the class S, which includes all functions $f \in \mathcal{A}$ that are univalent (injective) in the unit disk $\mathscr{O}$. Let P represent the collection of functions within $\mathcal{A}$ that possess positive real parts, defined as follows:

$$\mathsf{p}(z) = 1 + \sum_{n=1}^{\infty} \mathsf{p}_n z^n = 1 + \mathsf{p}_1 z + \mathsf{p}_2 z^2 + \mathsf{p}_3 z^3 + \ldots, \tag{2}$$

where

$$|\mathsf{p}_n| \leq 2, \quad \text{for all} \quad n \geq 1. \tag{3}$$

This is in accordance with the renowned Carathéodory's Lemma (for more details, see [1]). Essentially, $\varphi \in \mathsf{P}$ if and only if $\varphi(z) \prec (1+z)(1-z)^{-1}$ for $z \in \mathscr{O}$.

As the foundation upon which many important subclasses of analytic functions are built, the class P is crucial to the study of analytic functions. For any function f in the subfamily S of $\mathcal{A}$, there exists an inverse function denoted as f^{-1} and defined by

$$z = f^{-1}(f(z)) \text{ and } \xi = f(f^{-1}(\xi)), \quad (\mathsf{r}_0(f) \geq 0.25; \ |\xi| < \mathsf{r}_0(f); z \in \mathscr{O}). \tag{4}$$

where

$$\eta(\xi) = f^{-1}(\xi) = \xi - \alpha_2 \xi^2 + \left(2\alpha_2^2 - \alpha_3\right) \xi^3 - \left(5\alpha_2^3 + \alpha_4 - 5\alpha_3\alpha_2\right) \xi^4 + \cdots. \tag{5}$$

function $f \in \mathsf{S}$ is said to be bi-univalent if its inverse function $f^{-1} \in \mathsf{S}$. The subclass of S denoted by $\sum$ contains all bi-univalent functions in $\mathscr{O}$. A table illustrating certain functions within the class $\sum$ and their inverse functions is provided below.

Rabotnov-type kernels, originally introduced by Rabotnov [2] within the framework of linear viscoelasticity, provide a powerful tool for modeling hereditary phenomena such as creep and relaxation. These kernels are typically expressed in terms of convolution operators involving Mittag–Leffler-type functions and thus offer a rigorous representation of fractional-order operators in constitutive equations [3]. Due to their remarkable flexibility, Rabotnov kernels have become a standard device in the mathematical description of stress–strain relations with memory effects in mechanics and engineering. Moreover, their deep connection with fractional calculus has established them as a central component in the modern analysis of viscoelastic materials and complex dynamical systems [4–10].

Table 1: Lists several of the starlike classes defined by the subordination principle.

f	f^{-1}
$f_1(z) = \dfrac{z}{1+z}$	$f_1^{-1}(z) = \dfrac{z}{1-z}$
$f_2(z) = -\log(1-z)$	$f_1^{-1}(z) = \dfrac{e^{2z}-1}{e^{2z}+1}$
$f_3(z) = \frac{1}{2}\log\left(\dfrac{1+z}{1-z}\right)$	$f_1^{-1}(z) = \dfrac{e^{z}-1}{e^{z}}$

Definition 1. *[11] Let $\varrho, \varphi, \vartheta \in \mathbb{C}$ with $\Re e(\varrho) > 0$, $\Re e(\varphi) > 0$, $\Re e(\vartheta)$, and $|q| < 1$. The generalized q–Mittag–Leffler function $\mathrm{E}^{\varphi}_{\varrho,\varphi}$ is defined by*

$$\mathrm{E}^{\vartheta}_{\varrho,\varphi}(z;q) = \sum_{n=0}^{\infty} \frac{(q^{\vartheta};q)_n}{(q;q)_n} \frac{z^n}{\Gamma_q(\varrho n + \varphi)}, \tag{6}$$

where Γ_q denotes the q–gamma function.

The q–gamma function Γ_q, which serves as the q–analogue of Euler's gamma function, is defined recursively (see [12, 13]) by

$$\Gamma_q(\kappa+1) \;=\; \frac{1-q^{\kappa}}{1-q}\,\Gamma_q(\kappa) = [\kappa]_q\,\Gamma_q(\kappa),$$

where

$$[\kappa]_q = \begin{cases} \frac{1-q^{\kappa}}{1-q}, & 0<q<1,\ \kappa \in \mathbb{C}^* = \mathbb{C}\setminus\{0\} \\ 1, & q \mapsto 0^+,\ \kappa \in \mathbb{C}^* \\ \kappa, & q \mapsto 1^-,\ \kappa \in \mathbb{C}^* \\ q^{\gamma-1} + q^{\gamma-2} + \cdots + q + 1 = \sum\limits_{n=0}^{\gamma-1} q^n, & 0<q<1,\ \kappa = \gamma \in \mathbb{N}, \end{cases}$$

This formulation ensures that Γ_q retains many of the structural properties of the classical gamma function while encoding the discrete deformation induced by the parameter q.

The q–analogue of the Pochhammer symbol, also known as the q–shifted factorial, is given by (see [13])

$$(\kappa;q)_n = \begin{cases} (1-\kappa)(1-\kappa\,q)\cdots(1-\kappa\,q^{n-1}), & n = 1,2,3,\ldots, \\ 1, & n = 0, \end{cases}$$

and admits the representation

$$(\kappa;q)_n \;=\; \frac{(1-q)^n\,\Gamma_q(\kappa+n)}{\Gamma_q(\kappa)}, \qquad n > 0,$$

which highlights its intrinsic connection to the q–gamma function.

Remark 1. *In the limiting case $q \to 1^-$, the q–Mittag–Leffler function $\mathrm{E}^{\varphi}_{\varrho,\varphi}$ reduces to the classical generalized Mittag–Leffler function, thereby bridging the discrete q–framework with its continuous counterpart. This makes the function an effective tool in geometric function theory, particularly when exploring analytic classes generated by fractional and q–calculus operators.*

Definition 2. *Let $\varrho \in \mathbb{C}$ with $\Re(\varrho) > 0$, $\lambda > 0$, and $|q| < 1$. We define the function $\Phi^{\varphi}_{\varrho,\lambda}(z;q)$ by*

$$\Phi^{\varphi}_{\varrho,\lambda}(z;q) = z^{\varrho} \sum_{n=0}^{\infty} \frac{(q^{\varphi};q)_n}{(q;q)_n} \frac{[\lambda]_q^n}{\Gamma_q\left((n+1)(1+\varrho)\right)} z^{n(1+\varrho)}. \tag{7}$$

When $q \to 1^-$, the function $\Phi^{\varphi}_{\varrho,\lambda}(z;q)$ reduces to the classical Rabotnov function $\Phi_{\varrho,\lambda}(z)$ (see [2]). Since $\Phi^{\varphi}_{\varrho,\lambda}(z;q)$ is not normalized, we adopt the following normalized form:

$$\begin{aligned} \mathbb{R}^{\varphi}_{\varrho,\lambda}(z;q) &= z^{\frac{1}{1+\varrho}+1}\Gamma_q(1+\varrho)\,\Phi^{\varphi}_{\varrho,\lambda}(z^{\frac{1}{1+\varrho}};q) \\ &= z + \sum_{n=2}^{\infty} \frac{(q^{\varphi};q)_{n-1}}{(q;q)_{n-1}} \frac{[\lambda]_q^{n-1}\Gamma_q(1+\varrho)}{\Gamma_q((1+\varrho)n)} z^n, \qquad z \in \mathbb{U}. \end{aligned} \tag{8}$$

Remark 2. *The function $\Phi^{\varphi}_{\varrho,\lambda}(z;q)$ may be regarded as a q–analogue of kernel-type generating functions frequently employed in the study of analytic and bi-univalent function classes. In particular, it encodes the combined influence of the q–Mittag–Leffler structure and the parameter λ, and for $q \to 1^-$, it reduces to its classical counterpart involving Euler's gamma function. Such kernels play a central role in the construction of subclasses of analytic functions defined through subordination, convolution, and fractional q–calculus operators.*

We now introduce a linear operator of Hadamard–convolution type associated with the q–Rabotnov kernel.

Definition 3. *For $\varrho, \varphi \in \mathbb{C}$ with $\Re(\varrho) > 0$ and $\lambda > 0$, the linear operator $\mathcal{F}^{\varphi}_{\varrho,\lambda} : \mathcal{A} \to \mathcal{A}$ is defined by*

$$\mathcal{F}^{\varphi}_{\varrho,\lambda}(f(z);q) = \mathbb{R}^{\varphi}_{\varrho,\lambda}(z;q) * f(z) = z + \sum_{n=2}^{\infty} \frac{(q^{\varphi};q)_{n-1}}{(q;q)_{n-1}} \frac{[\lambda]_q^{n-1}\,\Gamma_q(1+\varrho)}{\Gamma_q(n(1+\varrho))} \alpha_n z^n, \tag{9}$$

where f is of the form (1), and $*$ denotes the Hadamard (or coefficient-wise) product of power series.

Remark 3. *The operator $\mathcal{F}^{\varphi}_{\varrho,\lambda}$ generalizes classical convolution operators by incorporating q–Rabotnov kernels. Such operators play a crucial role in constructing and investigating subclasses of analytic and bi-univalent functions, particularly in deriving sharp coefficient bounds and Fekete–Szegö type inequalities.*

Alsoboh et al. [14], by employing the subordination mapping

$$\Upsilon(z;q) = \frac{1 + q\vartheta_q^2 z^2}{1 - \vartheta_q\, z - q\vartheta_q^2 z^2}, \tag{10}$$

introduced a new family of q–starlike functions. They also established a fundamental connection between the q–analogue of Fibonacci numbers ϑ_q and their associated Fibonacci polynomials

$$\vartheta_q = \frac{1 - \sqrt{4q+1}}{2q}. \tag{11}$$

In particular, they proved that if

$$\Upsilon(z;q) = 1 + \sum_{n=1}^{\infty} \widehat{p}_n\, z^n,$$

then the coefficients $\widehat{p}_n$ satisfy the recurrence relation

$$\widehat{p}_n = \begin{cases} \vartheta_q, & n = 1, \\ (2q+1)\vartheta_q^2, & n = 2, \\ (3q+1)\vartheta_q^3, & n = 3, \\ \big(\varphi_{n+1}(q) + q\,\varphi_{n-1}(q)\big)\vartheta_q^n, & s \geq 4, \end{cases} \tag{12}$$

where the q–Fibonacci polynomials are given by

$$\varphi_n(q) = \frac{(1 - q\vartheta_q)^n - \vartheta_q^{\,n}}{\sqrt{4q+1}}, \qquad n \in \mathbb{N}. \tag{13}$$

The advent of q-calculus has significantly advanced the study of analytic function theory by enabling the discovery of novel subclasses with intricate geometric and algebraic properties. These developments underscore the versatility of q-calculus, demonstrating its potential to enrich classical function theory and uncover new mathematical phenomena. The relevance of these findings extends to both theoretical and applied settings, providing a robust foundation for future research and innovation in the field [15–21].

2. Definition and example

Motivated by q-Fibonacci numbers and the q–Rabotnov operator, this section will now look at a novel subclasses of bi-univalent functions related to shell-like curves.

Definition 4. *A function $f \in \Sigma$ given by* (1)*. We say that f belongs to the class $\mathfrak{R}_{\Sigma_q}(\varrho, \varphi, \lambda)$ if the following subordinations hold:*

$$\partial_q \left(\mathcal{F}_{\varrho,\lambda}^{\varphi}(f(z);q)\right) \prec \Upsilon(z;q) = \frac{1 + q\vartheta_q^2 z^2}{1 - \vartheta_q\, z - q\vartheta_q^2 z^2}, \qquad (z \in \mathscr{O}) \tag{14}$$

and

$$\partial_q \left(\mathcal{F}^{\varphi}_{\varrho,\lambda}(\eta(\xi);q)\right) \prec \Upsilon(\xi;q) = \frac{1+q\vartheta_q^2\xi^2}{1-\vartheta_q\,\xi - q\vartheta_q^2\xi^2}, \qquad (\xi\in\mathscr{O}) \tag{15}$$

where $\eta = f^{-1}$ *is the inverse of* f, ∂_q *denotes the* q*–derivative, and* ϑ_q *is given by* (11).

By imposing suitable specializations on the parameter q, one can generate a number of well-known subclasses of the bi-univalent function class Σ. For clarity, we record below some representative examples which illustrate how the general class $\mathfrak{R}_{\Sigma_q}(\varrho,\varphi,\lambda)$ reduces to particular families under these parameter choices.

Example 1. *If* $q\to 1^-$, *then a function* $f\in\mathfrak{R}_{\Sigma}(\varrho,\varphi,\lambda)$ *is characterized as belonging to the family of bi-univalent functions* $f\in\Sigma$ *that satisfy the subordinations*

$$\left(\mathcal{F}^{\varphi}_{\varrho,\lambda}(f(z);q)\right)' \prec \frac{1+\vartheta^2z^2}{1-\vartheta\,z-\vartheta^2z^2},, \qquad (z\in\mathscr{O})$$

and

$$\left(\mathcal{F}^{\varphi}_{\varrho,\lambda}(\eta(\xi);q)\right)' \prec \frac{1+q\vartheta^2\xi^2}{1-\vartheta\,\xi-\vartheta^2\xi^2}, \qquad (\xi\in\mathscr{O})$$

where $\vartheta=\frac{1-\sqrt{5}}{2}$ *denotes the golden ratio, arising from the classical Fibonacci numbers.*

3. Main Results

In this section, we obtain the initial Taylor coefficients $|\alpha_2|$ and $|\alpha_3|$ for the bi-univalent starlike and convex subclass $\mathfrak{R}_{\Sigma_q}(\varrho,\varphi,\lambda)$.

Firstly, let $\mathsf{p}(z) = 1+\mathsf{p}_1z+\mathsf{p}_2z^2+\mathsf{p}_3z^3+\dots$, and $\mathsf{p}(z)\prec\Upsilon(z;q)$. Then there exist $\varphi\in\mathsf{P}$ such that $|\varphi(z)|<1$ in $\mathscr{O}$ and $\mathsf{p}(z)=\Upsilon(\varphi(z);q)$, we have

$$\hbar(z) = (1+\varphi(z))(1-\varphi(z))^{-1} = 1+\ell_1z+\ell_2z^2+\cdots\in\mathsf{P} \qquad (z\in\mathscr{O}). \tag{16}$$

It follows that

$$\varphi(z) = \frac{\ell_1 z}{2} + \left(\ell_2-\frac{\ell_1^2}{2}\right)\frac{z^2}{2} + \left(\ell_3-\ell_1\ell_2-\frac{\ell_1^3}{4}\right)\frac{z^3}{2}+\cdots. \tag{17}$$

Moreover, by expanding $\Upsilon(\varphi(z);q)$ into its Taylor–Maclaurin series, we obtain

$$\begin{aligned}
\Upsilon(\varphi(z);q) &= 1+\widehat{\mathsf{p}_1}\left[\frac{\ell_1 z}{2} + \left(\ell_2-\frac{\ell_1^2}{2}\right)\frac{z^2}{2} + \left(\ell_3-\ell_1\ell_2-\frac{\ell_1^3}{4}\right)\frac{z^3}{2}+\cdots\right]\\
&\quad+\widehat{\mathsf{p}_2}\left[\frac{\ell_1 z}{2} + \left(\ell_2-\frac{\ell_1^2}{2}\right)\frac{z^2}{2} + \left(\ell_3-\ell_1\ell_2-\frac{\ell_1^3}{4}\right)\frac{z^3}{2}+\cdots\right]^2\\
&\quad+\widehat{\mathsf{p}_3}\left[\frac{\ell_1 z}{2} + \left(\ell_2-\frac{\ell_1^2}{2}\right)\frac{z^2}{2} + \left(\ell_3-\ell_1\ell_2-\frac{\ell_1^3}{4}\right)\frac{z^3}{2}+\cdots\right]^3+\cdots\\
&= 1+\frac{\widehat{\mathsf{p}_1}\ell_1}{2}z+\frac{1}{2}\left[\left(\ell_2-\frac{\ell_1^2}{2}\right)\widehat{\mathsf{p}_1}+\frac{\ell_1^2}{2}\widehat{\mathsf{p}_2}\right]z^2\\
&\quad+\frac{1}{2}\left[\left(\ell_3-\ell_1\ell_2+\frac{\ell_1^3}{4}\right)\widehat{\mathsf{p}_1}+\ell_1\left(\ell_2-\frac{\ell_1^2}{2}\right)\widehat{\mathsf{p}_2}+\frac{\ell_1^3}{4}\widehat{\mathsf{p}_3}\right]z^3+\cdots.
\end{aligned} \tag{18}$$

And similarly, there exists an analytic function ν such that $|\nu(\xi)| < 1$ in $\mathscr{O}$ and $\mathsf{p}(\xi) = \Upsilon(\nu(\xi); q)$. Therefore, the function

$$\kappa(\xi) = (1 + \nu(\xi))(1 - \nu(\xi))^{-1} = 1 + \tau_1\xi + \tau_2\xi^2 + \cdots \in \mathsf{P}. \tag{19}$$

It follows that

$$\nu(\xi) = \frac{\tau_1\xi}{2} + \left(\tau_2 - \frac{\tau_1^2}{2}\right)\frac{\xi^2}{2} + \left(\tau_3 - \tau_1\tau_2 - \frac{\tau_1^3}{4}\right)\frac{\xi^3}{2} + \cdots, \tag{20}$$

and

$$\begin{aligned}\Upsilon(\nu(\xi); q) = 1 &+ \frac{\widehat{\mathsf{p}_1}\tau_1}{2}\xi + \frac{1}{2}\left[\left(\tau_2 - \frac{\tau_1^2}{2}\right)\widehat{\mathsf{p}_1} + \frac{\tau_1^2}{2}\widehat{\mathsf{p}_2}\right]\xi^2 \\ &+ \frac{1}{2}\left[\left(\tau_3 - \tau_1\tau_2 + \frac{\tau_1^3}{4}\right)\widehat{\mathsf{p}_1} + \tau_1\left(\tau_2 - \frac{\tau_1^2}{2}\right)\widehat{\mathsf{p}_2} + \frac{\tau_1^3}{4}\widehat{\mathsf{p}_3}\right]\xi^3 + \cdots.\end{aligned} \tag{21}$$

In the following theorem we determine the initial Taylor coefficients $|\alpha_2|$ and $|\alpha_2|$ for the class $\mathfrak{R}_{\Sigma_q}(\varrho, \varphi, \lambda)$. Later we will reduce these bounds to other classes for special cases.

Theorem 1. *Let f given by* (1) *be in the class $\mathfrak{R}_{\Sigma_q}(\varrho, \varphi, \lambda)$. Then* $|\alpha_2| \leq$

$$\min\left\{\begin{array}{l}\dfrac{\vartheta_q^2\,\Gamma_q^2\big(2(1+\varrho)\big)}{[2]_q^2\,[\varphi]_q^2\,[\lambda]_q^2\,\Gamma_q^2(1+\varrho)}, \\ \sqrt{\dfrac{2\,|\vartheta_q|\,[2]_q\,\Gamma_q\big(3(1+\varrho)\big)\,\Gamma_q^2\big(2(1+\varrho)\big)}{\left|2[3]_q\,[\varphi]_q\,[\varphi+1]_q\,[\lambda]_q^2\,\Gamma_q(1+\varrho)\,\Gamma_q^2\big(2(1+\varrho)\big) - \left(4q + 2 - \frac{2}{\vartheta_q}\right)[2]_q^3\,[\varphi]_q^2\,[\lambda]_q^2\,\Gamma_q^2(1+\varrho)\,\Gamma_q\big(3(1+\varrho)\big)\right|}}\end{array}\right\},$$

and

$$\big|\alpha_3\big| = \frac{\vartheta_q^2\,\Gamma_q^2\,\big(2(1+\varrho)\big)}{[2]_q^2\,[\varphi]_q^2\,[\lambda]_q^2\,\Gamma_q^2(1+\varrho)} + \frac{[2]_q\,\Gamma_q\,\big(3(1+\varrho)\big)\big|\vartheta_q\big|}{[3]_q\,[\varphi]_q\,[\varphi+1]_q\,[\lambda]_q^2\,\Gamma_q(1+\varrho)}.$$

Proof. Let $f \in \mathfrak{R}_{\Sigma_q}(\varrho, \varphi, \lambda)$ and $\eta = f^{-1}$. Considering (14) and (15) we have

$$\partial_q\left(\mathcal{F}_{\varrho,\lambda}^{\varphi}(f(z); q)\right) = \Upsilon(\varphi(z); q), \qquad (z \in \mathscr{O}), \tag{22}$$

and

$$\partial_q\left(\mathcal{F}_{\varrho,\lambda}^{\varphi}(\eta(\xi); q)\right) = \Upsilon(\nu(\xi); q), \qquad (\xi \in \mathscr{O}). \tag{23}$$

Using (8), we have

$$\partial_q\left(\mathcal{F}_{\varrho,\lambda}^{\varphi}(f(z); q)\right) = 1 + \frac{[2]_q\,[\varphi]_q\,[\lambda]_q\,\Gamma_q(1+\varrho)}{\Gamma_q\,\big(2(1+\varrho)\big)}\,\alpha_2\,z + \frac{[3]_q\,[\varphi]_q\,[\varphi+1]_q\,[\lambda]_q^2\,\Gamma_q(1+\varrho)}{[2]_q\,\Gamma_q\,\big(3(1+\varrho)\big)}\,\alpha_3\,z^2 + O(z^3), \tag{24}$$

and

$$\partial_q\left(\mathcal{F}^{\varphi}_{\varrho,\lambda}(\eta(\xi);q)\right)=1-\frac{[2]_q\,[\varphi]_q\,[\lambda]_q\,\Gamma_q(1+\varrho)}{\Gamma_q\left(2(1+\varrho)\right)}\,\alpha_2\,\xi+\frac{[3]_q\,[\varphi]_q\,[\varphi+1]_q\,[\lambda]_q^2\,\Gamma_q(1+\varrho)}{[2]_q\,\Gamma_q\left(3(1+\varrho)\right)}\,\left(2\,\alpha_2^2-\alpha_3\right)\,\xi^2+O(\xi^3)\,. \tag{25}$$

By comparing (22) and (24), along (18), yields

$$\begin{aligned}&\frac{[2]_q\,[\varphi]_q\,[\lambda]_q\,\Gamma_q(1+\varrho)}{\Gamma_q\left(2(1+\varrho)\right)}\,\alpha_2\,z+\frac{[3]_q\,[\varphi]_q\,[\varphi+1]_q\,[\lambda]_q^2\,\Gamma_q(1+\varrho)}{[2]_q\,\Gamma_q\left(3(1+\varrho)\right)}\,\alpha_3\,z^2+\cdots\\&=\frac{\widehat{\mathsf{p}_1}\ell_1}{2}z+\frac{1}{2}\left[\left(\ell_2-\frac{\ell_1^2}{2}\right)\widehat{\mathsf{p}_1}+\frac{\ell_1^2}{2}\widehat{\mathsf{p}_2}\right]z^2+\cdots\,.\end{aligned} \tag{26}$$

Besied that, by comparing (18) and (25), along (21), yields

$$\begin{aligned}&-\frac{[2]_q\,[\varphi]_q\,[\lambda]_q\,\Gamma_q(1+\varrho)}{\Gamma_q\big(2(1+\varrho)\big)}\,\alpha_2\,\xi+\frac{[3]_q\,[\varphi]_q\,[\varphi+1]_q\,[\lambda]_q^2\,\Gamma_q(1+\varrho)}{[2]_q\,\Gamma_q\big(3(1+\varrho)\big)}\,\left(2\alpha_2^2-\alpha_3\right)\xi^2+\cdots\\&=\frac{\widehat{\mathsf{p}_1}\,\tau_1}{2}\,\xi+\frac{1}{2}\left[\left(\tau_2-\tfrac{\tau_1^2}{2}\right)\widehat{\mathsf{p}_1}+\tfrac{\tau_1^2}{2}\,\widehat{\mathsf{p}_2}\right]\xi^2+\cdots\,.\end{aligned} \tag{27}$$

Equating the pertinent coefficient in (26) and (27), we obtain

$$\frac{[2]_q\,[\varphi]_q\,[\lambda]_q\,\Gamma_q(1+\varrho)}{\Gamma_q\left(2(1+\varrho)\right)}\,\alpha_2=\frac{\widehat{\mathsf{p}_1}\ell_1}{2} \tag{28}$$

$$-\frac{[2]_q\,[\varphi]_q\,[\lambda]_q\,\Gamma_q(1+\varrho)}{\Gamma_q\left(2(1+\varrho)\right)}\,\alpha_2=\frac{\widehat{\mathsf{p}_1}\tau_1}{2} \tag{29}$$

$$\frac{[3]_q\,[\varphi]_q\,[\varphi+1]_q\,[\lambda]_q^2\,\Gamma_q(1+\varrho)}{[2]_q\,\Gamma_q\left(3(1+\varrho)\right)}\,\alpha_3=\frac{1}{2}\left[\left(\ell_2-\frac{\ell_1^2}{2}\right)\widehat{\mathsf{p}_1}+\frac{\ell_1^2}{2}\widehat{\mathsf{p}_2}\right] \tag{30}$$

$$\frac{[3]_q\,[\varphi]_q\,[\varphi+1]_q\,[\lambda]_q^2\,\Gamma_q(1+\varrho)}{[2]_q\,\Gamma_q\left(3(1+\varrho)\right)}\,\left(2\,\alpha_2^2-\alpha_3\right)=\frac{1}{2}\left[\left(\tau_2-\frac{\tau_1^2}{2}\right)\widehat{\mathsf{p}_1}+\frac{\tau_1^2}{2}\widehat{\mathsf{p}_2}\right] \tag{31}$$

From (28) and (29), we have

$$\ell_1=-\tau_1\quad\Longleftrightarrow\quad\ell_1^2=\tau_1^2, \tag{32}$$

and using (12), we have

$$\alpha_2^2=\frac{\vartheta_q^2\,\Gamma_q^2\left(2(1+\varrho)\right)}{8\,[2]_q^2\,[\varphi]_q^2\,[\lambda]_q^2\,\Gamma_q^2(1+\varrho)}\,\left(\ell_1^2+\tau_1^2\right)\,, \tag{33}$$

or equivalent to

$$\ell_1^2+\tau_1^2=\frac{8}{\vartheta_q^2}\left(\frac{[2]_q\,[\varphi]_q\,[\lambda]_q\,\Gamma_q(1+\varrho)}{\Gamma_q\left(2(1+\varrho)\right)}\right)^2\,\alpha_2^2\,. \tag{34}$$

Now, by summing (30) and (31), we obtain

$$\frac{2\,[3]_q\,[\varphi]_q\,[\varphi+1]_q\,[\lambda]_q^2\,\Gamma_q(1+\varrho)}{[2]_q\,\Gamma_q\left(3(1+\varrho)\right)}\alpha_2^2=\frac{(\ell_2+\tau_2)\vartheta_q}{2}+\left[\frac{(2q+1)\vartheta_q^2}{4}-\frac{\vartheta_q}{4}\right]\left(\ell_1^2+\tau_1^2\right). \tag{35}$$

By putting (33) in (35), we obtain

$$\alpha_2^2=\frac{\frac{\vartheta_q}{2}\left(\ell_2+\tau_2\right)[2]_q\,\Gamma_q\left(3(1+\varrho)\right)\Gamma_q\left(2(1+\varrho)\right)^2}{2\,[3]_q\,[\varphi]_q\,[\varphi+1]_q\,[\lambda]_q^2\,\Gamma_q(1+\varrho)\,\Gamma_q\left(2(1+\varrho)\right)^2-\left(4q+2-\frac{2}{\vartheta_q}\right)[2]_q^3\,[\varphi]_q^2\,[\lambda]_q^2\,\Gamma_q(1+\varrho)^2\,\Gamma_q\left(3(1+\varrho)\right)}. \tag{36}$$

Using (3) for (36), we have

$$|\alpha_2|\leq\sqrt{\frac{2|\vartheta_q|\,[2]_q\,\Gamma_q\left(3(1+\varrho)\right)\Gamma_q\left(2(1+\varrho)\right)^2}{\left|2\,[3]_q\,[\varphi]_q\,[\varphi+1]_q\,[\lambda]_q^2\,\Gamma_q(1+\varrho)\,\Gamma_q\left(2(1+\varrho)\right)^2-\left(4q+2-\frac{2}{\vartheta_q}\right)[2]_q^3\,[\varphi]_q^2\,[\lambda]_q^2\,\Gamma_q(1+\varrho)^2\,\Gamma_q\left(3(1+\varrho)\right)\right|}}. \tag{37}$$

Besided that, from (33)

$$\left|\alpha_2\right|\leq\frac{\vartheta_q^2\,\Gamma_q^2\left(2(1+\varrho)\right)}{[2]_q^2\,[\varphi]_q^2\,[\lambda]_q^2\,\Gamma_q^2(1+\varrho)}.$$

Now, so as to find the bound on $|\alpha_3|$, let's subtract from (30) and (31) along (33), we obtain

$$\alpha_3=\alpha_2^2+\frac{[2]_q\,\Gamma_q\left(3(1+\varrho)\right)\vartheta_q\left(\ell_2-\tau_2\right)}{4\,[3]_q\,[\varphi]_q\,[\varphi+1]_q\,[\lambda]_q^2\,\Gamma_q(1+\varrho)}. \tag{38}$$

Hence, we get

$$\left|\alpha_3\right|=\left|\alpha_2\right|^2+\frac{[2]_q\,\Gamma_q\left(3(1+\varrho)\right)\left|\vartheta_q\right|}{[3]_q\,[\varphi]_q\,[\varphi+1]_q\,[\lambda]_q^2\,\Gamma_q(1+\varrho)}. \tag{39}$$

Then, in view of (33), we obtain

$$\left|\alpha_3\right|\leq\frac{\vartheta_q^2\,\Gamma_q^2\left(2(1+\varrho)\right)}{[2]_q^2\,[\varphi]_q^2\,[\lambda]_q^2\,\Gamma_q^2(1+\varrho)}+\frac{[2]_q\,\Gamma_q\left(3(1+\varrho)\right)\left|\vartheta_q\right|}{[3]_q\,[\varphi]_q\,[\varphi+1]_q\,[\lambda]_q^2\,\Gamma_q(1+\varrho)}. \tag{40}$$

In the following theorem, we find the Fekete-Szegö functional for $f\in\mathfrak{R}_{\Sigma_q}(\varrho,\varphi,\lambda)$.

Theorem 2. *Let f given by (1) be in the class $\mathfrak{R}_{\Sigma_q}(\varrho,\varphi,\lambda)$ and $\rho\in\mathbb{R}$. Then we have*

$$\left|\alpha_3-\rho\,\alpha_2^2\right|\leq\begin{cases}\frac{[2]_q\,\Gamma_q\left(3(1+\varrho)\right)\left|\vartheta_q\right|}{[3]_q\,[\varphi]_q\,[\varphi+1]_q\,[\lambda]_q^2\,\Gamma_q(1+\varrho)}, & 0\leq\left|\mathscr{K}(\rho)\right|\leq\frac{[2]_q\,\Gamma_q\left(3(1+\varrho)\right)\left|\vartheta_q\right|}{4\,[3]_q\,[\varphi]_q\,[\varphi+1]_q\,[\lambda]_q^2\,\Gamma_q(1+\varrho)}\\ 4\left|\mathscr{K}(\rho)\right|, & \left|\mathscr{K}(\rho)\right|\geq\frac{[2]_q\,\Gamma_q\left(3(1+\varrho)\right)\left|\vartheta_q\right|}{4\,[3]_q\,[\varphi]_q\,[\varphi+1]_q\,[\lambda]_q^2\,\Gamma_q(1+\varrho)}\end{cases},$$

where

$$\mathscr{K}(\rho)=\frac{(1-\rho)\frac{\vartheta_q}{2}\,[2]_q\,\Gamma_q\left(3(1+\varrho)\right)\Gamma_q\left(2(1+\varrho)\right)^2}{2\,[3]_q\,[\varphi]_q\,[\varphi+1]_q\,[\lambda]_q^2\,\Gamma_q(1+\varrho)\,\Gamma_q\left(2(1+\varrho)\right)^2-\left(4q+2-\dfrac{2}{\vartheta_q}\right)[2]_q^3\,[\varphi]_q^2\,[\lambda]_q^2\,\Gamma_q(1+\varrho)^2\,\Gamma_q\left(3(1+\varrho)\right)}. \tag{41}$$

Proof. Let $f\in\mathfrak{R}_{\Sigma_q}(\varrho,\varphi,\lambda)$, from (36) and (38) we have $\alpha_3-\rho\,\alpha_2^2=$

$$\frac{[2]_q\,\Gamma_q\left(3(1+\varrho)\right)\vartheta_q\left(\ell_2-\tau_2\right)}{4\,[3]_q\,[\varphi]_q\,[\varphi+1]_q\,[\lambda]_q^2\,\Gamma_q(1+\varrho)}$$

$$+\frac{(1-\rho)\left(\ell_2+\tau_2\right)\frac{\vartheta_q}{2}\,[2]_q\,\Gamma_q\left(3(1+\varrho)\right)\Gamma_q\left(2(1+\varrho)\right)^2}{2\,[3]_q\,[\varphi]_q\,[\varphi+1]_q\,[\lambda]_q^2\,\Gamma_q(1+\varrho)\,\Gamma_q\left(2(1+\varrho)\right)^2-\left(4q+2-\dfrac{2}{\vartheta_q}\right)[2]_q^3\,[\varphi]_q^2\,[\lambda]_q^2\,\Gamma_q(1+\varrho)^2\,\Gamma_q\left(3(1+\varrho)\right)}$$

$$=\left(\mathscr{K}(\rho)+\frac{[2]_q\,\Gamma_q\left(3(1+\varrho)\right)\vartheta_q}{4\,[3]_q\,[\varphi]_q\,[\varphi+1]_q\,[\lambda]_q^2\,\Gamma_q(1+\varrho)}\right)\ell_2+\left(\mathscr{K}(\rho)-\frac{[2]_q\,\Gamma_q\left(3(1+\varrho)\right)\vartheta_q}{4\,[3]_q\,[\varphi]_q\,[\varphi+1]_q\,[\lambda]_q^2\,\Gamma_q(1+\varrho)}\right)\tau_2\,, \tag{42}$$

$$\frac{[2]_q\,\Gamma_q\left(3(1+\varrho)\right)\left|\vartheta_q\right|}{4\,[3]_q\,[\varphi]_q\,[\varphi+1]_q\,[\lambda]_q^2\,\Gamma_q(1+\varrho)}$$

where $\mathscr{K}(\rho)$ is given by (41).

Then, by taking modulus of (42), we conclude that

$$\left|\alpha_3-\rho\,\alpha_2^2\right|\le\begin{cases}\frac{[2]_q\,\Gamma_q\left(3(1+\varrho)\right)\left|\vartheta_q\right|}{[3]_q\,[\varphi]_q\,[\varphi+1]_q\,[\lambda]_q^2\,\Gamma_q(1+\varrho)}, & 0\le\left|\mathscr{K}(\rho)\right|\le\frac{[2]_q\,\Gamma_q\left(3(1+\varrho)\right)\left|\vartheta_q\right|}{4\,[3]_q\,[\varphi]_q\,[\varphi+1]_q\,[\lambda]_q^2\,\Gamma_q(1+\varrho)}\\ 4\left|\mathscr{K}(\rho)\right|, & \left|\mathscr{K}(\rho)\right|\ge\frac{[2]_q\,\Gamma_q\left(3(1+\varrho)\right)\left|\vartheta_q\right|}{4\,[3]_q\,[\varphi]_q\,[\varphi+1]_q\,[\lambda]_q^2\,\Gamma_q(1+\varrho)}\end{cases}.$$

Corollary 1. *Let f given by (1) be in the class $\mathfrak{R}_\Sigma(\varrho,\varphi,\lambda)$ and let $\rho\in\mathbb{R}$. Then, we have*

$$|\alpha_2|\le\min\left\{\frac{\vartheta^2\,\Gamma^2\big(2(1+\varrho)\big)}{4\,\varphi^2\,\lambda^2\,\Gamma^2(1+\varrho)},\sqrt{\frac{4\,|\vartheta|\,\Gamma\big(3(1+\varrho)\big)\,\Gamma^2\big(2(1+\varrho)\big)}{\left|6\,\varphi(\varphi+1)\,\lambda^2\,\Gamma(1+\varrho)\,\Gamma^2\big(2(1+\varrho)\big)-\left(6-\frac{2}{\vartheta}\right)8\,\varphi^2\lambda^2\,\Gamma^2(1+\varrho)\,\Gamma\big(3(1+\varrho)\big)\right|}}\right\}.$$

$$\left|\alpha_3\right|=\frac{\vartheta^2\,\Gamma^2\big(2(1+\varrho)\big)}{4\,\varphi^2\,\lambda^2\,\Gamma^2(1+\varrho)}+\frac{2\,\Gamma\big(3(1+\varrho)\big)\,|\vartheta|}{3\,\varphi(\varphi+1)\,\lambda^2\,\Gamma(1+\varrho)},$$

and

$$\left|\alpha_3-\rho\,\alpha_2^2\right|\le\begin{cases}\dfrac{2\,\Gamma\big(3(1+\varrho)\big)\,|\vartheta|}{3\,\varphi(\varphi+1)\,\lambda^2\,\Gamma(1+\varrho)}, & 0\le\left|\mathscr{K}(\rho)\right|\le\dfrac{\Gamma\big(3(1+\varrho)\big)\,|\vartheta|}{6\,\varphi(\varphi+1)\,\lambda^2\,\Gamma(1+\varrho)}\\ 4\left|\mathscr{K}(\rho)\right|, & \left|\mathscr{K}(\rho)\right|\ge\dfrac{\Gamma\big(3(1+\varrho)\big)\,|\vartheta|}{6\,\varphi(\varphi+1)\,\lambda^2\,\Gamma(1+\varrho)}\end{cases}$$

where

$$\mathscr{K}(\rho)=\frac{(1-\rho)\,\vartheta\,\Gamma\big(3(1+\varrho)\big)\,\Gamma\big(2(1+\varrho)\big)^2}{6\,\varphi(\varphi+1)\,\lambda^2\,\Gamma(1+\varrho)\,\Gamma\big(2(1+\varrho)\big)^2-\left(6-\frac{2}{\vartheta}\right)8\,\varphi^2\lambda^2\,\Gamma(1+\varrho)^2\,\Gamma\big(3(1+\varrho)\big)}.$$

4. Conclusion

Conclusion

In this paper, we have introduced and studied a new subclass of bi-univalent functions associated with shell-like curves defined via the q-Rabotnov function and the q-analogue of Fibonacci numbers. By applying the subordination principle, we derived coefficient estimates for the initial Taylor–Maclaurin coefficients, namely $|a_2|$ and $|a_3|$, and established sharp Fekete–Szegö type inequalities for the proposed class.

The obtained results not only extend and refine several existing contributions in the theory of bi-univalent functions but also emphasize the rich interplay between q-special functions, shell-like domains, and analytic inequalities. In particular, the connections drawn between the q-Rabotnov function, q-Fibonacci numbers, and geometric function theory provide new insights that deepen the structural understanding of bi-univalent functions.

We anticipate that the techniques and findings presented herein will serve as a foundation for further developments in q-calculus and its applications to operator theory, approximation theory, and other areas of complex analysis. Future work may focus on exploring additional subclasses defined via other q-special functions and extending the present results to multivalent or harmonic settings.

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