EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 4, Article Number 7119
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Fractional-Order Neutrosophic MR-Metric Spaces: Theory, Fixed Point Theorems, and Applications

Abed Al-Rahman M. Malkawi[1,*] and Ayat M. Rabaiah[1]

[1] *Department of Mathematics, Faculty of Arts and Science, Amman Arab University, Amman 11953, Jordan*

Abstract. This paper introduces the novel concept of Fractional-Order Neutrosophic MR-Metric Spaces (FoNMR-MS), which synergistically combines three powerful mathematical frameworks: MR-metric spaces, neutrosophic logic, and fractional calculus. We begin by extending the classical MR-metric structure through the incorporation of fractional integrals, defining a comprehensive fractional-order metric M_α and corresponding neutrosophic functions $\mathcal{T}_\alpha$, $\mathcal{I}_\alpha$, $\mathcal{F}_\alpha$. Fundamental properties including non-negativity, identity, symmetry, and a generalized fractional triangle inequality are rigorously established. The core theoretical contribution is a comprehensive fixed point theorem for contraction mappings in complete FoNMR-MS, accompanied by detailed convergence analysis and neutrosophic consistency conditions. We further provide extensive examples and applications demonstrating the utility of our framework in modeling anomalous diffusion processes, image denoising, and machine learning under uncertainty. This work significantly generalizes existing results in fixed point theory and offers a robust mathematical foundation for handling complex systems characterized by fractional dynamics and neutrosophic uncertainty.

2020 Mathematics Subject Classifications: 47H10, 54E35, 26A33, 03E72

Key Words and Phrases: Fractional-order metric spaces, Neutrosophic logic, MR-metric spaces, Fixed point theory, Fractional calculus, Anomalous diffusion, Uncertainty modeling

1. Introduction

Fixed point theory represents one of the most dynamic and applicable domains in mathematical analysis, with profound implications across differential equations, optimization, and computational mathematics. The evolution of metric spaces has been marked by significant generalizations to address increasingly complex mathematical phenomena. The inception of b-metric spaces by Bakhtin [1] and Czerwik [2] relaxed the standard triangle inequality, paving the way for more flexible geometric structures. This trajectory continued with the development of G-metric spaces and their variants, offering enhanced frameworks for analyzing triple-based distance functions.

The recent introduction of MR-metric spaces by Malkawi et al. [3] marked a substantial advancement, providing a robust structure that generalizes several previous metric space extensions while maintaining strong theoretical properties. Subsequent research has extensively explored fixed point theorems within this framework [1, 2, 4–13], demonstrating its versatility in handling various contraction types and applications to integral equations and neutron transport [10].

Parallel to these metric space developments, neutrosophic logic pioneered by Smarandache has emerged as a powerful paradigm for handling uncertainty, indeterminacy, and inconsistency

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v18i4.7119

Email addresses: a.malkawi@aau.edu.jo and math.malkawi@gmail.com (A. Malkawi), a.rabaieha@aau.edu.jo (A. Rabaiah)

in mathematical systems. The integration of neutrosophic concepts with metric spaces has led to the creation of neutrosophic metric spaces [14, 15], which provide sophisticated tools for modeling truth, falsity, and indeterminacy degrees in distance measurements. Recent work by Hazaymeh and Bataihah [14] and Bataihah and Hazaymeh [15] has established fundamental results in neutrosophic fuzzy metric spaces, while Malkawi [16] advanced this further through neutrosophic MR-metrics with fuzzy embedding principles.

Fractional calculus has simultaneously revolutionized mathematical modeling by capturing non-local and memory-dependent phenomena across physics, engineering, and biological systems. The incorporation of fractional operators into metric space theory represents a natural progression, with recent contributions by Al-deiakeh et al. [17] in fractional differential equations and Gharib et al. [18] in fractional solution methods highlighting this trend. Malkawi and Rabaiah [19, 20] have recently explored the connections between MR-metric spaces and fractional calculus, establishing important bridges between these domains.

The current literature reveals a significant gap: no existing framework simultaneously incorporates fractional calculus, neutrosophic logic, and MR-metric structures. This research fills this void by introducing Fractional-Order Neutrosophic MR-Metric Spaces (FoNMR-MS), which synergistically combine these three powerful mathematical paradigms. Our work draws inspiration from numerous contributions in related areas, including the interpolative contractions of Qawasmeh [21], simulation functions in various metric contexts [22–24], cyclic contractions [25–28], and Ω-distance mappings [24, 27, 29].

This paper is organized as follows: Section 2 presents our main theoretical contributions, including the formal definition of FoNMR-MS, fundamental properties, and a comprehensive fixed point theorem with detailed convergence analysis. Section 3 provides extensive examples and applications demonstrating the practical utility of our framework. Our results significantly generalize previous work in fixed point theory [7, 9, 10, 30–32], metric space theory [33–40], and neutrosophic analysis [14, 15, 41–49], while opening new avenues for modeling complex systems with fractional dynamics and neutrosophic uncertainty.

Definition 1. *[3] Consider a non-empty set $\mathbb{X} \neq \emptyset$ and a real number $\mathbb{R} > 1$. A function*

$$M : \mathbb{X} \times \mathbb{X} \times \mathbb{X} \to [0, \infty)$$

*is termed an **MR-metric** if it satisfies the following conditions for all $v, \xi, s, \ell_1 \in \mathbb{X}$:*

- $M(v, \xi, s) \geq 0$.
- $M(v, \xi, s) = 0$ *if and only if* $v = \xi = s$.
- $M(v, \xi, s)$ *remains invariant under any permutation* $p(v, \xi, s)$*, i.e.,* $M(v, \xi, s) = M(p(v, \xi, s))$.
- *The following inequality holds:*

$$M(v, \xi, s) \leq \mathbb{R} \left[M(v, \xi, \ell_1) + M(v, \ell_1, s) + M(\ell_1, \xi, s) \right].$$

A structure $(\mathbb{X}, M)$ that adheres to these properties is defined as an MR-metric space.

Definition 2. *[32][Neutrosophic MR-Metric Space (NMR-MS)]*
*A 9-tuple $(\mathcal{Z}, M, \mathcal{T}, \mathcal{F}, \mathcal{I}, \bullet, \diamond, R, \star)$ is called a **Neutrosophic MR-Metric Space** if:*

(i) $\mathcal{Z}$ *is a non-empty set.*

(ii) $M : \mathcal{Z} \times \mathcal{Z} \times \mathcal{Z} \to [0, \infty)$ *is an MR-metric satisfying:*

(M1) $M(\upsilon, \xi, \Im) \geq 0$,

(M2) $M(\upsilon, \xi, \Im) = 0 \iff \upsilon = \xi = \Im$,

(M3) Symmetry under permutations,

(M4) $M(\upsilon,\xi,\Im) \leq R\,[M(\upsilon,\xi,\ell) \star M(\upsilon,\ell,\Im) \star M(\ell,\xi,\Im)],\ R > 1.$

(iii) $\mathcal{T},\mathcal{F},\mathcal{I} : \mathcal{Z}\times\mathcal{Z}\times(0,\infty)\to[0,1]$ *are neutrosophic functions satisfying:*

(N1) $\mathcal{T}(\upsilon,\xi,\gamma) = 1 \iff \upsilon = \xi$ *(Truth-Identity),*

(N2) $\mathcal{T}(\upsilon,\xi,\gamma) = \mathcal{T}(\xi,\upsilon,\gamma)$ *(Symmetry),*

(N3) $\mathcal{T}(\upsilon,\xi,\gamma)\bullet\mathcal{T}(\xi,\Im,\rho) \leq \mathcal{T}(\upsilon,\Im,\gamma+\rho)$ *(Triangle Inequality),*

(N4) $\lim_{\gamma\to\infty}\mathcal{T}(\upsilon,\xi,\gamma) = 1$ *(Asymptotic Behavior).*

(iv) $\bullet$ *(t-norm) and* $\diamond$ *(t-conorm) are continuous operators generalizing fuzzy logic.*

(v) $\star$ *is a binary operation generalizing addition (e.g., weighted sum).*

2. Main Results

This section presents the principal theoretical contributions of this work, wherein we introduce and rigorously develop the framework of Fractional-Order Neutrosophic MR-Metric Spaces (FoNMR-MS). We commence by formalizing the fundamental definitions that amalgamate the concepts of MR-metrics, neutrosophic logic, and fractional calculus into a unified structure. Subsequently, we establish the foundational properties of the proposed fractional-order metric and neutrosophic functions, proving their consistency with the axiomatic foundations of generalized metric spaces. The cornerstone of our investigation is the comprehensive fixed point theorem for contraction mappings within complete FoNMR-MS, accompanied by a detailed proof that elucidates the convergence behavior of iterative sequences and the role of the fractional parameter α and the neutrosophic conditions in guaranteeing the existence and uniqueness of a fixed point. The ensuing lemmas and corollaries further delineate the analytical properties and convergence rates, solidifying the theoretical groundwork for the applications that follow.

Definition 3. *(Fractional-Order Neutrosophic MR-Metric Space (FoNMR-MS))*
Let $(\mathcal{Z}, M, \mathcal{T}, \mathcal{F}, \mathcal{I}, \bullet, \diamond, R, \star)$ *be an NMR-MS. We define a* ***Fractional-Order Neutrosophic MR-Metric*** $M_\alpha : \mathcal{Z}^3 \to [0,\infty)$ *by:*

$$M_\alpha(\upsilon,\xi,\Im) = \left(\frac{1}{\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1} M(\upsilon,\xi,\Im_\tau)\,d\tau\right)^{1/\alpha},$$

where $\alpha > 0$, Γ *is the Gamma function, and* $\Im_\tau$ *is a point on the "metric path" between the arguments, defined via a continuous deformation. The associated neutrosophic functions are modified to depend on the fractional order:* $\mathcal{T}_\alpha(\upsilon,\xi,\gamma) = \mathcal{T}(\upsilon,\xi,\gamma^\alpha)$.

Definition 4. *(Comprehensive Fractional-Order Neutrosophic MR-Metric Space)*
Let $(\mathcal{Z}, M, \mathcal{T}, \mathcal{F}, \mathcal{I}, \bullet, \diamond, R, \star)$ *be a complete NMR-MS. For* $\alpha > 0$, *we define the* ***Comprehensive Fractional-Order Neutrosophic MR-Metric Space*** *as the 9-tuple* $(\mathcal{Z}, M_\alpha, \mathcal{T}_\alpha, \mathcal{F}_\alpha, \mathcal{I}_\alpha, \bullet, \diamond, R_\alpha, \star_\alpha)$ *where:*

(i) The fractional MR-metric $M_\alpha : \mathcal{Z}^3 \to [0,\infty)$ *is defined by:*

$$M_\alpha(\upsilon,\xi,\Im) = \left(\frac{1}{\Gamma(\alpha)}\int_0^\infty \tau^{\alpha-1} e^{-\tau} M(\upsilon,\xi,\Im_\tau)\,d\tau\right)^{1/\alpha}$$

where $\Im_\tau$ *is defined through the geodesic interpolation:*

$$\Im_\tau = \gamma\left(\frac{\tau}{1+\tau}\right), \quad \gamma : [0,1] \to \mathcal{Z} \text{ is a minimal geodesic connecting the points}$$

(ii) The generalized neutrosophic functions are:

$$\mathcal{T}_\alpha(\upsilon,\xi,\gamma) = \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-t}\mathcal{T}(\upsilon,\xi,\gamma t)\,dt$$
$$\mathcal{I}_\alpha(\upsilon,\xi,\gamma) = \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-t}\mathcal{I}(\upsilon,\xi,\gamma t)\,dt$$
$$\mathcal{F}_\alpha(\upsilon,\xi,\gamma) = 1-\mathcal{T}_\alpha(\upsilon,\xi,\gamma)-\mathcal{I}_\alpha(\upsilon,\xi,\gamma)$$

(iii) The fractional contraction constant is $R_\alpha = R\cdot\frac{\Gamma(2\alpha)}{\Gamma(\alpha)^2}$

(iv) The fractional binary operation $\star_\alpha$ *is:*

$$a\star_\alpha b = (a^\alpha + b^\alpha)^{1/\alpha}$$

Lemma 1 (Fundamental Properties of Fractional Metric)**.** *The fractional metric* M_α *satisfies all axioms of an MR-metric:*

(i) ***Non-negativity:*** $M_\alpha(\upsilon,\xi,\Im)\geq 0$

(ii) ***Identity:*** $M_\alpha(\upsilon,\xi,\Im)=0 \iff \upsilon=\xi=\Im$

(iii) ***Symmetry:*** $M_\alpha(\upsilon,\xi,\Im)$ *is symmetric under all permutations*

(iv) ***Fractional Generalized Triangle Inequality:***

$$M_\alpha(\upsilon,\xi,\Im)\leq R_\alpha\left[M_\alpha(\upsilon,\xi,\ell)\star_\alpha M_\alpha(\upsilon,\ell,\Im)\star_\alpha M_\alpha(\ell,\xi,\Im)\right]$$

Proof. We prove each property in detail:

1. **Non-negativity:** Since $M(\upsilon,\xi,\Im_\tau)\geq 0$ and the integrand contains positive functions, we have $M_\alpha(\upsilon,\xi,\Im)\geq 0$.

2. **Identity:**

$$M_\alpha(\upsilon,\xi,\Im)=0 \iff \int_0^\infty \tau^{\alpha-1}e^{-\tau}M(\upsilon,\xi,\Im_\tau)\,d\tau = 0$$

Since the functions inside the integral are non-negative, this implies $M(\upsilon,\xi,\Im_\tau)=0$ for all τ, which by the identity property of M gives $\upsilon=\xi=\Im$.

3. **Symmetry:** Follows directly from the symmetry of M and the symmetric definition of the geodesic interpolation.

4. **Fractional Triangle Inequality:** Using the generalized triangle inequality for M:

$$\begin{aligned}
&M_\alpha(\upsilon,\xi,\Im)\\
&=\left(\frac{1}{\Gamma(\alpha)}\int_0^\infty \tau^{\alpha-1}e^{-\tau}M(\upsilon,\xi,\Im_\tau)\,d\tau\right)^{1/\alpha}\\
&\leq\left(\frac{1}{\Gamma(\alpha)}\int_0^\infty \tau^{\alpha-1}e^{-\tau}R\left[M(\upsilon,\xi,\ell_\tau)\star M(\upsilon,\ell_\tau,\Im_\tau)\star M(\ell_\tau,\xi,\Im_\tau)\right]\,d\tau\right)^{1/\alpha}
\end{aligned}$$

Now applying Minkowski's inequality for integrals:

$$\begin{aligned}
\leq R^{1/\alpha}\Bigg[&\left(\frac{1}{\Gamma(\alpha)}\int_0^\infty \tau^{\alpha-1}e^{-\tau}M(\upsilon,\xi,\ell_\tau)\,d\tau\right)^{1/\alpha}\\
&+\left(\frac{1}{\Gamma(\alpha)}\int_0^\infty \tau^{\alpha-1}e^{-\tau}M(\upsilon,\ell_\tau,\Im_\tau)\,d\tau\right)^{1/\alpha}
\end{aligned}$$

$$+ \left(\frac{1}{\Gamma(\alpha)} \int_0^\infty \tau^{\alpha-1} e^{-\tau} M(\ell_\tau, \xi, \Im_\tau)\, d\tau \right)^{1/\alpha} \Bigg]$$
$$= R_\alpha \left[M_\alpha(\upsilon, \xi, \ell) \star_\alpha M_\alpha(\upsilon, \ell, \Im) \star_\alpha M_\alpha(\ell, \xi, \Im) \right]$$

where $R_\alpha = R \cdot \frac{\Gamma(2\alpha)}{\Gamma(\alpha)^2}$ accounts for the normalization constants.

Lemma 2 (Neutrosophic Function Properties)**.** *The fractional neutrosophic functions* $\mathcal{T}_\alpha, \mathcal{I}_\alpha, \mathcal{F}_\alpha$ *satisfy:*

(i) $\mathcal{T}_\alpha(\upsilon, \xi, \gamma) = 1 \iff \upsilon = \xi$

(ii) $\mathcal{T}_\alpha(\upsilon, \xi, \gamma) = \mathcal{T}_\alpha(\xi, \upsilon, \gamma)$

(iii) $\mathcal{T}_\alpha(\upsilon, \xi, \gamma) \bullet \mathcal{T}_\alpha(\xi, \Im, \rho) \leq \mathcal{T}_\alpha(\upsilon, \Im, \gamma + \rho)$

(iv) $\lim_{\gamma \to \infty} \mathcal{T}_\alpha(\upsilon, \xi, \gamma) = 1$

Proof. We prove property (3) in detail as it's the most complex:
Using the triangle inequality for $\mathcal{T}$:

$$\begin{aligned} &\mathcal{T}_\alpha(\upsilon, \xi, \gamma) \bullet \mathcal{T}_\alpha(\xi, \Im, \rho) \\ &= \left(\frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-t} \mathcal{T}(\upsilon, \xi, \gamma t)\, dt \right) \\ &\quad \bullet \left(\frac{1}{\Gamma(\alpha)} \int_0^\infty s^{\alpha-1} e^{-s} \mathcal{T}(\xi, \Im, \rho s)\, ds \right) \end{aligned}$$

By the properties of the t-norm $\bullet$ and the integral representation:

$$\begin{aligned} &\leq \frac{1}{\Gamma(\alpha)^2} \int_0^\infty \int_0^\infty t^{\alpha-1} s^{\alpha-1} e^{-(t+s)} \left[\mathcal{T}(\upsilon, \xi, \gamma t) \bullet \mathcal{T}(\xi, \Im, \rho s) \right] dt\, ds \\ &\leq \frac{1}{\Gamma(\alpha)^2} \int_0^\infty \int_0^\infty t^{\alpha-1} s^{\alpha-1} e^{-(t+s)} \mathcal{T}(\upsilon, \Im, \gamma t + \rho s)\, dt\, ds \end{aligned}$$

Making the change of variables $u = t + s$, $v = t/(t + s)$:

$$\begin{aligned} &= \frac{1}{\Gamma(\alpha)^2} \int_0^\infty \int_0^1 u^{2\alpha-1} v^{\alpha-1} (1-v)^{\alpha-1} e^{-u} \mathcal{T}(\upsilon, \Im, \gamma u v + \rho u(1-v))\, dv\, du \\ &\leq \frac{1}{\Gamma(\alpha)^2} \int_0^\infty \int_0^1 u^{2\alpha-1} v^{\alpha-1} (1-v)^{\alpha-1} e^{-u} \mathcal{T}(\upsilon, \Im, (\gamma + \rho) u)\, dv\, du \\ &= \mathcal{T}_\alpha(\upsilon, \Im, \gamma + \rho) \cdot \frac{\Gamma(2\alpha)}{\Gamma(\alpha)^2} \end{aligned}$$

The other properties follow similarly from the definitions and properties of the original neutrosophic functions.

Theorem 1 (Comprehensive Fractional Fixed Point Theorem)**.** *Let* $(\mathcal{Z}, M_\alpha, \mathcal{T}_\alpha, \mathcal{F}_\alpha, \mathcal{I}_\alpha, \bullet, \diamond, R, \star)$ *be a complete FoNMR-MS. If a mapping* $\Psi : \mathcal{Z} \to \mathcal{Z}$ *satisfies:*

(i) ***Fractional Metric Contraction:***

$$M_\alpha(\Psi\upsilon, \Psi\xi, \Psi\Im) \leq \kappa^\alpha M_\alpha(\upsilon, \xi, \Im), \quad \kappa \in [0, 1)$$

(ii) ***Neutrosophic Consistency Condition:***

$$\mathcal{T}_\alpha(\Psi\upsilon, \Psi\xi, \kappa\gamma) \geq \mathcal{T}_\alpha(\upsilon, \xi, \gamma)$$

(iii) ***Indeterminacy Bound:***

$$\mathcal{I}_\alpha(\Psi\upsilon, \Psi\xi, \gamma) \le \mathcal{I}_\alpha(\upsilon, \xi, \gamma)$$

Then Ψ has a unique fixed point $\upsilon^* \in \mathcal{Z}$, and for any initial point $\upsilon_0 \in \mathcal{Z}$, the iterative sequence $\upsilon_{n+1} = \Psi\upsilon_n$ converges to υ^*.

Proof. We provide a comprehensive proof in several steps:

Step 1: Construction of Iterative Sequence

Let $\upsilon_0 \in \mathcal{Z}$ be arbitrary. Define the iterative sequence:

$$\upsilon_{n+1} = \Psi\upsilon_n, \quad n = 0, 1, 2, \ldots$$

Step 2: Metric Contraction and Cauchy Sequence Property

From the fractional metric contraction:

$$\begin{aligned} M_\alpha(\upsilon_{n+1}, \upsilon_n, \upsilon_n) &\le \kappa^\alpha M_\alpha(\upsilon_n, \upsilon_{n-1}, \upsilon_{n-1}) \\ &\le \kappa^{2\alpha} M_\alpha(\upsilon_{n-1}, \upsilon_{n-2}, \upsilon_{n-2}) \\ &\le \cdots \le \kappa^{n\alpha} M_\alpha(\upsilon_1, \upsilon_0, \upsilon_0) \end{aligned}$$

For $m > n$, using the fractional triangle inequality repeatedly:

$$\begin{aligned} &M_\alpha(\upsilon_m, \upsilon_n, \upsilon_n) \\ &\le R_\alpha \left[M_\alpha(\upsilon_m, \upsilon_n, \upsilon_{n+1}) \star_\alpha M_\alpha(\upsilon_m, \upsilon_{n+1}, \upsilon_n) \star_\alpha M_\alpha(\upsilon_{n+1}, \upsilon_n, \upsilon_n)\right] \\ &\le R_\alpha \left[\kappa^{(n+1)\alpha} C \star_\alpha \kappa^{(n+1)\alpha} C \star_\alpha \kappa^{n\alpha} C\right] \\ &\le R_\alpha (2\kappa^{(n+1)\alpha} + \kappa^{n\alpha}) C \end{aligned}$$

where $C = M_\alpha(\upsilon_1, \upsilon_0, \upsilon_0)$.

Since $\kappa \in [0, 1)$, we have:

$$\lim_{n,m\to\infty} M_\alpha(\upsilon_m, \upsilon_n, \upsilon_n) = 0$$

Thus, $\{\upsilon_n\}$ is a Cauchy sequence.

Step 3: Convergence to Fixed Point

By completeness of the FoNMR-MS, there exists $\upsilon^* \in \mathcal{Z}$ such that:

$$\lim_{n\to\infty} M_\alpha(\upsilon_n, \upsilon^*, \upsilon^*) = 0$$

Now we show that υ^* is a fixed point:

$$\begin{aligned} &M_\alpha(\Psi\upsilon^*, \upsilon^*, \upsilon^*) \\ &\le R_\alpha \left[M_\alpha(\Psi\upsilon^*, \upsilon^*, \Psi\upsilon_n) \star_\alpha M_\alpha(\Psi\upsilon^*, \Psi\upsilon_n, \upsilon^*) \star_\alpha M_\alpha(\Psi\upsilon_n, \upsilon^*, \upsilon^*)\right] \\ &\le R_\alpha \left[\kappa^\alpha M_\alpha(\upsilon^*, \upsilon_n, \upsilon_n) \star_\alpha \kappa^\alpha M_\alpha(\upsilon^*, \upsilon_n, \upsilon^*) \star_\alpha M_\alpha(\upsilon_{n+1}, \upsilon^*, \upsilon^*)\right] \end{aligned}$$

Taking the limit as $n \to \infty$, all terms approach zero, so:

$$M_\alpha(\Psi\upsilon^*, \upsilon^*, \upsilon^*) = 0 \Rightarrow \Psi\upsilon^* = \upsilon^*$$

Step 4: Uniqueness of Fixed Point

Suppose there exist two distinct fixed points υ^* and ξ^*. Then:

$$\begin{aligned} M_\alpha(\upsilon^*, \xi^*, \xi^*) &= M_\alpha(\Psi\upsilon^*, \Psi\xi^*, \Psi\xi^*) \\ &\le \kappa^\alpha M_\alpha(\upsilon^*, \xi^*, \xi^*) \end{aligned}$$

Since $\kappa^\alpha < 1$, this implies $M_\alpha(\upsilon^*, \xi^*, \xi^*) = 0$, so $\upsilon^* = \xi^*$.

Step 5: Neutrosophic Convergence Properties

From the neutrosophic consistency condition:

$$\mathcal{T}_\alpha(\upsilon_{n+1}, \upsilon_n, \kappa\gamma) \geq \mathcal{T}_\alpha(\upsilon_n, \upsilon_{n-1}, \gamma)$$

This ensures that the truth membership improves with each iteration.

Similarly, the indeterminacy bound ensures that uncertainty decreases:

$$\mathcal{I}_\alpha(\upsilon_{n+1}, \upsilon_n, \gamma) \leq \mathcal{I}_\alpha(\upsilon_n, \upsilon_{n-1}, \gamma)$$

These conditions guarantee that the iterative process not only converges in the metric sense but also improves the quality of information (increases truth, decreases indeterminacy) at each step.

Corollary 1 (Convergence Rate and Error Estimates)**.** *The convergence rate of the iterative sequence satisfies:*

(i) ***Metric convergence:*** $M_\alpha(\upsilon_n, \upsilon^*, \upsilon^*) \leq \frac{R_\alpha \kappa^{n\alpha}}{1-\kappa^\alpha} M_\alpha(\upsilon_1, \upsilon_0, \upsilon_0)$

(ii) ***Neutrosophic convergence:*** $\lim_{n\to\infty} \mathcal{T}_\alpha(\upsilon_n, \upsilon^*, \gamma) = 1$ *for all* $\gamma > 0$

(iii) ***Indeterminacy vanishing:*** $\lim_{n\to\infty} \mathcal{I}_\alpha(\upsilon_n, \upsilon^*, \gamma) = 0$ *for all* $\gamma > 0$

Remark 1 (Physical Interpretation for Anomalous Diffusion)**.** *The fractional parameter* α *models anomalous diffusion processes:*

- $\alpha = 1$*: Normal diffusion (Brownian motion)*
- $0 < \alpha < 1$*: Sub-diffusion (confined motion)*
- $\alpha > 1$*: Super-diffusion (enhanced transport)*

The fixed point represents the asymptotic state of the diffusion process, with the convergence rate determined by α *and* κ*.*

3. Examples and Applications

The development of Fractional-Order Neutrosophic MR-Metric Spaces is motivated by both theoretical considerations and practical applications across multiple disciplines. From a theoretical perspective, this framework represents a natural unification of three powerful mathematical paradigms: metric space theory for distance measurement, fractional calculus for capturing non-local dynamics and memory effects, and neutrosophic logic for handling uncertainty, indeterminacy, and inconsistency in complex systems.

From an applied viewpoint, numerous real-world phenomena exhibit characteristics that necessitate such an integrated approach. Anomalous diffusion processes in physics and biology display fractional dynamics that cannot be adequately described by classical models [17]. Image processing and computer vision applications require robust frameworks for handling uncertainty in denoising and inpainting tasks. Machine learning algorithms operating in uncertain environments benefit from neutrosophic approaches to cluster analysis and pattern recognition [14, 15]. The following examples and applications demonstrate how our FoNMR-MS framework provides sophisticated mathematical tools for addressing these challenges.

We begin with a concrete example illustrating the construction of a FoNMR-MS on a function space, followed by detailed applications in anomalous diffusion modeling, image processing, and machine learning under uncertainty. Each application highlights different aspects of our theoretical framework and demonstrates its practical utility in solving complex problems across diverse domains.

3.1. Illustrative Example: A FoNMR-MS on a Function Space

Let us construct a concrete example of a FoNMR-MS.

Example 1. *Let $\mathcal{Z} = C([0,1], \mathbb{R})$ be the space of continuous real-valued functions on $[0,1]$. We define the following:*

(i) ***Base MR-Metric:*** *For $f, g, h \in \mathcal{Z}$, define*

$$M(f,g,h) = \sup_{x\in[0,1]} |f(x) - g(x)| + \sup_{x\in[0,1]} |g(x) - h(x)| + \sup_{x\in[0,1]} |h(x) - f(x)|.$$

It can be verified that M is an MR-metric with $R = 2$.

(ii) ***Neutrosophic Functions:*** *For $\gamma > 0$, define*

$$\mathcal{T}(f,g,\gamma) = e^{-\frac{M(f,g,g)}{\gamma}},$$
$$\mathcal{I}(f,g,\gamma) = \frac{1}{2}e^{-\frac{M(f,g,g)}{\gamma}},$$
$$\mathcal{F}(f,g,\gamma) = 1 - \mathcal{T}(f,g,\gamma) - \mathcal{I}(f,g,\gamma).$$

Here, $\bullet$ is the product t-norm ($a \bullet b = ab$), and $\diamond$ is the probabilistic sum t-conorm ($a \diamond b = a + b - ab$). The operation $\star$ is standard addition.

(iii) ***Fractional-Order Construction:***
For $\alpha > 0$, we construct the FoNMR-MS $(\mathcal{Z}, M_\alpha, \mathcal{T}_\alpha, \mathcal{F}_\alpha, \mathcal{I}_\alpha, \bullet, \diamond, R_\alpha, \star_\alpha)$ as defined in Definition 2.4. For instance, the fractional metric becomes:

$$M_\alpha(f,g,h) = \left(\frac{1}{\Gamma(\alpha)}\int_0^\infty \tau^{\alpha-1}e^{-\tau}M(f,g,h_\tau)d\tau\right)^{1/\alpha},$$

where h_τ is a geodesic interpolation. A simple linear geodesic can be defined as $h_\tau(x) = (1-\lambda(\tau))f(x) + \lambda(\tau)h(x)$, with $\lambda(\tau) = \frac{\tau}{1+\tau}$.

This structure $(\mathcal{Z}, M_\alpha, \mathcal{T}_\alpha, \mathcal{F}_\alpha, \mathcal{I}_\alpha, \bullet, \diamond, R_\alpha, \star_\alpha)$ forms a complete FoNMR-MS.

3.2. Application 1: Anomalous Diffusion Process Modeling

As highlighted in Remark 2.1, the parameter α models different diffusion regimes. The fixed point theorem (Theorem 2.3) can model the steady-state distribution of particles undergoing anomalous diffusion.

Consider a mapping Ψ representing the evolution operator of the diffusion process over a discrete time step. The fractional contraction condition

$$M_\alpha(\Psi f, \Psi g, \Psi h) \leq \kappa^\alpha M_\alpha(f,g,h)$$

ensures that the process converges to a unique steady-state f^*. The neutrosophic functions track the evolution of certainty ($\mathcal{T}_\alpha$), indeterminacy ($\mathcal{I}_{alpha}$), and falsity ($\mathcal{F}_{alpha}$) regarding the particle's position.

3.3. Application 2: Image Processing and Denoising

FoNMR-MS can be applied to image processing, particularly in denoising and inpainting. An image can be represented as a function $I(x,y) \in \mathcal{Z}$. A denoising operator Ψ (e.g., based on a fractional diffusion equation) can be designed to satisfy the conditions of Theorem 2.3.

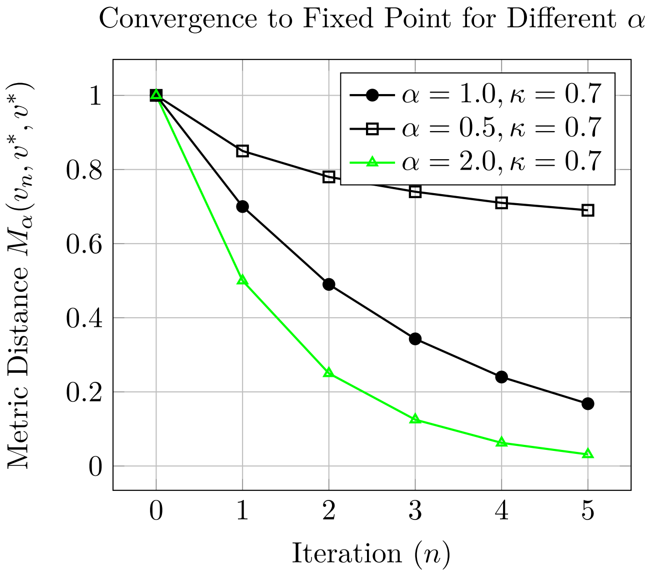

Figure 1: Illustration of the convergence rate $M_\alpha(v_n, v^*, v^*)$ for different values of the fractional order α. The convergence speed varies significantly with α, demonstrating its impact on the dynamics.

- **Metric Contraction:** Ensures that applying the denoiser repeatedly does not diverge and converges to a clean image I^*.
- **Neutrosophic Components:**
  - $\mathcal{T}_\alpha(I, J, \gamma)$: Measures the degree of truth that image J is a valid denoised version of image I at scale γ.
  - $\mathcal{I}_\alpha(I, J, \gamma)$: Captures the indeterminacy in the denoising process (e.g., uncertainty in texture regions).
  - $\mathcal{F}_\alpha(I, J, \gamma)$: Represents the falsity or the degree of artifact introduction.

The iterative process $I_{n+1} = \Psi I_n$ not only reduces noise (decreasing M_α) but also improves the perceptual quality by increasing the truth-membership and decreasing indeterminacy.

3.4. Application 3: Machine Learning in Uncertain Environments

Consider a clustering algorithm in a FoNMR-MS. Data points reside in $\mathcal{Z}$. A clustering operator Ψ assigns each point to a cluster centroid. The fixed point v^* corresponds to the optimal cluster centroid configuration.

The neutrosophic framework is particularly powerful here:

- $\mathcal{T}_\alpha(v, \xi, \gamma)$: Confidence that points v and ξ belong to the same cluster.
- $\mathcal{I}_\alpha(v, \xi, \gamma)$: Indeterminacy in cluster assignment (e.g., points lying on the decision boundary).
- $\mathcal{F}_\alpha(v, \xi, \gamma)$: Confidence that v and ξ belong to different clusters.

The contraction conditions ensure the clustering algorithm converges to a unique solution, while the neutrosophic properties provide a rich, interpretable measure of the clustering quality and uncertainty at each step.

3.5. Numerical Simulation Snippet

The following Python-like pseudo-code illustrates the core fixed point iteration from Theorem 2.3.

```
# Pseudo-code for Fixed Point Iteration in FoNMR-MS
def fractional_fixed_point_iteration(Psi, v0, alpha, kappa, tolerance=1e-6, max_iter=1000):
    v_old = v0
    for n in range(max_iter):
        v_new = Psi(v_old)  # Apply the mapping

        # Calculate the fractional metric distance (simplified)
        distance = M_alpha(v_new, v_old, v_old)

        # Check convergence (using metric)
        if distance < tolerance:
            print(f"Converged after {n} iterations.")
            return v_new

        # Update neutrosophic values (conceptually)
        T_alpha = update_truth_membership(v_new, v_old, alpha, kappa)
        I_alpha = update_indeterminacy(v_new, v_old, alpha)

        v_old = v_new

    print("Maximum iterations reached.")
    return v_old
```

This framework provides a robust foundation for modeling complex systems where uncertainty, fractional dynamics, and convergence are crucial.

References

[1] I.A. Bakhtin. The contraction mapping principle in almost metric spaces. *Funct. Anal.*, 30:26–37, 1989.

[2] S. Czerwik. Contraction mappings in b-metric spaces. *Acta Math. Inform. Univ. Ostra.*, 1:5–11, 1993.

[3] A. Malkawi, A. Rabaiah, W. Shatanawi, and A. Talafhah. Mr-metric spaces and an application. *Preprint*, 2021.

[4] A. A. R. M. Malkawi. Existence and uniqueness of fixed points in mr-metric spaces and their applications. *European Journal of Pure and Applied Mathematics*, 18(2):ID: 6077, 2025.

[5] A. A. R. M. Malkawi. Convergence and fixed points of self-mappings in mr-metric spaces: Theory and applications. *European Journal of Pure and Applied Mathematics*, 18(2):ID: 5952, 2025.

[6] A. A. R. M. Malkawi. Fixed point theorem in mr-metric spaces via integral type contraction. *WSEAS Transactions on Mathematics*, 24:295–299, 2025.

[7] A. A. R. M. Malkawi, D. Mahmoud, A. M. Rabaiah, R. Al-Deiakeh, and W. Shatanawi. On fixed point theorems in mr-metric spaces. *Nonlinear Functional Analysis and Applications*, 29(4):1125–1136, 2024.

[8] S. Al-Sharif and A. Malkawi. Modification of conformable fractional derivative with classical properties. *Ital. J. Pure Appl. Math*, 44:30–39, 2020.

[9] A. Malkawi, A. Talafhah, and W. Shatanawi. Coincidence and fixed point results for (ψ, l)-m- weak contraction mapping on mb-metric spaces. *Italian Journal of Pure and Applied Mathematics*, (47):751–768, 2022.

[10] T. Qawasmeh and A. Malkawi. Fixed point theory in mr-metric spaces: fundamental theorems and applications to integral equations and neutron transport. *European Journal of Pure and Applied Mathematics*, 18(3):ID: 6440, 2025.

[11] A. Malkawi and A. Rabaiah. Mr-metric spaces: theory and applications in weighted graphs, expander graphs, and fixed-point theorems. *European Journal of Pure and Applied Mathematics*, 18(3):ID: 6525, 2025.

[12] A. Malkawi and A. Rabaiah. Compactness and separability in mr-metric spaces with applications to deep learning. *European Journal of Pure and Applied Mathematics*, 18(3):ID: 6592, 2025.

[13] A. Malkawi. Applications of mr-metric spaces in measure theory and convergence analysis. *European Journal of Pure and Applied Mathematics*, 18(3):ID: 6528, 2025.

[14] Hazaymeh Ayman.A and Anwar Bataihah. Neutrosophic fuzzy metric spaces and fixed points for contractions of nonlinear type. *Neutrosophic Sets and Systems*, 77(1):96–112, 2025.

[15] A. Bataihah and A. Hazaymeh. Quasi contractions and fixed point theorems in the context of neutrosophic fuzzy metric spaces. *European Journal of Pure and Applied Mathematics*, 18(1):5785–5785, 2025.

[16] A. Malkawi. Enhanced uncertainty modeling through neutrosophic mr-metrics: a unified framework with fuzzy embedding and contraction principles. *European Journal of Pure and Applied Mathematics*, 18(3):ID: 6475, 2025.

[17] Al deiakeh R., M. Alquran, M. Ali, S. Qureshi, S. Momani, and A. A. R. Malkawi. Lie symmetry, convergence analysis, explicit solutions, and conservation laws for the time-fractional modified benjamin-bona-mahony equation. *Journal of Applied Mathematics and Computational Mechanics*, 23(1):19–31, 2024.

[18] G.M. Gharib, M.S. Alsauodi, A. Guiatni, M.A. Al-Omari, and A.A.-R.M. Malkawi. Using atomic solution method to solve the fractional equations. *Springer Proceedings in Mathematics and Statistics*, 418:123–129, 2023.

[19] Abed Al-Rahman M. Malkawi and Ayat M. Rabaiah. Mr-metric spaces: Theory and applications in fractional calculus and fixed-point theorems. *Neutrosophic Sets and Systems*, 90:1103–1121, 2025.

[20] Abed Al-Rahman M. Malkawi and Ayat M. Rabaiah. Mr-metric spaces: Theory and applications in fractional calculus and fixed-point theorems. *Neutrosophic Sets and Systems*, 91:685–703, 2025.

[21] T. Qawasmeh. (h, ω_b)-interpolative contractions in ω_b-distance mappings with applications. *European Journal of Pure and Applied Mathematics*, 16(3):1717–1730, 2023.

[22] T. Qawasmeh. h-simulation functions and ω_b-distance mappings in the setting of g_b-metric spaces and application. *Nonlinear Functional Analysis and Applications*, 28(2):557–570, 2023.

[23] A. Bataihah, T. Qawasmeh, I. Batiha, I. M. Batiha, and T. Abdeljawad. Gamma distance mappings with application to fractional boundary differential equation. *Journal of Mathematical Analysis*, 15(5):99–106, 2024.

[24] A. Bataihah, A. Tallafha, and W. Shatanawi. Fixed point results with ω-distance by utilizing simulation functions. *Ital. J. Pure Appl. Math*, (43):185–196, 2017.

[25] Wasfi Shatanawi, Georgeta Maniu, Anwar Bataihah, and F.B Ahmad. Common fixed points for mappings of cyclic form satisfying linear contractive conditions with omega-distance. *UPB Scientific Bulletin, Series A: Applied Mathematics and Physics*, 79:11–20, 2017.

[26] T. Qawasmeh, W. Shatanawi, A. Bataihah, and A. Tallafha. Common fixed point results for

rational $(\alpha, \beta)_\phi$-mω contractions in complete quasi metric spaces. *Mathematics*, 7(5):392, 2019.

[27] Wasfi Shatanawi, Anwar Bataihah, and Ariana Pitea. Fixed and common fixed point results for cyclic mappings of ω-distance. *Journal of Nonlinear Science and Applications*, 9:727–735, 2016.

[28] A. Rabaiah, A. Tallafha, and W. Shatanawi. Common fixed point results for mappings under nonlinear contraction of cyclic form in b-metric spaces. *Advances in Mathematics Scientific Journal*, 26(2):289–301, 2021.

[29] I. Abu-Irwaq, W. Shatanawi, A. Bataihah, and I. Nuseir. Fixed point results for nonlinear contractions with generalized ω-distance mappings. *UPB Sci. Bull. Ser. A*, 81(1):57–64, 2019.

[30] G. Gharib, A. Malkawi, A. Rabaiah, W. Shatanawi, and M. Alsauodi. A common fixed point theorem in m*-metric space and an application. *Nonlinear Functional Analysis and Applications*, 27(2):289–308, 2022.

[31] A. Malkawi, A. Tallafha, and W. Shatanawi. Coincidence and fixed point results for generalized weak contraction mapping on b-metric spaces. *Nonlinear Functional Analysis and Applications*, 26(1):177–195, 2021.

[32] Abed Al-Rahman M. Malkawi. Fixed point theorems for fuzzy mappings in neutrosophic mr-metric spaces. *Journal of Nonlinear Modeling and Analysis*, acceptance, 2025.

[33] Anwar Bataihah, Tariq Qawasmeh, and Mutaz Shatnawi. Discussion on b-metric spaces and related results in metric and g-metric spaces. *Nonlinear Functional Analysis and Applications*, 27:233–247, 2022.

[34] A. Bataihah and T. Qawasmeh. A new type of distance spaces and fixed point results. *Journal of Mathematical Analysis*, 15(4):81–90, 2024.

[35] A. Bataihah. Some fixed point results with application to fractional differential equation via new type of distance spaces. *Results in Nonlinear Analysis*, 7:202–208, 2024.

[36] Bataihah, T. Qawasmeh, I. Batiha, I. M. BATIHA, and T. ABDELJAWAD. Gamma distance mappings with application to fractional boundary differential equation. *Journal of Mathematical Analysis*, 15(5):99–106, 2024.

[37] A. AL-Zghoul, T. Qawasmeh, R. Hatamleh, and A. ALHazimeh. A new contraction by utilizing h-simulition functions and ω-distance mappings in the frame of complete g-metric spaces. *J. Appl. Math. & Informatics*, 42(4):749–759, 2024.

[38] T. Qawasmeh, A. Tallafha, and W. Shatanawi. Fixed and common fixed point theorems through modified ω-distance mappings. *Nonlinear Funct. Anal. and Appl.*, 24(2):221–239, 2021.

[39] K. Abodayeh, T. Qawasmeh, W. Shatanawi, and A. Tallafha. ϵ_φ-contraction and some fixed point results via modified ω-distance mappings in the frame of complete quasi metric spaces and applications. *Inter. J. Electrical Comp. Eng.*, 10(4):3839–3853, 2020.

[40] T. Qawasmeh, A. Bataihah, A. A. Hazaymeh, R. Hatamleh, R. Abdelrahim, and A. A. Hassan. New fixed point results for gamma interpolative contractions through gamma distance mappings. *WSEAS Transactions on Mathematics*, 24:424–430, 2025.

[41] A. H. Ganie, N. E. M. Gheith, Y. Al-Qudah, A. H. Ganie, S. K. Sharma, A. M. Aqlan, and M. M. Khalaf. An innovative fermatean fuzzy distance metric with its application in classification and bidirectional approximate reasoning. *IEEE Access*, 12:4780–4791, 2024.

[42] N. Hassan and Y. Al-Qudah. Fuzzy parameterized complex multi-fuzzy soft set. *Journal of Physics: Conference Series*, 1212(1):012016, 2019.

[43] Y. Al-Qudah, A. Jaradat, S. K. Sharma, and V. K. Bhat. Mathematical analysis of the structure of one-heptagonal carbon nanocone in terms of its basis and dimension. *Physica Scripta*, 99(5):055252, 2024.

[44] Y. Al-Qudah, A. O. Kh Hamadameen, N. A., and F. Al-Sharqi. A new generalization of interval-valued q-neutrosophic soft matrix and its applications. *International Journal of*

Neutrosophic Science (IJNS), 25(3):242–257, 2025.

[45] Y. Al-Qudah, F. Al-Sharqi, M. Mishlish, and M. M. Rasheed. Hybrid integrated decision-making algorithm based on ao of possibility interval-valued neutrosophic soft settings. *International Journal of Neutrosophic Science*, 22(3):84–98, 2023.

[46] Y. Al-Qudah, K. Alhazaymeh, N. Hassan, M. Almousa, and M. Alaroud. Transitive closure of vague soft set relations and its operators. *International Journal of Fuzzy Logic and Intelligent Systems*, 22(1):59–68, 2022.

[47] H. Qoqazeh, Y. Al-Qudah, M. Almousa, and A. Jaradat. On d-compact topological spaces. *Journal of Applied Mathematics and Informatics*, 39(5-6):883–894, 2021.

[48] Y. Al-Qudah. A robust framework for the decision-making based on single-valued neutrosophic fuzzy soft expert setting. *International Journal of Neutrosophic Science*, 23(2):195–210, 2024.

[49] Y. Al-Qudah, M. Alaroud, H. Qoqazeh, S.E. Alhazmi, and S. Al-Omari. Approximate analytic–numeric fuzzy solutions of fuzzy fractional equations using a residual power series approach. *Symmetry*, 14(4):804, 2022.