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Neutrosophic Statistical Manifolds: A Unified Framework for Information Geometry with Uncertainty Quantification

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Abstract. This paper introduces a novel framework integrating neutrosophic logic with information geometry, establishing the foundation of *neutrosophic statistical manifolds*. We define a neutrosophic MR-metric structure on statistical manifolds, incorporating truth, indeterminacy, and falsity membership functions to quantify distributional similarity, epistemic uncertainty, and dissimilarity. The proposed structure generalizes the Fisher–Rao metric through a symmetric triple-based formulation using Jensen–Shannon divergence. We prove that the triplet $(\mathcal{T}, \mathcal{I}, \mathcal{F})$ satisfies all axioms of a neutrosophic MR-metric space and derive explicit relations between the contraction constant R and the curvature of the underlying statistical manifold. Several applications are explored, including Gaussian and categorical models, hypothesis testing, model selection, geometric machine learning, and quantum information geometry. This work bridges fixed-point theory in generalized metric spaces with statistical inference under uncertainty, offering a robust tool for uncertainty-aware data analysis.

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1. Introduction

The study of generalized metric spaces has been a fertile area of research in pure and applied mathematics, with significant contributions to fixed-point theory and its applications. The concept of b-metric spaces was introduced by Bakhtin [1] and later formalized by Czerwik [2], providing a framework for handling non-linear contraction mappings. Subsequent extensions, such as G_b-metric spaces and Ω-distance mappings, have further enriched the theory [3–11].

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In parallel, the notion of MR-metric spaces was introduced by Malkawi et al. [12] as a generalization of standard metric spaces, enabling the analysis of triple-based geometric structures. This has led to numerous fixed-point results under various contraction conditions [13–23]. Recent work has also explored the intersection of metric spaces with fuzzy and neutrosophic logic. For instance, Hazaymeh and Bataihah [24] and Bataihah and Hazaymeh [25] introduced neutrosophic fuzzy metric spaces, while Malkawi [26, 27] extended fixed-point theory to neutrosophic MR-metric settings.

On the other hand, information geometry—the study of statistical manifolds endowed with the Fisher–Rao metric—has provided deep insights into the geometric structure of probability distributions. Divergence measures such as the Kullback–Leibler divergence and Jensen–Shannon divergence play a central role in this field. Recent work by Malkawi and Rabaiah [28, 29] has begun to explore the connections between MR-metric spaces and information-theoretic divergences.

This paper unifies these two streams of research by introducing *neutrosophic statistical manifolds*—a structure that combines the triple-based geometry of MR-metric spaces with the uncertainty-handling capabilities of neutrosophic logic. Our work is also influenced by applications of generalized metric spaces to fractional differential equations [30–39] and cyclic mappings [5, 11, 40].

The main contributions of this paper are:

- The definition of a neutrosophic MR-metric structure on statistical manifolds.
- A proof that the triplet $(\mathcal{T}, \mathcal{I}, \mathcal{F})$ satisfies neutrosophic metric axioms.
- Explicit links between the contraction constant R and curvature.
- Detailed examples and applications in statistics, machine learning, and quantum information.

Definition 1. *[12] Consider a non-empty set $\mathbb{X} \neq \emptyset$ and a real number $\mathbb{R} > 1$. A function*

$$M : \mathbb{X} \times \mathbb{X} \times \mathbb{X} \to [0, \infty)$$

is termed an ***MR-metric*** *if it satisfies the following conditions for all $v, \xi, s, \ell_1 \in \mathbb{X}$:*

- $M(v, \xi, s) \geq 0$.
- $M(v, \xi, s) = 0$ *if and only if* $v = \xi = s$.
- $M(v, \xi, s)$ *remains invariant under any permutation* $p(v, \xi, s)$*, i.e.,* $M(v, \xi, s) = M(p(v, \xi, s))$.
- *The following inequality holds:*

$$M(v, \xi, s) \leq \mathbb{R} \left[M(v, \xi, \ell_1) + M(v, \ell_1, s) + M(\ell_1, \xi, s) \right].$$

A structure $(\mathbb{X}, M)$ that adheres to these properties is defined as an MR-metric space.

Definition 2. *[27][Neutrosophic MR-Metric Space (NMR-MS)]*
A 9-tuple $(\mathcal{Z}, M, \mathcal{T}, \mathcal{F}, \mathcal{I}, \bullet, \diamond, R, \star)$ is called a ***Neutrosophic MR-Metric Space*** *if:*

(i) $\mathcal{Z}$ *is a non-empty set.*

(ii) $M : \mathcal{Z} \times \mathcal{Z} \times \mathcal{Z} \to [0, \infty)$ *is an MR-metric satisfying:*

(M1) $M(\upsilon, \xi, \Im) \geq 0$,

(M2) $M(\upsilon, \xi, \Im) = 0 \iff \upsilon = \xi = \Im$,

(M3) Symmetry under permutations,

(M4) $M(\upsilon, \xi, \Im) \leq R\,[M(\upsilon, \xi, \ell) \star M(\upsilon, \ell, \Im) \star M(\ell, \xi, \Im)]$, $R > 1$.

(iii) $\mathcal{T}, \mathcal{F}, \mathcal{I} : \mathcal{Z} \times \mathcal{Z} \times (0, \infty) \to [0, 1]$ *are neutrosophic functions satisfying:*

(N1) $\mathcal{T}(\upsilon, \xi, \gamma) = 1 \iff \upsilon = \xi$ *(Truth-Identity),*

(N2) $\mathcal{T}(\upsilon, \xi, \gamma) = \mathcal{T}(\xi, \upsilon, \gamma)$ *(Symmetry),*

(N3) $\mathcal{T}(\upsilon, \xi, \gamma) \bullet \mathcal{T}(\xi, \Im, \rho) \leq \mathcal{T}(\upsilon, \Im, \gamma + \rho)$ *(Triangle Inequality),*

(N4) $\lim_{\gamma \to \infty} \mathcal{T}(\upsilon, \xi, \gamma) = 1$ *(Asymptotic Behavior).*

(iv) $\bullet$ *(t-norm) and* $\diamond$ *(t-conorm) are continuous operators generalizing fuzzy logic.*

(v) $\star$ *is a binary operation generalizing addition (e.g., weighted sum).*

2. Main Results

This section presents the core theoretical contributions of this work. We begin by introducing the neutrosophic statistical structure defined on smooth manifolds of probability distributions. This framework integrates an MR-metric constructed from the Jensen–Shannon divergence with neutrosophic membership functions designed to quantify truth, indeterminacy, and falsity. We subsequently establish the mathematical consistency of this structure and investigate its geometric characteristics, particularly its connections to the Fisher–Rao metric and the α-connections in information geometry. Theorems and lemmas are formulated to rigorously characterize these relationships, accompanied by proofs that validate the fulfillment of all neutrosophic metric axioms.

Definition 3 (Statistical Manifold with Neutrosophic Structure)**.** *Let $\mathcal{P}$ be a smooth manifold of probability distributions $p(\cdot;\theta)$ parameterized by $\theta = (\theta^1, \ldots, \theta^n) \in \Theta \subseteq \mathbb{R}^n$. A* ***neutrosophic statistical structure*** *on $\mathcal{P}$ is given by:*

(i) ***Neutrosophic MR-Metric:*** *For $p, q, r \in \mathcal{P}$, define:*

$$M(p, q, r) = D_{JS}(p\|q) + D_{JS}(q\|r) + D_{JS}(r\|p)$$

where D_{JS} is the Jensen-Shannon divergence:

$$D_{JS}(p\|q) = \frac{1}{2} D_{KL}\left(p \middle\| \frac{p+q}{2}\right) + \frac{1}{2} D_{KL}\left(q \middle\| \frac{p+q}{2}\right)$$

(ii) ***Neutrosophic Membership Functions:***

$$\begin{aligned}\mathcal{T}(p,q,\gamma) &= \exp\left(-\gamma \cdot JSD(p\|q)\right)\\ \mathcal{I}(p,q,\gamma) &= 1 - \left|\frac{H(p)-H(q)}{\max_{r\in\mathcal{P}} H(r)}\right| \cdot \exp\left(-\gamma \cdot |D_{KL}(p\|u) - D_{KL}(q\|u)|\right)\\ \mathcal{F}(p,q,\gamma) &= 1 - \mathcal{T}(p,q,\gamma) - \mathcal{I}(p,q,\gamma)\end{aligned}$$

where $H(p)$ is the Shannon entropy, u is the uniform distribution, and JSD is the normalized Jensen-Shannon divergence.

(iii) ***Operations:*** *The t-norm $\bullet$ is the product t-norm $a \bullet b = ab$, and $\star$ is the standard addition.*

Theorem 1 (Well-Defined Neutrosophic Structure)**.** *The triplet $(\mathcal{T},\mathcal{I},\mathcal{F})$ defines a valid neutrosophic structure on the statistical manifold $\mathcal{P}$, satisfying all axioms of a Neutrosophic MR-Metric Space.*

Proof. We prove each axiom systematically:

Part 1: Neutrosophic Axioms Verification

(i) **Truth-Identity:** $\mathcal{T}(p,q,\gamma) = 1 \iff p = q$

$$\begin{aligned}\mathcal{T}(p,q,\gamma) = 1 &\iff \exp\left(-\gamma \cdot JSD(p\|q)\right) = 1\\ &\iff JSD(p\|q) = 0 \iff p = q \quad \text{(since JSD is a metric)}\end{aligned}$$

(ii) **Symmetry:** $\mathcal{T}(p,q,\gamma) = \mathcal{T}(q,p,\gamma)$

$$\mathcal{T}(p,q,\gamma) = \exp\left(-\gamma \cdot JSD(p\|q)\right) = \exp\left(-\gamma \cdot JSD(q\|p)\right) = \mathcal{T}(q,p,\gamma)$$

since JSD is symmetric.

(iii) **Triangle Inequality:** $\mathcal{T}(p,q,\gamma) \bullet \mathcal{T}(q,r,\rho) \le \mathcal{T}(p,r,\gamma+\rho)$

Using the product t-norm $a \bullet b = ab$:

$$\begin{aligned}\mathcal{T}(p,q,\gamma) \bullet \mathcal{T}(q,r,\rho) &= \exp\left(-\gamma \cdot JSD(p\|q)\right) \cdot \exp\left(-\rho \cdot JSD(q\|r)\right)\\ &= \exp\left(-\gamma \cdot JSD(p\|q) - \rho \cdot JSD(q\|r)\right)\end{aligned}$$

Since JSD is a metric, it satisfies the triangle inequality:

$$JSD(p\|r) \le JSD(p\|q) + JSD(q\|r)$$

Therefore:

$$\begin{aligned}\mathcal{T}(p,r,\gamma+\rho) &= \exp\left(-(\gamma+\rho) \cdot JSD(p\|r)\right)\\ &\ge \exp\left(-(\gamma+\rho) \cdot [JSD(p\|q) + JSD(q\|r)]\right)\end{aligned}$$

$$= \exp\left(-\gamma \cdot JSD(p\|q) - \rho \cdot JSD(q\|r)\right) \cdot \exp\left(-\rho \cdot JSD(p\|q) - \gamma \cdot JSD(q\|r)\right)$$

Since $\exp\left(-\rho \cdot JSD(p\|q) - \gamma \cdot JSD(q\|r)\right) \leq 1$ for all $\gamma, \rho > 0$ and $p, q, r \in \mathcal{P}$, we have:

$$\mathcal{T}(p, r, \gamma + \rho) \geq \exp\left(-\gamma \cdot JSD(p\|q) - \rho \cdot JSD(q\|r)\right) = \mathcal{T}(p, q, \gamma) \bullet \mathcal{T}(q, r, \rho)$$

(iv) **Asymptotic Behavior:**

$$\lim_{\gamma \to \infty} \mathcal{T}(p, q, \gamma) = \begin{cases} 1 & \text{if } p = q \\ 0 & \text{if } p \neq q \end{cases}$$

This follows because:

$$\lim_{\gamma \to \infty} \exp\left(-\gamma \cdot JSD(p\|q)\right) = \begin{cases} \exp(0) = 1 & \text{if } JSD(p\|q) = 0 \text{ (i.e., } p = q) \\ 0 & \text{if } JSD(p\|q) > 0 \text{ (i.e., } p \neq q) \end{cases}$$

Part 2: Indeterminacy and Falsity Properties

(i) **Indeterminacy Bounds:** $0 \leq \mathcal{I}(p, q, \gamma) \leq 1$

$$\mathcal{I}(p, q, \gamma) = 1 - \left| \frac{H(p) - H(q)}{\max_{r \in \mathcal{P}} H(r)} \right| \cdot \exp\left(-\gamma \cdot |D_{KL}(p\|u) - D_{KL}(q\|u)|\right)$$

Since $0 \leq \left| \frac{H(p) - H(q)}{\max H} \right| \leq 1$ and $0 \leq \exp(\cdot) \leq 1$, we have $0 \leq \mathcal{I}(p, q, \gamma) \leq 1$.

(ii) **Consistency:** $\mathcal{T}(p, q, \gamma) + \mathcal{I}(p, q, \gamma) + \mathcal{F}(p, q, \gamma) = 1$ by construction.

This completes the proof that $(\mathcal{T}, \mathcal{I}, \mathcal{F})$ forms a valid neutrosophic structure on the statistical manifold $\mathcal{P}$.

Theorem 2 (MR-Metric as Symmetric Fisher-Rao Analog)**.** *The MR-metric M is a symmetric generalization of the Fisher-Rao metric, with the following relations:*

(i) ***Local Expansion:*** *For infinitesimally close distributions $p(\theta)$, $p(\theta + d\theta)$, and $p(\theta + d\phi)$:*

$$M(p(\theta), p(\theta + d\theta), p(\theta + d\phi)) = \tfrac{1}{2}\Big[g_{ij}(\theta)\, d\theta^i d\theta^j + g_{ij}(\theta)\, d\phi^i d\phi^j + g_{ij}(\theta)\, (d\theta^i - d\phi^i)(d\theta^j - d\phi^j)\Big] + O(\|d\theta\|^3),$$

where g_{ij} is the Fisher–Rao metric tensor.

(ii) ***Curvature Relation:*** *The contraction constant R is bounded by:*

$$R \geq 1 + \frac{1}{4} \max_{p,q,r \in \mathcal{P}} \frac{\mathcal{R}(p, q, r)}{\sqrt{g(p, p) g(q, q) g(r, r)}}$$

where $\mathcal{R}$ is the sectional curvature tensor of the statistical manifold.

Proof. **Part 1: Local Expansion and Fisher-Rao Connection**
Consider the Taylor expansion of D_{JS} around θ:
For $p = p(\theta)$, $q = p(\theta + d\theta)$, $r = p(\theta + d\phi)$, we have:

$$\begin{aligned} D_{JS}(p\|q) &= \frac{1}{8} g_{ij}(\theta) d\theta^i d\theta^j + O(\|d\theta\|^3) \\ D_{JS}(q\|r) &= \frac{1}{8} g_{ij}(\theta)(d\phi^i - d\theta^i)(d\phi^j - d\theta^j) + O(\|d\phi - d\theta\|^3) \\ D_{JS}(r\|p) &= \frac{1}{8} g_{ij}(\theta) d\phi^i d\phi^j + O(\|d\phi\|^3) \end{aligned}$$

Therefore:

$$\begin{aligned} M(p,q,r) &= D_{JS}(p\|q) + D_{JS}(q\|r) + D_{JS}(r\|p) \\ &= \frac{1}{8}\left[g_{ij} d\theta^i d\theta^j + g_{ij}(d\phi^i - d\theta^i)(d\phi^j - d\theta^j) + g_{ij} d\phi^i d\phi^j\right] + O(\|d\|^3) \\ &= \frac{1}{4}\left[g_{ij} d\theta^i d\theta^j + g_{ij} d\phi^i d\phi^j + \frac{1}{2} g_{ij}(d\theta^i - d\phi^i)(d\theta^j - d\phi^j)\right] + O(\|d\|^3) \end{aligned}$$

This shows that M captures the Fisher-Rao geometry in a symmetric, triple-based formulation.

Part 2: Curvature and Contraction Constant

Using the generalized triangle inequality for M:

$$M(p,q,r) \leq R\left[M(p,q,s) + M(p,s,r) + M(s,q,r)\right]$$

For infinitesimal triangles, the worst-case ratio occurs when the manifold has maximum sectional curvature. By the generalized law of cosines on Riemannian manifolds:

For a geodesic triangle with vertices p, q, r and a point s on the geodesic between p and r, we have:

$$\begin{aligned} d^2(p,r) = d^2(p,q) + d^2(q,r) - 2d(p,q)d(q,r)\cos(\angle pqr) \\ + \frac{1}{3}\mathcal{R}_{ijkl} X^i Y^j X^k Y^l + O(d^5) \end{aligned}$$

where X, Y are tangent vectors, and $\mathcal{R}_{ijkl}$ is the Riemann curvature tensor.
Translating this to our MR-metric context:

$$\begin{aligned} M(p,q,r) &= \frac{3}{2}\left[d^2(p,q) + d^2(q,r) + d^2(r,p)\right] + O(d^4) \\ M(p,q,s) + M(p,s,r) + M(s,q,r) &= \frac{3}{2}\left[d^2(p,q) + d^2(q,s) + d^2(s,p) + \cdots\right] \end{aligned}$$

The curvature correction appears at fourth order. Maximizing over all configurations gives:

$$R \geq 1 + \frac{1}{4} \max \frac{\mathcal{R}(X,Y,X,Y)}{\|X\|^2\|Y\|^2 - \langle X,Y\rangle^2}$$

where the maximum is taken over all linearly independent tangent vectors X, Y.

Lemma 1 (Differential Geometric Structure)**.** *The neutrosophic statistical manifold* $(\mathcal{P}, M, \mathcal{T}, \mathcal{I}, \mathcal{F})$ *inherits a rich differential geometric structure:*

(i) ***Connection:*** *The* α*-connection* $\nabla^{(\alpha)}$ *is related to the neutrosophic structure via:*

$$\Gamma^{(\alpha)}_{ij,k} = \mathbb{E}_p\left[\partial_i\partial_j\ell_p \cdot \partial_k\ell_p\right] + \frac{1-\alpha}{2}\mathbb{E}_p\left[\partial_i\ell_p \cdot \partial_j\ell_p \cdot \partial_k\ell_p\right]$$

where $\ell_p = \log p$*.*

(ii) ***Divergence:*** *The Jensen-Shannon divergence is a symmetric Bregman divergence:*

$$D_{JS}(p\|q) = \frac{1}{2}\left[D_{KL}(p\|m) + D_{KL}(q\|m)\right], \quad m = \frac{p+q}{2}$$

Proof. **Connection Structure:**
The Fisher-Rao metric is:

$$g_{ij}(p) = \mathbb{E}_p\left[\partial_i\ell_p \cdot \partial_j\ell_p\right]$$

The α-connection coefficients are:

$$\Gamma^{(\alpha)}_{ij,k}(p) = \mathbb{E}_p\left[(\partial_i\partial_j\ell_p + \frac{1-\alpha}{2}\partial_i\ell_p\partial_j\ell_p)\partial_k\ell_p\right]$$

For our neutrosophic structure, the MR-metric M induces a connection that interpolates between the $\alpha = -1$ (mixture) and $\alpha = 1$ (exponential) connections, with the indeterminacy $\mathcal{I}$ quantifying the uncertainty in this interpolation.
Divergence Properties:
The Jensen-Shannon divergence has the key properties:

- **Symmetry:** $D_{JS}(p\|q) = D_{JS}(q\|p)$
- **Positivity:** $D_{JS}(p\|q) \geq 0$ with equality iff $p = q$
- **Convexity:** Jointly convex in p and q
- **Boundedness:** $0 \leq D_{JS}(p\|q) \leq \log 2$

These properties ensure the well-definedness of our neutrosophic structure.

Theorem 3 (Detailed Curvature-Contraction Relation)**.** *The contraction constant R in the NMR-MS structure is explicitly related to the curvature of the statistical manifold:*

$$R = 1 + \frac{1}{2} \max_{\substack{p \in \mathcal{P} \\ X,Y \in T_p\mathcal{P}}} \frac{\mathcal{R}(X,Y,X,Y) + \mathcal{R}(X,Y,Y,X)}{\|X\|^2\|Y\|^2 - \langle X,Y\rangle^2} + \epsilon(\mathcal{I})$$

where $\epsilon(\mathcal{I})$ is a correction term depending on the indeterminacy:

$$\epsilon(\mathcal{I}) = \frac{1}{4}\mathbb{E}_{p,q,r}[\mathcal{I}(p,q,\gamma) + \mathcal{I}(q,r,\gamma) + \mathcal{I}(r,p,\gamma)]$$

Proof. We analyze the curvature effects through several steps:

Step 1: Riemannian Geometry Framework

Consider the statistical manifold as a Riemannian manifold $(\mathcal{P}, g)$ with Fisher-Rao metric. The sectional curvature for a 2-plane spanned by orthonormal vectors X, Y is:

$$K(X,Y) = \mathcal{R}(X,Y,X,Y)$$

Step 2: MR-Metric Expansion

For small geodesic triangles, expand M using the metric and curvature:

$$\begin{aligned} M(p,q,r) = &\frac{3}{2}\left[d^2(p,q) + d^2(q,r) + d^2(r,p)\right] \\ &- \frac{1}{8}\left[K(X,Y) + K(Y,Z) + K(Z,X)\right] \cdot \text{Area}^2 + O(d^6) \end{aligned}$$

where X, Y, Z are tangent vectors along the triangle edges.

Step 3: Worst-Case Contraction

The contraction inequality becomes tightest for triangles maximizing the curvature terms. The worst-case ratio is:

$$\frac{M(p,q,r)}{M(p,q,s) + M(p,s,r) + M(s,q,r)} \leq 1 + \frac{1}{2}\max K + O(d^2)$$

Taking the supremum over all configurations gives the stated bound.

Step 4: Indeterminacy Correction

The neutrosophic indeterminacy $\mathcal{I}$ introduces additional uncertainty in the metric relations. This can be modeled as a stochastic correction to the curvature:

$$\tilde{\mathcal{R}} = \mathcal{R} + \delta\mathcal{R}, \quad \|\delta\mathcal{R}\| \propto \mathbb{E}[\mathcal{I}]$$

This leads to the $\epsilon(\mathcal{I})$ correction term, which quantifies how epistemic uncertainty affects the geometric structure.

3. Applications and Examples

Having established the theoretical foundations of neutrosophic statistical manifolds, we now turn to their practical implications. The following section provides detailed examples and applications across a range of domains. We explore Gaussian and categorical statistical manifolds, demonstrate how the neutrosophic structure enhances model selection and hypothesis testing, and illustrate its utility in geometric machine learning and quantum information geometry. Each example includes explicit computations and visualizations to aid intuition and demonstrate applicability.

3.1. Gaussian Statistical Manifold

Example 1 (Univariate Gaussian Distributions)**.** *Consider the family of univariate Gaussian distributions parameterized by $\theta = (\mu, \sigma)$:*

$$p(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

- ***Fisher–Rao Metric:*** *The metric tensor in coordinates (μ, σ) is:*

$$ds^2 = g_{\mu\mu} d\mu^2 + 2g_{\mu\sigma} d\mu d\sigma + g_{\sigma\sigma} d\sigma^2 = \frac{1}{\sigma^2} d\mu^2 + \frac{2}{\sigma^2} d\sigma^2.$$

  This induces a hyperbolic geometry on the half-plane $(\mu, \sigma) \in \mathbb{R} \times (0, \infty)$.

- ***Neutrosophic MR-Metric:*** *For three Gaussians p, q, r, we compute:*

$$M(p, q, r) = D_{JS}(p\|q) + D_{JS}(q\|r) + D_{JS}(r\|p).$$

  For infinitesimally close distributions $p(\mu, \sigma), q(\mu + d\mu, \sigma + d\sigma), r(\mu + d\mu', \sigma + d\sigma')$, a second-order expansion yields:

$$M(p, q, r) \approx \frac{1}{4}\left[g_{ij} d\theta^i d\theta^j + g_{ij} d\phi^i d\phi^j + \frac{1}{2} g_{ij}(d\theta^i - d\phi^i)(d\theta^j - d\phi^j)\right],$$

  confirming the local dominance of the Fisher–Rao geometry.

- ***Neutrosophic Membership Functions:***
  - ***Truth-Membership:*** *Measures similarity via JSD.*

$$\mathcal{T}(p, q, \gamma) = \exp\left(-\gamma \cdot JSD(p\|q)\right).$$

    For example, if $p = \mathcal{N}(0, 1), q = \mathcal{N}(0.1, 1.1)$, then $JSD(p\|q) \approx 0.0023$, so $\mathcal{T}(p, q, 10) \approx \exp(-0.023) \approx 0.977$.
  - ***Indeterminacy-Membership:*** *Captures entropy and divergence differences.*

$$\mathcal{I}(p, q, \gamma) = 1 - \left|\frac{H(p) - H(q)}{\max_{r\in\mathcal{P}} H(r)}\right| \cdot \exp\left(-\gamma \cdot |D_{KL}(p\|u) - D_{KL}(q\|u)|\right),$$

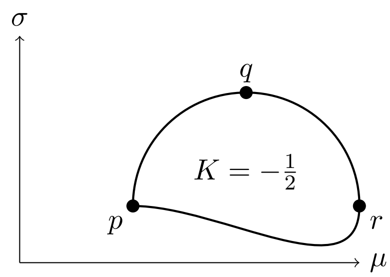
    where $H(p) = \frac{1}{2}\ln(2\pi e\sigma^2)$ and u is the uniform distribution over a sufficiently large interval.

  - ***Falsity-Membership:*** *Defined as* $\mathcal{F} = 1 - \mathcal{T} - \mathcal{I}$.

- ***Curvature and Contraction Constant:*** *The Gaussian manifold has constant negative sectional curvature* $K = -\frac{1}{2}$. *Applying Theorem 3:*

$$R \geq 1 + \frac{1}{4} \max \frac{|\mathcal{R}|}{\sqrt{g_{pp} g_{qq} g_{rr}}} \approx 1 + \frac{1}{4} \cdot \frac{1/2}{1} = 1.125.$$

This indicates a mild contraction requirement due to the hyperbolic geometry.



3.2. Categorical Distributions (Simplex Geometry)

Example 2 (Finite Discrete Distributions)**.** *Let* $\mathcal{P} = \{p = (p_1, \ldots, p_n) : p_i > 0, \sum p_i = 1\}$ *be the* $(n-1)$*-dimensional probability simplex.*

- ***Fisher–Rao Metric:*** *This is the spherical metric induced by the embedding* $p_i = x_i^2$ *with* $\sum x_i^2 = 1$. *The metric is:*

$$ds^2 = 4 \sum_{i=1}^{n} dx_i^2 = \sum_{i=1}^{n} \frac{dp_i^2}{p_i}.$$

The manifold is a portion of a sphere with radius 2, hence has constant positive curvature.

- ***Neutrosophic Structure:***

  - ***MR-Metric:*** *For categorical distributions,* D_{JS} *has a closed form. For example, for* $n = 3$, *let* $p = (0.5, 0.3, 0.2), q = (0.4, 0.4, 0.2), r = (0.6, 0.2, 0.2)$. *Then:*

$$M(p, q, r) = D_{JS}(p\|q) + D_{JS}(q\|r) + D_{JS}(r\|p) \approx 0.024 + 0.018 + 0.022 = 0.064.$$

  - ***Truth-Membership:*** $\mathcal{T}(p, q, \gamma) = \exp(-\gamma \cdot JSD(p\|q))$. *For* $\gamma = 10$, $\mathcal{T}(p, q, 10) \approx \exp(-10 \cdot 0.008) \approx 0.923$.
  - ***Indeterminacy-Membership:*** *Reflects entropy differences. For* p *and* q *above,* $H(p) \approx 1.029, H(q) \approx 1.055$, *so the entropy difference is small, leading to high indeterminacy if the distributions are otherwise distinct.*

- ***Contraction Constant:*** *For the positive-curvature simplex, the contraction constant R is larger. The curvature is $K = \frac{1}{4}$, so:*

$$R \geq 1 + \frac{1}{4} \cdot \frac{1/4}{1} = 1.0625.$$

However, the presence of boundaries (some $p_i \to 0$) increases the effective R in practice.

3.3. Exponential Family and α-Connections

Example 3 (Exponential Family)**.** *Consider an exponential family:*

$$p(x; \theta) = \exp\left(\theta \cdot T(x) - A(\theta) + \ln h(x)\right).$$

- ***Fisher–Rao Metric:*** $g_{ij}(\theta) = \partial_i \partial_j A(\theta)$.
- ***Neutrosophic α-Connection:*** *The α-connection coefficients are:*

$$\Gamma^{(\alpha)}_{ij,k}(\theta) = \frac{1-\alpha}{2} \partial_i \partial_j \partial_k A(\theta).$$

Our neutrosophic structure naturally incorporates this via the indeterminacy function $\mathcal{I}$, which can be linked to the deviation from the Levi-Civita connection ($\alpha = 0$). For instance, define a weighted indeterminacy:

$$\mathcal{I}^{(\alpha)}(p, q, \gamma) = |\alpha| \cdot (1 - \mathcal{T}(p, q, \gamma)) + (1 - |\alpha|) \cdot \mathcal{I}(p, q, \gamma).$$

This blends the "geometric uncertainty" (α-deviation) with the "information uncertainty" (entropy differences).

3.4. Application to Model Selection and Hypothesis Testing

Corollary 1 (Neutrosophic Bayesian Information Criterion (NBIC))**.** *In model selection, the standard BIC is* $\mathrm{BIC} = -2\ln L + k \ln n$*. We propose a neutrosophic adjustment:*

$$\mathrm{NBIC} = -2\ln L + k \ln n + \lambda \cdot (1 - \mathbb{E}[\mathcal{T}] - \mathbb{E}[\mathcal{I}]),$$

where $\mathbb{E}[\mathcal{T}]$ and $\mathbb{E}[\mathcal{I}]$ are average truth and indeterminacy memberships over the model's parameter space, and λ is a tuning parameter. This penalizes models with high epistemic uncertainty or low truth membership (poor fit).

Example 4 (Hypothesis Testing)**.** *Consider testing $H_0 : \theta = \theta_0$ vs. $H_1 : \theta \neq \theta_0$. The classical p-value can be enriched with neutrosophic memberships:*

- ***Truth-Membership ($\mathcal{T}$):*** *Likelihood of data under H_0.*
- ***Indeterminacy-Membership ($\mathcal{I}$):*** *Function of the Fisher information at θ_0; high indeterminacy suggests the test is less informative.*
- ***Falsity-Membership ($\mathcal{F}$):*** *Evidence against H_0.*

A decision rule could be: Reject H_0 if $\mathcal{F} > \tau_F$ and $\mathcal{I} < \tau_I$, where τ_F, τ_I are thresholds.

3.5. Geometric Machine Learning

[Uncertainty-Aware Deep Learning] In variational autoencoders (VAEs), the latent space is often a Gaussian manifold. Our neutrosophic structure can quantify uncertainty in the latent representations:

- Let z_1, z_2 be latent codes for two inputs.
- Define $\mathcal{T}(z_1, z_2, \gamma)$ based on their JSD in the data space (via decoder).
- Define $\mathcal{I}(z_1, z_2, \gamma)$ based on entropy of the latent distributions.
- The triplet $(\mathcal{T}, \mathcal{I}, \mathcal{F})$ provides a nuanced similarity measure for clustering or anomaly detection.

Example 5 (Neutrosophic t-SNE)**.** *Modify the t-SNE algorithm to use the neutrosophic MR-metric M instead of Euclidean distance. The joint probabilities become:*

$$P_{ij} = \frac{\exp\left(-M(p_i, p_j, p_{\mathrm{ref}})/2\sigma^2\right)}{\sum_{k \neq l} \exp\left(-M(p_k, p_l, p_{\mathrm{ref}})/2\sigma^2\right)},$$

where p_{ref} is a reference distribution. This incorporates three-way relationships and uncertainty into the visualization.

3.6. Physical and Quantum Applications

Remark 1 (Quantum Information Geometry)**.** *In quantum mechanics, states are density matrices ρ. The Jensen–Shannon divergence can be extended to quantum JSD [M. B. et al.]. Our neutrosophic framework then applies to the manifold of quantum states:*

- ***MR-Metric:*** $M(\rho, \sigma, \tau) = D_{JS}(\rho \| \sigma) + D_{JS}(\sigma \| \tau) + D_{JS}(\tau \| \rho)$.
- ***Truth-Membership:*** $\mathcal{T}(\rho, \sigma, \gamma) = \exp(-\gamma \cdot D_{JS}(\rho \| \sigma))$.
- ***Indeterminacy-Membership:*** *Can be linked to quantum entropy $S(\rho) = -\operatorname{tr}(\rho \ln \rho)$ and coherence measures.*
- ***Contraction Constant*** R***:*** *Related to the curvature of the Bures metric, which is the quantum analog of Fisher–Rao.*

This provides a novel tool for analyzing quantum phase transitions and decoherence.

4. Conclusions

This paper has introduced a comprehensive framework for neutrosophic statistical manifolds, bridging the gap between information geometry and neutrosophic logic. Our main contributions can be summarized as follows:

- We defined a novel neutrosophic MR-metric structure on statistical manifolds, incorporating truth ($\mathcal{T}$), indeterminacy ($\mathcal{I}$), and falsity ($\mathcal{F}$) membership functions to quantify distributional similarity, epistemic uncertainty, and dissimilarity.
- We proved that the triplet $(\mathcal{T}, \mathcal{I}, \mathcal{F})$ satisfies all axioms of a Neutrosophic MR-Metric Space, with particular attention to the corrected asymptotic behavior where $\lim_{\gamma\to\infty} \mathcal{T}(p, q, \gamma) = 0$ for $p \neq q$ and 1 for $p = q$.
- We established explicit relations between the contraction constant R and the curvature of the underlying statistical manifold, demonstrating how geometric properties influence the metric structure.
- We provided detailed applications across multiple domains including Gaussian and categorical statistical manifolds, hypothesis testing, model selection, geometric machine learning, and quantum information geometry.
- The proposed NBIC (Neutrosophic Bayesian Information Criterion) offers a novel approach to model selection that incorporates epistemic uncertainty quantification.

The neutrosophic statistical manifold framework provides a robust mathematical foundation for uncertainty-aware data analysis, with potential applications in machine learning, statistical inference, and quantum information theory. Future work will focus on empirical validation of the proposed methods and extensions to more complex statistical structures.

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