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Fixed Point Theorems for (β, φ)-Expansive Mappings in Controlled Metric Space

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Abstract. In the present manuscript, we have introduced a new notion of (β, φ)-expansive mappings in controlled metric space. In addition to this, some fixed point results are also proved with the help of this notion. Some results from the literature are also deduced from our main results. An example is also provided to prove the validity of our result. As an application an integral equation is also solved with the help of our main result.

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1. Introduction

In 1922, Banach [1] proved an interesting result in fixed point theory known as Banach contraction principle. This principle proved like a boom in the area of non-linear analysis and its applications.

In 1989, Bakhtin [2] presented b-metric space, an extension of metric space and numerous desirable fixed point solutions for expansive mappings were examined in b-metric space. Also in 1993, Czerwik [3] expanded on the findings of b-metric space as well. The idea of expansive mapping of Wang *et al.* [4] as follows "Let (X, d) be a metric space. A mapping $T : \Omega \to \Omega$ on $(\Omega, \bar{d})$ such that $\forall\ u, e \in\ \Omega : \bar{d}(Tu, Te) \geq d(u, e)$". After

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getting the motivation of this, Shahi *et al.* [5] proved fixed point theorems and gave some applications for (ξ, α)-expansive mappings in complete metric spaces.

Recently, many research studies were conducted on b-metric space under different expansive conditions.

After that, many authors used different types of contractive conditions including (α, ψ)-expansive mappings in different metric and metric like spaces (see [6–14]).

In 2017, Kamran *et al.* [8] introduced an extended generalization of the b-metric space and many results in the literature were generalized by their study. Mlaiki *et al.* [15] produced a controlled metric type space, which is an extension of the extended b-metric space.

2. Preliminaries

Definition 1 ([8])**.** *Let Ω be a non-empty set and define the mappings $\mu : \Omega \times \Omega \to [1, \infty)$ and $\bar{d} : \Omega \times \Omega \to [0, \infty)$ such that for all $u, e, f \in \Omega$*

[label=(i),itemsep=-.16em,topsep=2pt]

(i) $\bar{d}(u, e) = 0 \Longleftrightarrow u = e$,

(ii) $\bar{d}(u, e) = \bar{d}(e, u)$,

(iii) $\bar{d}(u, e) \leq \mu(u, e)[\bar{d}(u, f) + \bar{d}(f, e)]$.

Then the pair $(\Omega, \bar{d})$ is known as an extended b-metric space. Further, Mlaiki *et al.* [15] introduced the notion of controlled metric type space which is defined as follows:

Definition 2 ([15])**.** *On a non-empty set Ω, define the mappings $\mu : \Omega \times \Omega \to [1, \infty)$ and $\bar{d} : \Omega \times \Omega \to [0, \infty)$ such that for all $u, e, f \in \Omega$ the following condition holds*

[label=(s_2),itemsep=-.16em,topsep=2pt]

(i) $\bar{d}(u, e) = 0 \Longleftrightarrow u = e$,

(ii) $\bar{d}(u, e) = \bar{d}(e, u)$,

(iii) $\bar{d}(u, e) \leq \mu(u, f)\bar{d}(u, f) + \mu(f, e)\bar{d}(f, e)$.

Then the pair $(\Omega, \bar{d})$ is called a controlled metric type space.

To prove the above definition, we provide the following examples.

Example 1 ([15])**.** *Choose $\Omega = \{1, 2, 3, \dots\}$. Take $\bar{d} : \Omega \times \Omega \to [0, \infty)$ such that*

$$\bar{d}(u, e) = \begin{cases} 0 \Leftrightarrow u = e \\ \dfrac{1}{u}, & \text{if } u = 2j \text{ and } e = 2j + 1, \\ \dfrac{1}{e}, & \text{if } u = 2j + 1 \text{ and } e = 2j, \\ 1, & \text{otherwise.} \end{cases} \tag{1}$$

Consider $\mu : \Omega \times \Omega \to [1, \infty)$ *as*

$$\mu(u,e) = \begin{cases} u, & \text{if } u = 2j \text{ and } e = 2j+1, \\ e, & \text{if } u = 2j+1 \text{ and } e = 2j, \\ 1, & \text{otherwise.} \end{cases} \tag{2}$$

It is clear that the conditions (s_1) *and* (s_2) *are satisfied.*
Now we investigate (s_3)*.*

[*label=Case* I:,itemsep=-.16em,topsep=2pt,itemindent=-1.75em,align=left]

(i) If $f = u$ or $f = e$, (s_3) is satisfied.

(ii) If $f \neq u$ and $f \neq e$. (s_3) holds when $u = e$.

Now we may assume that $u \neq e$*. Then, we have* $u \neq e \neq f$*. It is clear that* (s_3) *holds in each of the following subcases:*

[*label*=(1),itemsep=-.16em,topsep=2pt]

(i) u, f are even and $e = 2j + 1$.

(ii) $u = 2j$ are and e, f are odd.

(iii) u, f are odd and $e = 2j$.

(iv) u, e, f are even.

(v) u, e are even and $f = 2j + 1$.

(vi) u, e are odd and $f = 2j$.

(vii) u, f are odd.

Thus $\bar{d}$ *is a controlled metric type space.*
Moreover for $j = 2, 3, \ldots$ *we have*

$$\bar{d}(2j+1, 4j+1) = 1 \ > \frac{1}{j}\mu(2j+1, 4j+1)[\bar{d}(2j+1, 2j) + \bar{d}(2j, 4j+1)]. \tag{3}$$

Therefore $\bar{d}$ *is not an extended b-metric space.*

Example 2 ([15])**.** *Take* $\Omega = \{0, 1, 2\}$*. Consider the function* $\bar{d}$ *given as*

$$\begin{aligned} &\bar{d}(0,0) = \bar{d}(1,1) = \bar{d}(2,2) = 0, \\ &\bar{d}(0,1) = \bar{d}(1,0), \\ &\bar{d}(0,2) = \bar{d}(2,0) = \frac{1}{2}, \\ &\bar{d}(1,2) = \bar{d}(2,1) = \frac{2}{5}. \end{aligned} \tag{4}$$

Define a symmetric function $\mu : \Omega \times \Omega \to [1, \infty)$ *such that*

$$\begin{aligned} &\mu(0,0) = \mu(1,1) = \mu(2,2) = \mu(0,2) = 1, \\ &\mu(1,2) = \frac{5}{4}, \\ &\mu(0,1) = \frac{11}{10}\,. \end{aligned} \tag{5}$$

One can easily verify that $\bar{d}$ *is a controlled metric type space.*

Since

$$\bar{d}(0,1) = 1 > \frac{99}{100}\mu(0,1)[\bar{d}(0,2) + \bar{d}(2,1)]\,. \tag{6}$$

$\bar{d}$ *is not an extended b-metric.*

The concept of Cauchy and convergent sequences in controlled metric type spaces are defined as follows:

Definition 3 ([15])**.** *Let* $(\Omega, \bar{d})$ *be a controlled metric type space and* $\{u_j\}_{j\geq 0}$ *be a sequence in* Ω.

[label=(iii),itemsep=-.16em,topsep=2pt]

(i) The sequence $\{u_j\}$ converges to some $u \in \Omega$ if for all $\varepsilon > 0$ $\exists$ $P = P(\varepsilon) \in \mathbb{N}$ such that $\bar{d}(u_j, u) < \varepsilon$ for all $j \geq P$, we write $\lim\limits_{j\to 0} u_j = u$.

(ii) We say that $\{u_j\}$ is Cauchy if for all $\varepsilon > 0$ $\exists$ $P = P(\varepsilon) \in \mathbb{N}$ such that $\bar{d}(u_i, u_j) < \varepsilon$ for all $j \geq P$.

(iii) If every Cauchy sequence is convergent then the space $(\Omega, \bar{d})$ is called complete.

Definition 4 ([15])**.** *Let* $(\Omega, \bar{d})$ *be a controlled metric type space. Let* $u \in \Omega$ *and* $\varepsilon > 0$, $C(u, \varepsilon)$ *is defined as*

[label=(iv),itemsep=-.16em,topsep=2pt]

(i) The open ball

$$C(u, \varepsilon) = \{e \in \Omega, \bar{d}(u, e) < \varepsilon\}. \tag{7}$$

(ii) A self mapping T on Ω is said to be continuous at $u \in \Omega$ if $\forall$ $\varepsilon > 0$ $\exists$ $\delta > 0$ such that $T(C(u, \delta)) \subseteq C(Tu,\ \varepsilon)$.

Remark 1. *If for all* u, e *in* Ω, $\mu(u, e) = t \geq 1$, *then it is a b-metric space. Therefore, we conclude that every b-metric space is controlled metric space. However the converse is not always true.*

Clearly, If a mapping T *is continuous at* u *in the controlled metric type space then* $u_j \to u \Rightarrow Tu_j \to u$ *as* $j \to \infty$

Definition 5 ([16])**.** *Let Φ denote the set of all functions, $\varphi : [0,\infty) \to [0,\infty)$ such that*

[label=(v),itemsep=-.16em,topsep=2pt]

(i) φ is non-decreasing.

(ii) For all $q > 0$ where φ^j is the jth iterate of φ.

Now, we recall the following lemma.

Lemma 1 ([5])**.** *If $\varphi \in \Phi$ then $\varphi(q) < q$ for all $q \in (0,\infty)$.*

Next, Mlaiki *et al.* [**?**] introduced the following notion.

Definition 6 ([15])**.** *Let Ω be a non-empty set and $\mu : \Omega \times \Omega \to [1,\infty)$ be a mapping. A function $\varphi : [0,\infty) \to [0,\infty)$ is said to be controlled comparison function if φ satisfies the following conditions*

[label=(vi),itemsep=-.16em,topsep=2pt]

(i) φ is non-decreasing.

(ii) $\sum\limits_{j=1}^{\infty} \varphi^j(q) \prod\limits_{m=1}^{j} \mu(u_m, u_i)\mu(u_j, u_{j+1}) < \infty$ and $\lim\limits_{j\to 0} \varphi^j(q)\mu(u_j, u_{j+1}) < \infty$ for any sequence $\{u_j\}$ in Ω for all $\varepsilon > 0$ and non-negative integer i, φ^i is the ith iterate of φ.

The set of all controlled comparison functions is denoted by Φ which is an extension of b-comparison function of Berinde.

Note that if $\varphi \in \Phi$ then we have $\sum_{j=1}^{\infty} \varphi^j < \infty$, since $\sum\limits_{j=1}^{\infty} \varphi^j \prod\limits_{m=1}^{j} \mu(u_m, u_i) \geq \varphi^j(q)$ $\forall$ $q \geq 0$. Hence by Lemma 1, we have $\varphi(q) < q$.

To show that the family Φ is a non-empty set. We present the following example.

Example 3 ([15])**.** *Consider the controlled b-metric space $(\Omega, \bar{d})$ which was defined in Example 2.*

Define the mapping [17] $\varphi : [0,\infty) \to [0,\infty)$ by $\varphi(q) = \left(\frac{rq}{2}\right)$ where $r < 1$.

Note that $\mu(u, e) \leq 2$. Then, we have

$$\varphi^j(q) \prod_{m=1}^{j} \mu(u_m, u_i)\mu(u_j, u_{j+1}) \leq \left(\frac{r^j q}{2^j}\right) \cdot 2^{j+1} = 2r^j q.$$

Therefore,

$$\sum_{j=1}^{\infty} \varphi^j(q) \prod_{m=1}^{j} \mu(u_m, u_i)\mu(u_j, u_{j+1}) \leq \sum_{j=1}^{\infty} 2r^j q < \infty\,.$$

Similarly, it is not difficult to see that $\lim\limits_{j\to 0} \varphi^j(q)\mu(u_j, u_{j+1}) < \infty$.

3. Main Result

In this section, we shall introduce a new notion of (β, φ)-expansive mapping in controlled metric space and proved some fixed point result by making use of this notion.

Definition 7 ([16]). *Let Φ denote all functions, $\varphi : [0, \infty) \to [0, \infty)$ which satisfy the following properties*

[label=(vii),itemsep=-.16em,topsep=2pt]

(i) φ is non-decreasing.

(ii) $\sum_{j=1}^{\infty} \varphi^j(a) < \infty$ for each $a > 0$ where φ^j is the jth iterate of φ.

Lemma 2 ([16]). *If $\varphi : [0, \infty) \to [0, \infty)$ is a non-decreasing function then for each $a > 0$,*

$$\lim_{j \to 0} \varphi^j(a) = 0$$
$$\Rightarrow \qquad \varphi(a) < a$$

Definition 8 ([5]). *Let $(\Omega, \bar{d})$ be a metric space and $T : \Omega \to \Omega$ be a given mapping. We say that T is an (β, φ)-expansive mapping if $\exists$ two functions $\varphi \in \Phi$ and $\beta : \Omega \times \Omega \to [0, \infty)$ such that*

$$\varphi(\bar{d}(Tu, Te)) \geq \beta(u, e)\bar{d}(u, e), \quad \forall\ u, e \in \Omega \tag{8}$$

Definition 9 ([5]). *Let $T : \Omega \to \Omega$ and $\beta : \Omega \times \Omega \to [0, \infty)$. We say T is said to be β admissible if $\forall\ u, e \in \Omega$ with $\beta(u, e) \geq 1$ then $\beta(Tu, Te) \geq 1$.*

Now we prove our first result.

Theorem 1. *Let $(\Omega, \bar{d})$ be a complete controlled metric type space and $T : \Omega \to \Omega$ be a bijective and (β, φ)-expansive mapping satisfying the following conditions:*

[label=(i),itemsep=-.16em,topsep=2pt]

(i) T^{-1} is β admissible,

(ii) $\exists\ u_0 \in \Omega$ such that $\beta(u_0, T^{-1}u_0) \geq 1$,

(iii) T is continuous.

Then T has a fixed point, i.e. $\exists\ u \in \Omega$ such that $Tu = u$.

Moreover, if for any two fixed points of T in Ω say a, b, we have $\beta(a, b) \geq 1$ then T has a unique fixed point in Ω.

Proof. Let u_0 be the point in Ω satisfying condition (ii) in our theorem.

Define a sequence $\{u_j\}$ in Ω by $Tu_j = u_{j-1}\ \forall\ j \in \mathbb{N}$.

First of all, note if $\exists\ j$ such that $u_j = u_{j+1}$, then we are done and u_j is the fixed point of T.

So we may assume that $u_j \neq u_{j+1}\ \forall\ j\ \geq 0$.

Also from the hypothesis of our theorem, we know that $\beta(u_0, T^{-1}u_0) = \beta(u_0, u_1) \geq 1$, using the result T^{-1} is β admissible. We can easily deduce that for all $j \geq 0$ $\beta(u_j, u_{j+1}) \geq 1$.

Now using the fact that T is (β, φ)-expansive mapping.

We deduce that

$$\begin{aligned}\bar{d}(u_j, u_{j+1}) &\leq \beta(u_j, u_{j+1})\bar{d}(u_j, u_{j+1})\\ &\leq \varphi(\bar{d}(Tu_j, Tu_{j+1}))\\ &= \varphi(\bar{d}(u_{j-1}, Tu_j)).\end{aligned}$$

By repeating the same process, we get

$$\bar{d}(u_j, u_{j+1}) \leq \varphi^j t(\bar{d}(u_0, u_1)), \quad \forall\ j \in \mathbb{N} \tag{9}$$

Hence for all $i, j \in \mathbb{N}$ with $i > j$, we have

$$\begin{aligned}\bar{d}(u_j, u_i) &\leq \mu(u_j, u_{j+1})\bar{d}(u_j, u_{j+1}) + \mu(u_{j+1}, u_i)\bar{d}(u_{j+1}, u_i)\\ &\leq \mu(u_j, u_{j+1})\bar{d}(u_j, u_{j+1}) + \mu(u_{j+1}, u_i)\mu(u_{j+1}, u_{j+2})\bar{d}(u_{j+1}, u_{j+2})\\ &\quad + \mu(u_{j+1}, u_i)\mu(u_{j+2}, u_i)\bar{d}(u_{j+2}, u_i)\\ &\leq \mu(u_j, u_{j+1})\bar{d}(u_j, u_{j+1}) + \mu(u_{j+1}, u_i)\mu(u_{j+1}, u_{j+2})\bar{d}(u_{j+1}, u_{j+2})\\ &\quad + \mu(u_{j+1}, u_i)\mu(u_{j+2}, u_i)\mu(u_{j+2}, u_{j+3})\bar{d}(u_{j+2}, u_{j+3})\\ &\quad + \mu(u_{j+1}, u_i)\mu(u_{j+2}, u_i)\mu(u_{j+3}, u_i)\bar{d}(u_{j+3}, u_i)\\ &\leq \mu(u_j, u_{j+1})\varphi^j\bar{d}(u_0, u_1) + \mu(u_{j+1}, u_{j+2})\mu(u_{j+1}, u_i)\varphi^{j+1}\bar{d}(u_0, u_1)\\ &\quad + \mu(u_{j+1}, u_i)\mu(u_{j+2}, u_i)\mu(u_{j+2}, u_{j+3})\varphi^{j+2}\bar{d}(u_0, u_1) + \dots\\ &\quad + \mu(u_{j+1}, u_i)\mu(u_{j+2}, u_i)\mu(u_{j+3}, u_i)\dots\mu(u_{i-2}, u_{i-1})\varphi^{i-1}\bar{d}(u_0, u_1)\\ &= \mu(u_j, u_{j+1})\varphi^j\bar{d}(u_0, u_1) + \sum_{n=j+1}^{i-2}\varphi^n\bar{d}(u_0, u_1)\prod_{m=j+1}^{n}\mu(u_m, u_i)\mu(u_n, u_{n+1})\\ &= \mu(u_j, u_{j+1})\varphi^j\bar{d}(u_0, u_1) + \sum_{n=1}^{i-2}\varphi^n\bar{d}(u_0, u_1)\prod_{m=1}^{n}\mu(u_m, u_i)\mu(u_n, u_{n+1})\\ &\quad - \sum_{n=1}^{j}\varphi^n\bar{d}(u_0, u_1)\prod_{m=1}^{n}\mu(u_m, u_i)\mu(u_n, u_{n+1})\\ &= S_{i-2} - S_j\end{aligned} \tag{10}$$

where

$$S_j = \sum_{n=1}^{j}\varphi^n\bar{d}(u_0, u_1)\prod_{m=1}^{n}\mu(u_m, u_i)\mu(u_n, u_{n+1}). \tag{11}$$

Since $\varphi \in \Phi$, we deduce that $\lim_{i,j\to\infty}[S_{i-2}, S_j] = 0$

And

$$\lim_{j\to\infty}[\mu_j, \mu_{j+1}]\varphi^j \bar{d}(u_0, u_1) < \infty\,. \tag{12}$$

Thus the sequence $\{u_j\}_{j\geq 0}$ is a Cauchy sequence in Ω and $(\Omega, \bar{d})$ being the complete controlled metric type space. The $\{u_j\}$ converges to some $u \in \Omega$.

Also note that

$$\begin{aligned}\bar{d}(u, T^{-1}) &\leq \mu(u, u_{j+1})\bar{d}(u, u_{j+1}) + \mu(u_{j+1}, T^{-1}u)\bar{d}(u_{j+1}, T^{-1}u)\\ &= \mu(u, u)\bar{d}(u, u) + \mu(u, T^{-1}u)\bar{d}(u, T^{-1}u)\,.\end{aligned} \tag{13}$$

Proceeding the limit as $j \to \infty$, we get $\bar{d}(u, u_{j+1}) = 0$.

And being T is continuous, we conclude that $\bar{d}(u, T^{-1}u) = 0$, i.e.,

$$\begin{aligned} & T^{-1}u = u\\ \Rightarrow\quad & T(T^{-1}u) = Tu\\ \Rightarrow\quad & (TT^{-1})u = Tu\\ \Rightarrow\quad & u = Tu.\end{aligned}$$

Thus u is a fixed point T.

Now suppose that T has two fixed points a, b such that $\beta(a, b) \geq 1$.

Now using the fact that T is an (β, φ)-expansive mapping and β-admisible, we obtain,

$$\begin{aligned}\bar{d}(a, b) &= \bar{d}(T^{-1}a, T^{-1}b)\\ &\leq \beta(a, b)\bar{d}(T^{-1}a, T^{-1}b)\\ &\leq \varphi\bar{d}(a, b)\\ &\vdots\\ &\leq \varphi^j \bar{d}(a, b)\,.\end{aligned} \tag{14}$$

Since $\varphi \in \Phi$, taking the limit in the above inequality, we deduce that $\bar{d}(a, b) = 0$ which implies that $a = b$.

Thus T has a unique fixed point.

In the next theorem, we shall replace the continuity of T by the following weaker condition.

If $\{u_j\}_{j=1}^{\infty}$ is a sequence in Z such that $\beta(u_j, u_{j+1}) \geq 1$ for all j and $u_j \to u$ as $j \to \infty$ then

$$\beta(u_j, u) \geq 1 \quad \forall\ j.$$

Now we present an example to prove the validity of our result.

Example 4. *Let* $(\Omega, \bar{d})$ *be the controlled metric type space on* $\Omega = \{0, 1, 2\}$.
Define the function $\beta : \Omega \times \Omega \to (-\infty, \infty)$ *such that*

$$\beta(u,e) = \begin{cases} 1, & \text{if } (u,e) = (1,1), \\ \frac{1}{20}, & \text{if } (u,e) \neq (1,1). \end{cases} \tag{15}$$

Define the self mapping T *on* Ω *and* $\Omega = \{0, 1, 2\}$ *by* $T(0) = 2$, $T(1)1$, $T(2) = 0$ *and the function* $\varphi(q) = \frac{1}{8}q$ *and* $\bar{d} : \Omega \times \Omega \to [0, \infty)$ *defined by*

$$\begin{aligned} &\bar{d}(0,0) = \bar{d}(1,1) = \bar{d}(2,2) = 0, \\ &\bar{d}(0,1) = \bar{d}(1,0), \\ &\bar{d}(0,2) = \bar{d}(2,0) = \frac{1}{2}, \\ &\bar{d}(1,2) = \bar{d}(2,1) = \frac{2}{5}. \end{aligned}$$

We want to verify that T *satisfies the conditions of Theorem 1.*
It is clear that T *is continuous for* $u_0 = 1$.
We have $\beta(1, T^{-1}(1)) = \beta(1, 1) = 1 \geq 1$.
So T *is* β*-admissible.*
Now we verify that T *is* (β, φ)*-expansive mapping.*
Note that $\beta(u, e) = \beta(e, u)$

[*label*=(iv),itemsep=-.16em,topsep=2pt]

(i) $\beta(u,u)\bar{d}(Tu,Tu) = 0 \leq \varphi(\bar{d}(u,u)) = \varphi(0) = 0 \ \forall\ u \in \Omega$,

(ii) $\beta(1,0)\bar{d}(T1,T0) = \frac{1}{20}(\bar{d}(1,2)) = \frac{1}{50} \leq \varphi\bar{d}(1,2) = \varphi\left(\frac{2}{5}\right) = \frac{1}{20}$.

(iii) $\beta(1,2)\bar{d}(T1,T2) = \frac{1}{20}(\bar{d}(1,0)) = \frac{1}{20} \leq \varphi\bar{d}(1,0) = \varphi(1) = \frac{1}{8}$.

(iv) $\beta(2,0)\bar{d}(T2,T0) = \frac{1}{20}(\bar{d}(0,2)) = \frac{1}{40} \leq \varphi\bar{d}(0,2) = \varphi\left(\frac{1}{2}\right) = \frac{1}{16}$.

Therefore, T *satisfies the conditions in Theorem 1 and hence it has a unique fixed point* $u = 1$.

Theorem 2. *Let* $(\Omega, \bar{d})$ *be a complete Controlled metric type space and be a* $T : \Omega \to \Omega$ *be a bijective and* (β, φ)*-expansive mapping for some* $\varphi \in \Phi$. *Suppose that the following condition holds:*

[*label*=(ii),itemsep=-.16em,topsep=2pt]

(i) T^{-1} is β-admissible,

(ii) $\exists\ u_0 \in \Omega$ such that $\beta(u_0, T^{-1}u_0) \geq 1$,

(iii) If $\{u_j\}_{j=1}^{\infty}$ is a sequence in Ω such that $\beta(u_j, u_{j+1}) \geq 1$ for all j and $\{u_j\} \to u$ as $j \to \infty$ then $\beta(T^{-1}u_j, T^{-1}u) \geq 1\ \forall\ j$.

Then T has a fixed point.

Proof. In proving the result, we follow the same steps as in the proof of Theorem 1 to construct a sequence $\{u_j\}_{j=1}^{\infty}$ that converges to a point u, $u \in \Omega$. The constructed sequence has the property $\beta(u_j, u_{j+1}) \geq 1$, for all natural numbers j. The last assumption of the result implies that $\beta(T^{-1}u_j, T^{-1}u) \geq 1$.

Now, we will prove that u is a fixed point of T.

The triangle inequality implies that

$$\begin{aligned}\bar{d}(T^{-1}u, u) &\leq \mu(T^{-1}u, u_j)\bar{d}(T^{-1}u, u_j) + \mu(u_j, u)\bar{d}(u_j, u)\\ &= \mu(T^{-1}u, u_j)\bar{d}(T^{-1}u, u_j) + \mu(u, u)\bar{d}(u, u)\\ &= \mu(T^{-1}u, u_j)\bar{d}(T^{-1}u, u_j) + 0.\end{aligned}$$

Thus, we get

$$\begin{aligned}\bar{d}(T^{-1}u, u) &\leq \mu(T^{-1}u, u_j)\bar{d}(T^{-1}u, u_j)\\ &\leq \mu(T^{-1}u, u_j)\bar{d}(T^{-1}u, u_j)\beta(T^{-1}u, u_j)\\ &\leq \mu(T^{-1}u, u_j)\varphi(\bar{d}(u, Tu_j))\\ &\leq \mu(T^{-1}u, u)\varphi(\bar{d}(u, Tu)). \qquad (16)\end{aligned}$$

Continuity of φ at $u = 0$ implies that $\bar{d}(T^{-1}u, u) = 0$ as $j \to +\infty$ i.e. $T^{-1}u = u$.

Consider $Tu = T(T^{-1}u) = (TT^{-1})u = u$.

Hence the result is proved.

Example 5. *Let $\Omega = [0, \infty)$ with controlled metric space $\bar{d} : \Omega \times \Omega \to [0, \infty)$ defined by $\bar{d} = |u - e|$ $\forall$ $u, e \in \Omega$ along with the mapping $\mu : \Omega \times \Omega \to [1, \infty)$ and define $T : \Omega \to \Omega$ is (β, φ)-expansive mapping with $\varphi \in \Phi$ by*

$$T(u) = \begin{cases} u^3, & u \geq \frac{1}{2}, \\ \frac{u}{2}, & 0 \leq u < \frac{1}{2} \end{cases}$$

and

$$\beta(u, e) = \begin{cases} 0, & u, e \geq \left[0, \frac{1}{2}\right), \\ 1, & \textit{otherwise.} \end{cases}$$

Clearly T is not continuous at $\frac{1}{2}$ and T is (β, φ)-expansive mapping with $\varphi(r) = \frac{r}{2}$ $\forall$ $r \geq 0$.

Moreover for all $u, e \in \Omega$, we have

$$\frac{1}{2}(\bar{d}(Tu, Te)) \geq \beta(u, e)\bar{d}(u, e).$$

Also there exist $u_0 \in \Omega$ such that $\beta(u_0, T^{-1}u_0) \geq 1$. Infact for $u_0 = 1$, we have $\beta(1, T^{-1}1) = 1$.

Now, let $u, e \in \Omega$ such that $\beta(u, e) \geq 1$. This implies that $u \geq 1$, $e \geq 1$.

And by definition of T^{-1} and β, we have

$$T^{-1}u = u^{\frac{1}{3}} \geq 1, \quad T^{-1}e = e^{\frac{1}{3}} \geq 1 \quad \textit{and} \quad \beta(T^{-1}u, T^{-1}) = 1$$

i.e. T^{-1} is β-admissible.

Finally, let $\{u_j\}$ be a sequence in Ω such that $\beta(u_j, u_{j+1}) \geq 1 \; \forall \; j$ and $\{u_j\} \to u \in \Omega$ as $j \to \infty$.

Since $\beta(u_j, u_{j+1}) \geq 1 \; \forall \; j$ by the definition of β, we have $u_j \geq 1 \; \forall \; j$ and $u \geq 1$ then $\beta(T^{-1} \; u_j, T^{-1}u) = 1$.

Therefore, all the conditions of Theorem 2 are satisfied. So that T has the fixed points. Here $0, 1$ are two fixed points of T.

To ensure the uniqueness of the fixed point in Theorem 2, we consider the following condition:

(N): For all $g, e \in \Omega$, there exist $w \in \Omega$ such that $\beta(g, w) \geq 1$ and $\beta(e, w) \geq 1$.

Theorem 3. *Adding the hypothesis of Theorem 2 to the condition (N), we yield the uniqueness of the fixed point of T.*

Proof. In this theorem, we will the prove of uniqueness of fixed points, i.e.

$$T(g) = g, \quad T(e) = e, \quad \forall \; g, e \in \Omega.$$

From the condition (N) there exist $w \in \Omega$ such that $\beta(g, w) \geq 1$ and $\beta(e, w) \geq 1$.

Using the β-admissible property of T^{-1}, we get

$$\beta(g, T^{-1}w) \geq 1 \quad \text{and} \quad \beta(e, T^{-1}w) \geq 1.$$

Therefore by repeatedly using β-admissible property of T^{-1}, we get

$$\beta(g, T^{-j}w) \geq 1 \text{ and } \beta(e, T^{-j}w) \geq 1 \quad \forall \; j \in \mathbb{N}. \tag{17}$$

Using the inequality of equations (8) and (17), we get

$$\begin{aligned}
(\bar{d}(g, T^{-j}w)) &\leq \beta(g, T^{-j}w)\bar{d}(g, T^{-j}w) \\
&\leq \varphi(\bar{d}(Tg, T^{-j+1}w)) \\
&= \varphi(\bar{d}(g, T^{-j+1}w)) \\
&\leq \varphi(\bar{d}(g, T^{-j+1}w)).
\end{aligned}$$

By the repetition of above inequality, we get

$$(\bar{d}(g, T^{-j}w)) \leq \varphi(\bar{d}(g, w)) \quad \forall \; j \in \mathbb{N}.$$

Thus we have,

$$T^{-j}w \to g \quad \text{as } j \to \infty.$$

Similarly we may obtain,

$$T^{-j}w \to e \quad \text{as } j \to \infty.$$

From the uniqueness of the limit of $T^{-j}w$, we get

$$g = e.$$

Hence the proof.

4. Consequences

Now we present the following as an immediate consequence of Theorem 1.

Corollary 1. *Let $(\Omega, \bar{d})$ be a complete controlled metric type spaces and $T : \Omega \to \Omega$ be a bijective and $\beta : \Omega \times \Omega \to [0, \infty)$ mappings satisfying the following conditions:*

[label=(i),itemsep=-.16em,topsep=2pt]

(i) T is continuous.

(ii) $\exists\ \varphi \in \Phi$ such that $\varphi(\bar{d}(Tu, Te)) \geq \beta(u, e)\bar{d}(u, e)\ \forall\ u, e \in \Omega$.

Then T has a unique fixed point.

Proof. Define the function $\beta : \Omega \times \Omega \to [0, \infty)$ by $\beta(u, e) = 1$.
Note that T^{-1} is β-admissible. Moreover, T satisfies all the conditions of Theorem 1, so T has a unique fixed point.

5. Application

Consider the non-linear Volterra Integral Equation

$$u(q) = \int_0^q \frac{u(l)}{1 + |u(l)|} dl, \quad q \in [0, 2].$$

Let $\Omega = C([0, 2], \mathbb{R})$ space of continuous real-valued functions on $[0, 2]$ and

$$\bar{d}(u, e) = \sup |u(q), e(q)|, \quad \forall\ q \in [0, 2].$$

Define $T : \Omega \to \Omega$ by

$$(Tu)q = \int_0^q \frac{u(l)}{1 + |u(l)|} dl, \quad q \in [0, 2].$$

Define $\beta : \Omega \times \Omega \to [0, \infty)$ by $\beta(u, e) = \frac{1}{4}$ and $\varphi : [0, \infty) \to [0, \infty)$ by $\varphi(q) = \frac{q}{4}$.
Define a symmetric function $\mu : \Omega \times \Omega \to [1, \infty)$ by $\mu(u, e) = 2\ \forall\ u, e \in [0, 2]$.
Clearly $\varphi(q)$ is an increasing and continuous function of q and $\varphi(0) = 0$ and $\varphi(q) < q$ $\forall\ q > 0$.

Now

$$
\begin{aligned}
\bar{d}(u,e) &= \sup|u(q)-e(q)| \qquad \forall\ q\in[0,2]\\
&= \sup\left|\left(\int_0^q \frac{u(l)}{1+|u(l)|}dl \int_0^q \frac{e(l)}{1+|e(l)|}dl\right)\right|\\
&= \sup|(Tu)(q)(Te)(q)|\\
&\leq |(Tu)(q)(Te)(q)|\\
&= \bar{d}(Tu,Te)\\
\Rightarrow \qquad \bar{d}(u,e) &\leq \bar{d}(Tu,Te).
\end{aligned}
$$

Now

$$
\begin{aligned}
\varphi(\bar{d}(Tu,Te)) &= \frac{\bar{d}(Tu,Te)}{4}\\
&\geq \frac{\bar{d}(u,e)}{4}\\
&= \beta(u,e)\ \bar{d}(u,e)\\
\Rightarrow \quad \varphi(\bar{d}(Tu,Te)) &\geq \beta(u,e)\bar{d}(u,e).
\end{aligned}
$$

Thus $T:\Omega\to\Omega$ is (β,φ)-expansive mapping. Thus T has atleast one fixed point. Clearly $T(0)=0$ and hence 0 is the unique fixed point of T.

6. Conclusion

Fixed point results in the setting of controlled metric space using (β,φ)-expansive mappings have been established in this manuscript. The derived results have been supplemented with suitable non trivial example and the result is applied to find analytical solution of integral equation. It will be open problem to extend the result in the generalised forms of controlled metric and metric like spaces and find application to integro-differential equations.

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References

[1] Banach S. Sur les op´erations dans les ensembles abstraits et leurs applications aux equations integrales. *Fund Math,*, 3:133–181, 1922.

[2] Bakhtin I A. The contraction mapping principle in almost metric spaces. *Funct Anal,*, 30:26–37, 1989.

[3] Czerwik S. Contraction mappings in b-metric spaces,. *Acta Math. Inform. Univ. Ostrav.,*, 30:5–11, 1993.

[4] Gao Z M. Wang S Z., Li B Y. and Iseki K. Some fixed point theorems on expansion mappings,. *Math. Japon,*, 29:631–636, 1984.

[5] Kaur J. Shahi, P. and S.S. Bhatia. Fixed point theorems for (α, ψ)-expansive mappings in complete metric spaces,. *Fixed Point Theory and Applications,*, 2012:157, 2012.

[6] Bhaskar T G and Lakshmikantham V. Fixed point theory in partially ordered metric spaces and applications,. *Non Linear Anal,*, 65:1379–1393, 2006.

[7] Mani G. Ege O. Gnanaprakasam AJ., Prakasam S K. Fixed point results via OF contraction and applications to Fredholm and integro-differential equations,. *Filomat,*, 38(29), 2024.

[8] Samreen M. Kamran T. and UL Ain Q. A generalization of b-metric space and some fixed point theorems,. *Mathematics,*, 5(2):19, 2017.

[9] Petrusel A. Karapinar E. and Petrusel G. On admissible hybrid Geraghty contractions,. *Carpathian Journal of Mathematics,*, 36(3):433–442, 2020.

[10] Jleli M and Samet B. A new generalization of the Banach contraction principle,. *Journal of Inequalities and Applications,*, 2014:38, 2014.

[11] Rohen Y. Souayah N. Mlaiki N., Mukheimer A. and Abdeljawad T. Fixed point theorems for α-ψ-contractive mapping in Sb-metric spaces,. *Journal of Mathematical Analysis,*, 8(5):40–46, 2017.

[12] Souayah N. and Mlaiki A. Some fixed point theorem in Sb metric spaces,. *J. Math. Computer Sci,*, 16:131–139, 2016.

[13] Abodayeh K. Shatanawi W. and Mukheimer A. Some fixed point theorems in extended b-metric spaces,. *UPB Scientific Bulletin.,*, 80, 2018.

[14] Gao Z M. Wang S Z., Li B Y. and Iseki K. On elliptic valued b-metric spaces and some new fixed point results with an application,. *AIMS Mathematics,*, 9(7), 2024.

[15] Souayah N. Mlaiki N., Aydi H. and Abdeljawad T. Controlled metric type spaces and the related contraction principle,. *Mathematics,*, 6(10):194, 2018.

[16] Vetro C. Samet B. and Vetro P. Fixed point theorems for α-ψ-contractive type mappings,. *Nonlinear Analysis: Theory, Methods Applications*, 75(4):2154–2165, 2012.

[17] Abodayeh K Mlaiki N Abuloha M, Rizk D and Abdeljawad. New results in controlled metric type spaces. *Journal of Mathematics*, 1:5575512, 2021.