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Generalized Branciari Metrics: Fixed Point Results and Open Problems

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Abstract. The notion of generalized metric, more commonly known as the rectangular metric, was introduced by Branciari in 2000, replacing the triangle inequality of metric spaces by the so-called rectangular inequality. In this paper we further generalize this notion by adding two more terms to the right-hand side of the rectangular inequality. We study basic properties of spaces endowed by such distance functions, and prove the modified versions of Banach and Kannan fixed point theorems on these spaces. The results are substantiated by examples. Finally, we state some open problems related to topological structure and fixed point theory on generalized Branciari metric spaces.

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Key Words and Phrases: Generalized metric spaces, fixed point theorems

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1. Introduction and preliminaries

Metric fixed point theory continues to be one of the most widely studied areas of modern mathematics, mostly motivated by numerous applications, while pertaining to relatively elementary theoretical apparatus. The study was initiated in 1922 by the famous Banach fixed point theorem [1]. This basic result was mainly generalized either by weakening the contractive condition, or weakening the metric structure, or combining these two approaches. However, most of the new "generalized metric" structures obtained in this way were shown to be metrizable, or the new fixed point results on these spaces can be reduced to their metric counterparts (see e.g. [2, 3] and references therein). One notable exception are the so-called generalized metric spaces of Branciari [4], more commonly referred to in the literature as the rectangular metric spaces (see [5]). We recall the definition of rectangular metric space as follows:

Definition 1.1 ([4]). Let X be a set and $d : X \times X \to [0, +\infty)$ a mapping such that for all $x, y \in X$ and for all $u, v \in X \setminus \{x, y\}$ with $u \neq v$ we have:

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(i) $d(x, y) = 0 \Leftrightarrow x = y$,

(ii) $d(x, y) = d(y, x)$, and

(iii) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$.

Then we will say that (X, d) is a generalized, or rectangular metric space (r.m.s. for short).

The notions of convergence, completeness, continuity, etc. are defined the same as in metric spaces. However, r.m.s. possess some aberrant properties different from standard metrics, for example:

- the limit of a convergent sequence needs not be unique;
- a convergent sequence needs not be Cauchy;
- rectangular metric needs not be a continuous function in any of the variables;

see [6]. Moreover, r.m.s. are metrizable only under certain special conditions ([7]). Therefore the fixed point results on r.m.s. cannot be obtained from the corresponding results on metric spaces. According to [8] this fact among other things is somewhat limiting the applicative potential of such results. Nevertheless, the study of fixed point theory, as well as toplogical properties on r.m.s. remains very active and fruitful, see [5] and references therein.

With this in mind, in the present paper we further generalize rectangular metrics to obtain so called generalized Branciari metric spaces. We notice that the rectangular inequality (iii) of Definition 1.1 does not contain all possible distances between four points x, y, u and v. By taking into account remaining distances, i.e. adding $d(x, v)$ and $d(y, u)$ to the sum on the right-hand side of the inequality, we obtain the desired generalization. The following section outlines the definition, examples and basic properties of these spaces which will be needed in the sequel. In the next section we state and prove two classical fixed point theorems of Banach [1] and Kannan [9] in generalized Branciari metric spaces as our main results. To conclude the paper, we list some open problems arising in the study of fixed point theory and topology of our novel spaces, also providing directions for further research.

2. Generalized Branciari metric spaces

We begin by giving the formal definition of generalized Branciari metrics.

Definition 2.1. Let X be a nonempty set. A mapping $d : X \times X \to [0, +\infty)$ satisfying the following conditions for all $x, y \in X$ and $u, v \in X \setminus \{x, y\}$ with $u \neq v$:

(i) $d(x, y) = 0 \Leftrightarrow x = y$,

(ii) $d(x, y) = d(y, x)$, and

(iii) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y) + d(x, v) + d(y, u)$

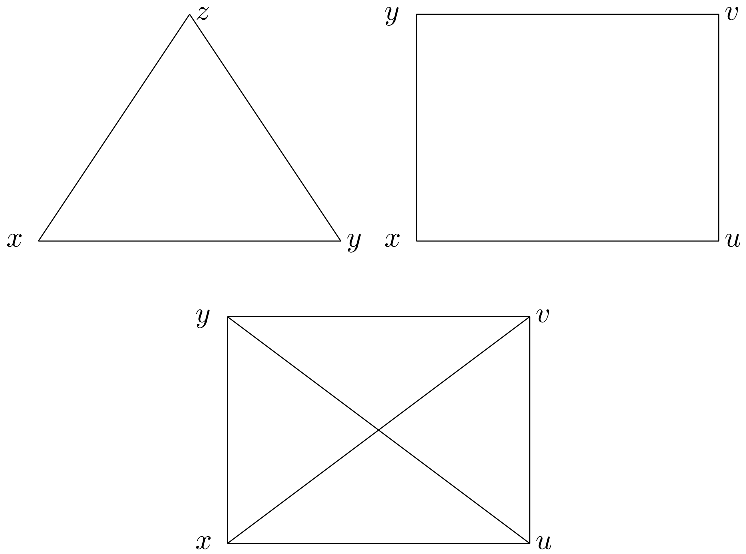

Figure 1: Graphical representation of metric space, r.m.s. and g.B.m.s. In each graph, the length of each edge is less than the sum of lengths of all remaining edges.

is called a **generalized Branciari metric** (**g.B.m.** for short) on X, while the pair (X, d) is a **generalized Branciari metric space** (**g.B.m.s.** in short).

Consider the four points x, y, u and v as vertices of a quadrilateral in a Euclidean space. Then the distances featuring in the rectangular inequality (iii) in Definition 1.1 can be thought of as the lengths of its sides. Hence the rectangular inequality states that the length of each side is not greater than the sum of lengths of remaining sides, as is well known from geometry. Therefore, our generalization in Definition 2.1 (iii) involves adding lengths of diagonals to the right-hand side of the rectangular inequality (see Figure 1 below). From the definition it is obvious that every r.m.s. is a g.B.m.s. However, the converse in general does not hold, as the following two examples show.

Example 2.1. Let $X = \{a_i : i = \overline{1,4}\}$ and define the mapping $d : X \times X \to [0, +\infty)$ as

$$d(a_i, a_j) = \begin{cases} 0, & i = j, \\ 4, & (i, j) \in \{(1, 2), (2, 1)\}, \\ 1, & \text{in all other cases.} \end{cases}$$

Then it is easy to check that (X, d) is a g.B.m.s. However, it is not a r.m.s. because $d(a_1, a_2) = 4 > 3 = d(a_1, a_3) + d(a_3, a_4) + d(a_4, a_2)$.

Example 2.2. Let $X = \mathbb{N}$ and $d : X \times X \to [0, +\infty)$ be defined by

$$d(x,y) = \begin{cases} 0, & x = y, \\ 5, & (x,y) \in \{(1,2),(2,1)\}, \\ \frac{1}{n}, & (x,y) \in \{(2,n),(n,2)\}_{n\geq 4}, \\ \frac{1}{2n}, & (x,y) \in \{(3,n),(n,3)\}_{n\geq 4}, \\ 2, & \text{otherwise} \end{cases}$$

for all $x, y \in X$. Then (X, d) is a g.B.m.s. but not a r.m.s. since, for example, $d(1,2) = 5 > 2\frac{3}{8} = d(1,3) + d(3,4) + d(4,2)$.

The basic notions such as convergence, completeness and continuity are defined in the same way as in standard metric spaces.

Since r.m.s. are a proper subclass of g.B.m.s. all the peculiar properties of r.m.s. mentioned in the previous section are also possessed by g.B.m.s. Indeed, in g.B.m.s. (X, d) from the Example 2.2:

- the sequence $a_n = n$ converges to both 2 and 3;
- the sequence $a_n = n$ is not Cauchy, as $\lim_{n\to+\infty} d(n, n+p) = 2$ for all $p \in \mathbb{N}$;
- the function d is not continuous, since $\lim_{n\to+\infty} d(1, n) = 2 \neq 5 = d(1, 2)$.

The limit of a convergent sequence in r.m.s. is unique if the sequence is Cauchy and all its members are pairwise distinct (see e.g. [5, Lemma 3.1]), a property which is very useful in proving fixed point results. The next Lemma states that the same is true in g.B.m.s. as well.

Lemma 2.1. *Let (X, d) be a g.B.m.s. and let $\{x_n\}$ be a convergent sequence in X. If the sequence $\{x_n\}$ is Cauchy, and $x_n \neq x_m$ for all $n, m \in \mathbb{N}$ with $m \neq n$, then the limit of $\{x_n\}$ is unique.*

Proof. Suppose that $\{x_n\} \subseteq X$ is a sequence satisfying the conditions of the Lemma, and suppose that $\lim_{n\to+\infty} d(x_n, x) = \lim_{n\to+\infty} d(x_n, y) = 0$ for some $x, y \in X$. Because $x_n \neq x_m$ for all $n, m \in \mathbb{N}$ with $m \neq n$, there exists $p \in \mathbb{N}$ such that $x_n \in X \setminus \{x, y\}$ for all $n \geq p$. Then for $m > n \geq p$ we have:

$$d(x,y) \leq d(x,x_n) + d(x_n,x_m) + d(x_m,y) + d(x,x_m) + d(x_n,y) \to 0$$

as $n, m \to +\infty$. But this implies $d(x, y) = 0$, and hence $x = y$.

3. Main results

We are now ready to state and prove our main results in this section.

We begin with Banach's fixed point theorem on g.B.m.s.

Theorem 3.1. *Let (X, d) be a complete g.B.m.s. and let the mapping $T : X \to X$ satisfy the following condition:*

$$d(Tx, Ty) \leq qd(x, y), \text{ for all } x, y \in X \text{ and some } q \in [0, 1). \tag{3.1}$$

Then the mapping T has a unique fixed point $a \in X$, such that $\lim_{n\to\infty} T^n x = a$ for all $x \in X$.

Proof. Let $x \in X$ be an arbitrary point, and let $x_n = T^n x$ for all $n \in \mathbb{N}_0$. Then by the repeated use of the contractive condition (3.1) we get:

$$d(x_n, x_{n+1}) \leq qd(x_{n-1}, x_n) \leq \cdots \leq q^n d(x_0, x_1)$$

for all $n \in \mathbb{N}_0$. Hence,

$$\lim_{n\to+\infty} d(x_n, x_{n+1}) = 0.$$

If $x_n = x_{n+1}$ for some $n \in \mathbb{N}_0$ then $x_n = Tx_n$, i.e. x_n is a fixed point of mapping T. Therefore we can assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}_0$.

Then, without loss of generality, we can also suppose that $x_n \neq x_m$ for all $n, m \in \mathbb{N}_0$ with $m \neq n$. Indeed, let $x_n = x_m$ for some $n, m \in \mathbb{N}_0$ and let $m > n$. Then

$$0 < d(x_n, x_{n+1}) = d(x_m, x_{m+1}).$$

But on the other hand, by (3.1) we have

$$0 < d(x_m, x_{m+1}) \leq qd(x_{m-1}, x_m) \leq \cdots \leq q^{m-n} d(x_n, x_{n+1}) < d(x_n, x_{n+1}),$$

which is a contradiction.

Next, by two-step induction we will prove that

$$d(x_0, x_n) \leq c \sum_{k=0}^{n-1} F_{k+1} q^k \text{ for all } n \in \mathbb{N}, \tag{3.2}$$

where $c = d(x_0, x_1) + d(x_1, x_2) + d(x_0, x_2)$, and $\{F_k\}_{k\in\mathbb{N}}$ is the Fibonacci sequence, defined recursively by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$.

For $n = 1$ and $n = 2$, the statement (3.2) holds trivially. Now, for an arbitrary $n \geq 3$ suppose that (3.2) is true for $n - 1$ and $n - 2$. Then we obtain:

$$\begin{aligned} d(x_0, x_n) &\leq d(x_0, x_1) + d(x_1, x_2) + d(x_2, x_n) + d(x_0, x_2) + d(x_1, x_n) \\ &\leq d(x_0, x_1) + d(x_1, x_2) + d(x_0, x_2) + q^2 d(x_0, x_{n-2}) + qd(x_0, x_{n-1}) \\ &\leq c + cq^2 \sum_{k=0}^{n-3} F_{k+1} q^k + cq \sum_{k=0}^{n-2} F_{k+1} q^k \\ &= c + c \sum_{k=0}^{n-3} F_{k+1} q^{k+2} + c \sum_{k=0}^{n-2} F_{k+1} q^{k+1} \end{aligned}$$

$$= c + cq + c\sum_{k=2}^{n-1}(F_{k-1} + F_k)q^k$$
$$= c\sum_{k=0}^{n-1} F_{k+1}q^k$$

It is known that

$$\sum_{k=0}^{+\infty} F_{k+1}q^k = \frac{1}{1-q-q^2} \text{ for } |q| < \varphi^{-1}, \tag{3.3}$$

where $\varphi = \frac{\sqrt{5}+1}{2}$ is the "golden ratio" constant (see e.g. [10]). Hence, to finally prove that the sequence $\{x_n\}$ is Cauchy, we distinguish two cases.

1. $q \in [0, \varphi^{-1})$: Then by (3.1), (3.2) and (3.3), for all $m, n \in \mathbb{N}$ such that $m > n$, we have

$$d(x_n, x_m) \leq q^n d(x_0, x_{m-n}) \leq cq^n \sum_{k=0}^{m-n-1} F_{k+1}q^k \leq cq^n \sum_{k=0}^{+\infty} F_{k+1}q^k$$
$$= c\frac{q^n}{1-q-q^2} \to 0 \text{ as } m, n \to +\infty,$$

2. $q \in [\varphi^{-1}, 1)$: There exists a suitably large $n_0 \in \mathbb{N}$ such that $q^{n_0} < \varphi^{-1}$. Then by (3.1) we have that

$$d(T^{n_0}x, T^{n_0}y) \leq q^{n_0} d(x, y)$$

for all $x, y \in X$. Therefore, this case is reduced to the previous one by applying the same procedure to the mapping T^{n_0}.

Since (X, d) is a complete g.B.m.s. there exists $a \in X$ such that

$$\lim_{n\to+\infty} d(x_n, a) = 0.$$

By Lemma 2.1 the limit a is unique, as $\{x_n\}$ is a Cauchy sequence of pairwise distinct points. Then from

$$d(x_{n+1}, Ta) \leq qd(x_n, a) \to 0 \text{ as } n \to \infty$$

we conclude that $Ta = a$, i.e. a is a fixed point of the mapping T. Its uniqueness is easily shown by contradiction, using the contractive condition (3.1).

Our next main result is the Kannan fixed point theorem on g.B.m.s.

Theorem 3.2. *Let (X, d) be a complete g.B.m.s. and let $T : X \to X$ be a mapping such that*

$$d(Tx, Ty) \leq q(d(x, Tx) + d(y, Ty)) \text{ for all } x, y \in X \text{ and some } q \in \Big[0, \frac{1}{2}\Big). \tag{3.4}$$

Then T possesses a unique fixed point $a \in X$, such that $\lim_{n\to+\infty} d(T^n x, a) = 0$ for any $x \in X$.

Proof. Again, let $x \in X$ be any point, and let $x_n = T^n x$ for all $n \in \mathbb{N}_0$. By (3.4), for all $n \in \mathbb{N}_0$ we have

$$d(x_n, x_{n+1}) \leq q(d(x_{n-1}, x_n) + d(x_n, x_{n+1})),$$

i.e.

$$d(x_n, x_{n+1}) \leq \frac{q}{1-q} d(x_{n-1}, x_n).$$

Since $q \in [0, \frac{1}{2})$, we have $\frac{q}{1-q} \in [0, 1)$ and thus $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$, so we can use the same method as in the proof of Theorem 3.1 to show that $\{x_n\}$ is a Cauchy sequence in X. As is well known, if a mapping T satisfies the Kannan contractive condition (3.4), then the mapping T^n satisfies the same condition with the contractive constant $\frac{q^n}{(1-q)^{n-1}}$ for all $n \in \mathbb{N}$. Hence, notice we can now distinguish the cases $\frac{q}{1-q} \in [0, \varphi^{-1})$ and $\frac{q}{1-q} \in [\varphi^{-1}, 1)$.

Since (X, d) is complete, the sequence $\{x_n\}$ converges (uniquely) to some $a \in X$. From (3.4), for all $n \in \mathbb{N}$ we get

$$\begin{aligned} d(a, Ta) &\leq d(a, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, Ta) + d(a, x_{n+1}) + d(x_n, Ta) \\ &\leq d(a, x_n) + d(x_n, x_{n+1}) + q(d(x_n, x_{n+1}) + d(a, Ta)) \\ &+ d(a, x_{n+1}) + q(d(x_{n-1}, x_n) + d(a, Ta)). \end{aligned}$$

Therefore,

$$d(a, Ta) \leq \frac{1}{1-2q}(d(a, x_n) + (1+q)d(x_n, x_{n+1}) + d(a, x_{n+1}) + qd(x_{n-1}, x_n)) \to 0$$

as $n \to +\infty$, i.e. $d(a, Ta) = 0$ and so $a = Ta$. Uniqueness of the fixed point is again easy to check.

The following example illustrates Theorems 3.1 and 3.2.

Example 3.1. Let $X = \{1, 2, 3, 4, 5, 6\}$, and let $d : X \times X \to [0, +\infty)$ be defined as

$$d(i, j) = \begin{cases} 0, & i = j, \\ 3, & i = 1 \text{ and } j = 2, 3, \text{ or } i = 2, 3 \text{ and } j = 1, \\ 6, & i = 1 \text{ and } j = \overline{4, 6}, \text{ or } i = \overline{4, 6} \text{ and } j = 1, \\ 1, & \text{otherwise.} \end{cases}$$

Then it is easily checked that (X, d) is a g.B.m.s. that is not a r.m.s. The only convergent (respectively, Cauchy) sequences in X are eventually constant, so (X, d) is complete. Also, let the mapping $T : X \to X$ be defined as

$$T(i) = \begin{cases} 3, & i = 1, \\ 6, & i = \overline{2, 6}. \end{cases}$$

Then the mapping T is a Banach contraction on (X, d), i.e. it satisfies the condition (3.1) with $q = \frac{1}{3}$.

Indeed, we have

$$d(T(1),T(i)) = 1 \le 1 = \frac{1}{3}d(1,i), \text{ for } i = 2,3,$$

$$d(T(1),T(i)) = 1 \le 2 = \frac{1}{3}d(1,i), \text{ for } i = \overline{4,6}$$

and

$$d(T(i),T(j)) = 0 \le \frac{1}{3}d(i,j), \text{ for } i,j = \overline{2,6}.$$

Hence, Theorem 3.1 can be applied to conclude that T has a unique fixed point $a = 6$ in X.

Notice that the mapping T is not a Banach contraction with respect to the standard metric ϱ on X, because, for example

$$\varrho(T(1),T(2)) = |3-6| > |1-2| = \varrho(1,2).$$

We also have that

$$d(T(1),T(i)) = 1 \le \frac{4}{3} = \frac{1}{3}(d(1,T(1)) + d(i,T(i)), \text{ for } i = \overline{2,5},$$

$$d(T(1),T(6)) = 1 \le 1 = \frac{1}{3}(d(1,T(1)) + d(6,T(6))$$

and

$$d(T(i),T(j)) = 0 \le \frac{1}{3}(d(i,T(i)) + d(j,T(j)), \text{ for } i,j = \overline{2,6}.$$

Hence, T is also a Kannan contraction on (X,d) with the constant $q = \frac{1}{3}$, so by Theorem 3.2 it has a unique fixed point $a = 6$.

Again, T is not a Kannan contraction on (X,ϱ), as

$$\varrho(T(1),T(2)) = |3-6| = 3 = \frac{1}{2}(\varrho(1,T(1)) + \varrho(2,T(2)).$$

4. Conclusion and open problems

In this paper we have introduced the concept of generalized Branciari metric space (g.B.m.s.) and studied its basic properties and fixed point theorems on such spaces. These spaces are a generalization of rectangular metric spaces, which are generally not metrizable, and fixed point results on them cannot be obtained from corresponding results on metric spaces. Hence we believe that g.B.m.s. are a powerful and interesting addition to the study of fixed point theory and topology of distance spaces, as we will illustrate by various open problems listed in this section.

As generally in metric fixed point theory, further research scope could be focused on proving analogues of standard metric fixed point theorems, or further generalization of the underlying structure of g.B.m.s. (or both).

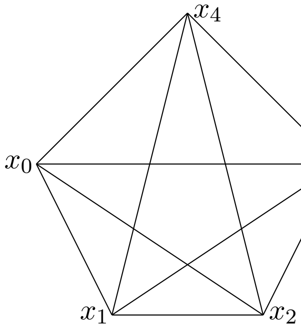

Figure 2: Graphical representation of 3-g.B.m.s. The length of each edge of the graph is less than the sum of lengths of all other edges.

Now we state a number of specific open problems and questions which arise already in the initial study of g.B.m.s.

1. Prove an analogue of the fixed point theorem of Chatterjea [11] on g.B.m.s. In particular, can the theorem be proved without restricting the domain of the contractive constant ($[0, \frac{1}{2})$) in the original version? If not, what is the minimal upper bound for the contractive constant? Does there exist a sharp upper bound, in the sense that the theorem remains true when the contractive constant attains that upper bound?

2. Prove analogues of Hardy-Rogers [12] and Ćirić [13] fixed point theorems on g.B.m.s. with same discussion as in the first problem.

3. Prove a version of fixed point theorem with "simulation functions" of Khojasteh et al. [14] in g.B.m.s.

4. An et al. [7] proved that a rectangular metric space is metrizable if and only if the limit of every convergent sequence in it is unique. Does the same result hold in g.B.m.s?

5. As in [4] we can further generalize g.B.m.s. by considering an extended "polygonal inequality" (for some $\nu \geq 3$) instead of (iii) in Definition 2.1 as follows:

$$(\nu\text{-iii})\ 2d(x_0, x_{\nu+1}) \leq \sum_{i=0}^{\nu} \sum_{j=i+1}^{\nu+1} d(x_i, x_j) \text{ for all } x_0, x_1, \ldots, x_\nu, x_{\nu+1} \in X$$
$$\text{such that } x_i \neq x_j \text{ for all } 1 \leq i < j \leq \nu \text{ and } x_1, \ldots, x_\nu \in X \setminus \{x_0, x_{\nu+1}\}$$

(coefficient 2 is used for convenience, since the same term appears in the sum on the right-hand side). By analogy with [4], we will say that (X, d) is a ν-generalized Branciari metric space (ν-g.B.m.s. in short) if it satisfies (ν-iii) alongside (i) and (ii) of Definition 2.1. Inequality (ν-iii) can be geometrically interpreted as stating that length of each side of a $(\nu + 2)$-gon is less than the sum of lengths of all its remaining sides and all its diagonals (see Figure 2 below).

Our next open problem concerns proving Banach and Kannan fixed point theorems on ν-g.B.m.s.

6. Suzuki et al. [15] proved that every 3-generalized metric space is metrizable. Does the same hold for 3-g.B.m.s? More generally, prove Suzuki-type [16] metrization results for ν-g.B.m.s.

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