EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 4, Article Number 7126
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Statistical Inference of Accelerated Ishita Model Based on Type-I Generalized Hybrid Censoring Data with Applications

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Abstract. In this paper, we adopt the Ishita lifetime distribution to analyze biomedical science and engineering lifetime data under an accelerated life test (ALT) model. This data is exposed concerning the mechanism of a type-I generalized hybrid censoring scheme under a partially step-stress ALT model. The model parameters and the parameters of life (survival and hazard rate function) are estimated using maximum likelihood and Bayesian estimation. Also, the interval estimators are formulated with respect to the normal distribution of the maximum likelihood estimate, two parametric bootstrap confidence techniques, and Bayesian credible intervals. Two real data sets are analyzed to illustrate the proposed methods. Monte Carlo simulation is used to compare various methods.

2020 Mathematics Subject Classifications: 62F10, 62F15, 62F40

Key Words and Phrases: Bayesian estimation, bootstrap confidence interval, classical estimation, generalized hybrid censoring scheme, Ishita distribution, MCMC

1. Introduction

In real-life applications such as engineering, medicine, insurance, and finance, the problem of modeling and analyzing real-life data is crucial. Exponential and Lindley distributions are the most important one-parameter lifetime distributions popular for modeling biomedical science and engineering lifetime data, see Lindley [1]. Shanker et al. [2] observed that exponential and Lindley distributions are not suitable for many lifetime data due to their nature of hazard rate functions, their shapes, and mean residual life. For searching for a better fit lifetime distribution than exponential and Lindley, Shanker [2] presented the Akash lifetime distribution. Shankar and Shukla [3] proposed a one-parameter model as a mixture of exponential(β) and gamma (3, β) distributions with mixing proportion $\frac{\beta^3}{\beta^3+2}$. This model is known as the Ishita distribution (ID) and is

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DOI: https://doi.org/10.29020/nybg.ejpam.v18i4.7126

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used for modeling the lifetime data in biomedical sciences and engineering. For modeling lifetime data in reliability and in terms of its hazard rate shapes, this distribution is flexible than the exponential, Lindley, and Akash distributions. The ID has the advantage of increasing and decreasing the hazard rate function, making it more flexible than the exponential, Akash, and Lindley distributions. Also, ID is considered a model of the exponential family. Different author developed new distributions related to ID, such as the Power Ishita distribution by Shukla and Shankar [4], the truncated lifetime test for an Ishita distribution by Al-Nasser et al. [5] Poisson Ishita distribution by Anwar et al. [6], and new size-biased Ishita distribution by Al-Omari et al. [7].

The random variable X is called Ishita's random variable if its probability density function (PDF) is given by

$$f(x) = \frac{\beta^3}{\beta^3 + 2}(\beta + x^2)\exp(-\beta x),\ x > 0,\ \beta > 0, \tag{1}$$

where β is scale parameter. The corresponding cumulative distribution function (CDF) and hazard failure rate function ($h(.)$) are given by

$$F(x) = 1 - \left(1 + \frac{\beta x(\beta x + 2)}{\beta^3 + 2}\right)\exp(-\beta x), \tag{2}$$

and

$$h(t) = \frac{\beta^4 + \beta^3 t^2}{\beta^2 t^2 + 2\beta t + \beta^3 + 2}. \tag{3}$$

For a different choice of the parameter β the graph representation of The PDF and CDF of the Ishita distribution are shown in Figures 1 and 2.

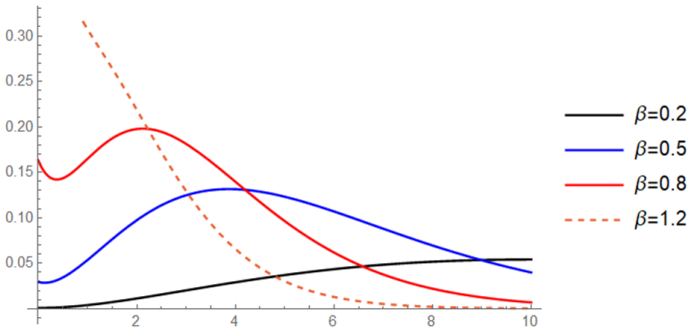

Figure 1: The graph of PDF of Ishita distribution.

Commonly, in reliability analysis or medical study, the lifetime data is collected for some, but not all, population units under test, which is known as a censoring scheme (CS). Type-I and type-II censoring schemes (CSs) are commonly simple CSs. The test time in type-I CS is constant prior to, but a random number of data points. In type-II CS, constant prior number data and a random test time are used. When both the total test time τ and the number of data points m are considered, the hybrid censoring

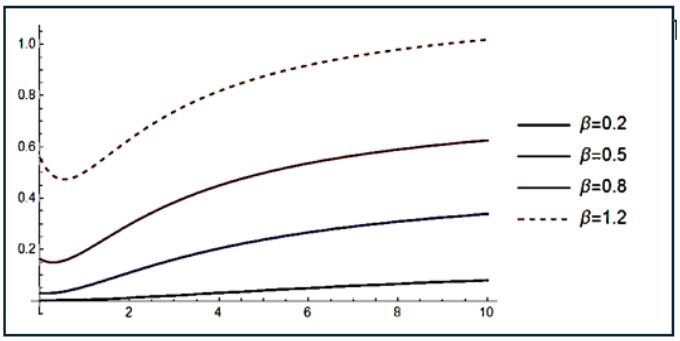

Figure 2: The graph of $h(.)$ of Ishita distribution.

scheme (HCS) was considered. The test is terminated at $\min(\tau, T_m)$ at type-I HCS and terminated at $\max(\tau, T_m)$ at type-II HCS, where T_m is $m-$th failure time and τ is the edial test time. For more information about type-I HCS see, Gupta and Kundu [8] and Kundu and Pradhan [9] and Childs et al. [10] and [11] for type-II HCS. All of these types of censoring schemes have the lack of memory that a small data size may be zero in type-I CS and type-I HCS. And, larger terminated test time in type-II CS and type-II HCS may be infinity. So that, authors can be avoid these schemes failure by considering the generalized hybrid censoring schemes (GHCSs), see [12]. In this paper, we adopted tyep-I GHCSs.

Designing the experiment according to type-I GHCS provides us with the guarantee of saving the minimum number of failures needed for statistical inference. Therefore, Let a random sample of size n is selected from the population units to run under type-I GHCS. The minimum and ideal numbers of failure are prior propose to be k and m. Also, the ideal test time is denoted by τ. When the experiment is running the failure time is recorded until the k-th failure time T_k is observed. If, the failure time T_k larger than the time τ then, the test terimenated at T_k. But, if the failure time T_k is smaller than τ the test terimenated at min (τ, T_m), see [13] and [14]. The schematic diagram of type-I GHCS is presented in Figur3. Suppose that, the observed data under type-I GHCS is denoted by $\mathbf{t} = (t_1 < t_2 < ... < t_r)$.The integer number $r = k$ for $\tau \leq t_k$, $r = m$ for $t_m \leq \tau$ and $k < r < m$ if $t_k < \tau < t_m$.

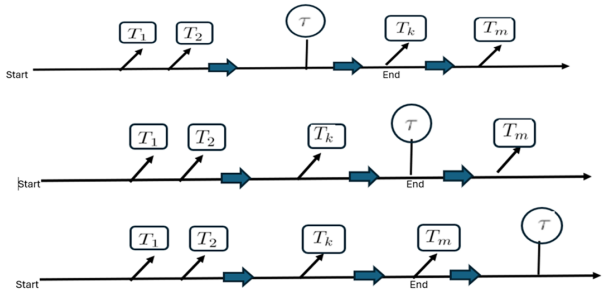

Figure 3: The schematic diagram of type-I GHCS.

The problem of obtaining of enough failure time data under normal conditions is more difficult, in manufacturing industries. Therefore, the ALTs presented more suitable method to solve this problem in a short period of time. According to [15], different type of ALTs are defined in literature. The first type of ALTs was called constant-stress ALT. In this type of ALTs, we kept the stress throughout the test at a constant level. Several authors considered constant-stress ALT, see [16], [17] and [18]. If the stress level is changed under fixed time or number of failure then, step-stress ALT is defined, see [19] and [20]. The third type of ALTs is called progressive-stress ALT, the stress in this type is continual increasing through the test, see [21]. In several cases of ALTs, some units were tested under use conditions, and other units were tested under stress conditions, which is known by partially ALTs model, see [22]. Partially ALTs model defined as partially constant-stress ALT model and partially step-stress ALT model, see Almalki et al [23] and Almarashi and Abd-Elmougod [24]. In the partially constant-stress ALT model, some units are tested under normal condition and other units are tested under accelerated conditions. But, in partially step-stress ALT model, all units are tested under normal conditions until fixed prior time or number is observed, and hence are tested under accelerated conditions. Statistical inference of ID under partially step-stress ALT is developed when data is obtained with respect to type-I generalized HSC. The estimation results of the model parameters formulated with the resected MLE, bootstrap confidence interval, and Bayes method are constructed.

The paper is described as follows: In Section 2, the essential assumptions and model formulation are presented. In Section 3, formulate the theoretical results of point estimation. In Section 4, formulate the theoretical results of interval estimation. The numerical study in the form of an analysis of two real data sets and a simulation study are formulated in Section 5. Finally, we conclude with some comments in Section 6.

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2. Assumptions and Model Formulation

Table 1: List of Abbreviations.

ID	Ishita distribution.	CDF	Cumulative distribution function.
PDF	Probability density function	ACI	Approximate confidence interval
CDF	Cumulative distribution function.	GHCS	Generalized hybrid censoring scheme
$h(.)$	Hazrd fialure rate function.	BTCI	Bootstrap-t confidence interval.
MH	Metropolis–Hastings.	CI	Credible intervals
ALT	Accelerated life tests	ME	Mean estimate
CI	Credible intervals	MSE	Mean squared error.
MIL	Mean interval length.	SEL	Square error loss.
PC	Probability coverage.	PBCI	Percentile bootstrap confidence interval.

Let a random sample of size n be selected from the population units to put under a life testing experiment. Also, suppose each of the lower and upper integer numbers of failures needed for statistical inference are prior proposed to be k and m. Without

loss of generality, the stress time change η is selected to be smaller than the ideal test time τ. With respect to partially step-stress ALTs, firstly, run the experiment under normal stress conditions with all units until the time η is observed. Then, it runs under stress conditions. If the k-th failure time $X_k < \tau$, we terminated the test at $\min(\tau, T_m)$ otherwise the test terminated at T_k. Therefore, the data with respect to type-I GHCS can be described as $\mathbf{X} = (X_1 < X_2 < ... < X_J < \eta < X_{J+1} < ... < X_r)$, where, $k \leq r \leq m$. The integer number J is observed under normal conditions, but $r - J$ under accelerated conditions conditions. The objective of changing to a higher stress level is to shorten the test time. If we denote the total test time by T, which passes through two stages, normal and accelerated conditions. Then, T with respect to partially step-stress the ALT model is described as

$$T = \begin{cases} X, & X < \eta \\ \eta + (\frac{X-\eta}{\lambda}), & X > \eta, \end{cases}, \tag{4}$$

where, η is the stress change time, the acceleration factor is defined by λ and X is the lifetime at normal conditions. When, the lifetime of unit has a PDF defined by (1) with parameter β, the PDF of total lifetime T can be defined by

$$f(t) = \begin{cases} f_1(t), & 0 < t \leq \eta \\ f_2(t), & t > \eta, \\ 0, & O.W. \end{cases}, \tag{5}$$

where $f_2(t)$ can be obtained from $f_1(t)$ in (1) by using transformation technique with respect to transformation (4) as

$$f_2(t) = \frac{\beta^3 \lambda}{\beta^3 + 2}(\beta + (\eta + \lambda(t - \eta))^2) \exp(-\beta(\eta + \lambda(t - \eta))),\ t > 0,\ \beta > 0, \lambda \geq 1, \tag{6}$$

The CDF, $F(.)$ under accelerated conditions are given by

$$F_2(t) = \left\{1 - \left(1 + \frac{\beta(\eta + \lambda(t - \eta))(\beta(\eta + \lambda(t - \eta)) + 2)}{\beta^3 + 2}\right) \exp(-\beta(\eta + \lambda(t - \eta)))\right., \tag{7}$$

Remarks 1:

(i) When $r = k = J$ the accelerated type-I GHCS reduce to normal stress case only.

(ii) When $r > J$ the observed accelerated type-I GHC data $\mathbf{t} = (t_1 < t_2 < ... < t_J < \tau < t_{J+1} < ... < t_r)$ and the joint likelihood function is given by

$$L(\mathbf{t}|\lambda, \beta) = \frac{n!}{(n-r)!} [1 - F_2(t_r)]^{n-r} \left(\prod_{i=1}^{J} f_1(t_i)\right) \left(\prod_{i=J+1}^{r} f_2(t_i)\right), \tag{8}$$

3. Point Estimation

In this section, for the given accelerated type-I GHC data the point ML estimation of the model parameters is discussed. Additionally, we discuss Bayes point estimation with gamma prior information for the ID parameter and non-informative prior information for the accelerated factor.

3.1. Point ML estimation

From (1) and (4), the joint likelihood function (8) for observed accelerated type-I GHC data $\mathbf{t}$ =(t_1, t_2, ..., t_J, t_{J+1}, ..., t_r) is given by

$$\begin{aligned} L(\lambda,\beta|\mathbf{t}) \propto & \left(\frac{\beta^3}{\beta^3+2}\right)^r \lambda^{r-J}\left(\prod_{i=1}^{J}(\beta+t_i^2)\right)\left(\prod_{i=J+1}^{r}(\beta+(\eta+\lambda(t_i-\eta))^2)\right) \\ \times & \left(1+\frac{\beta(\eta+\lambda(t_r-\eta))(\beta(\eta+\lambda(t_r-\eta))+2)}{\beta^3+2}\right)^{(n-r)} \\ \times & \exp\left\{-\beta\sum_{i=1}^{J}t_i-\beta\sum_{i=J+1}^{r}(\eta+\lambda(t_i-\eta)-\beta(n-r)(\eta+\lambda(t_r-\eta))\right\}. \end{aligned} \tag{9}$$

Taken the natural logarithm of the joint function (9) as

$$\begin{aligned} \ell(\lambda,\beta|\mathbf{t}) = & r\log\left[\frac{\beta^3}{\beta^3+2}\right]+(r-J)\log\lambda+\sum_{i=J+1}^{r}\log\left[\beta+(\eta+\lambda(t_i-\eta))^2\right] \\ & +(n-r)\left(\log\left[\beta^3+2+\beta(\eta+\lambda(t_r-\eta))(\beta(\eta+\lambda(t_r-\eta))+2)\right]-\log\left[\beta^3+2\right]\right) \\ & +\sum_{i=1}^{J}\log\left[\beta+t_i^2\right]-\beta\sum_{i=1}^{J}t_i-\beta\sum_{i=J+1}^{r}(\eta+\lambda(t_i-\eta)-\beta(n-r)(\eta+\lambda(t_i-\eta)). \end{aligned} \tag{10}$$

For simpilicty the log-likelihood function experssed as

$$\begin{aligned} \ell(\lambda,\beta|\mathbf{t}) = & r\log\left[\frac{\beta^3}{\beta^3+2}\right]+(r-J)\log\lambda+\sum_{i=1}^{J}\log\left[\beta+t_i^2\right]+\sum_{i=J+1}^{r}\log\left[\beta+z_i^2\right]-\beta\sum_{i=1}^{J}t_i \\ & -\beta\sum_{i=J+1}^{r}z_i-\beta(n-r)z_r+(n-r)\left(\log\left[\beta^3+2+\beta z_r(\beta z_r+2)\right]-\log\left[\beta^3+2\right]\right), \end{aligned} \tag{11}$$

where, $z_i=\eta+\lambda(t_i-\eta)$. The likelihood equations is obtained from (11) by taken the first partially derivatives respect to β and λ as follows

$$\begin{aligned} \frac{\partial\ell(\lambda,\beta|\mathbf{t})}{\partial\beta} = & \frac{6r}{\beta(\beta^3+2)}+\sum_{i=1}^{J}\frac{1}{\beta+t_i^2}+\log\sum_{i=J+1}^{r}\frac{1}{\beta+z_i^2}-\sum_{i=1}^{J}t_i-\sum_{i=J+1}^{r}z_i-(n-r)z_r \\ & +(n-r)\left(\frac{3\beta^2+z_r(\beta z_r+2)+\beta^2z_r^2}{\beta^3+2+\beta z_r(\beta z_r+2)}-\frac{3\beta^2}{\beta^3+2}\right)=0, \end{aligned} \tag{12}$$

and

$$\frac{\partial\ell(\lambda,\beta|\mathbf{t})}{\partial\lambda}=\frac{r-J}{\lambda}+\sum_{i=J+1}^{r}\frac{2z_i(t_i-\eta)}{\beta+z_i}-\beta\sum_{i=J+1}^{r}(t_i-\eta)-\beta(n-r)(t_r-\eta)$$

$$+ (n-r)\left(\frac{\beta(t_r-\eta)(\beta z_r+2)+\beta^2(t_r-\eta)z_r}{\beta^3+2+\beta z_r(\beta z_r+2)}\right)=0, \tag{13}$$

Equations (12) and (13) have shown that, the ML estimate of the parameters λ and β is defined in the form of two non-linear equations. Therefore, the estimated $\hat{\lambda}$ and $\hat{\beta}$ can be obtained by applied any iteration method such as Newton Raphson iteration.
Remarks (2):

(i) The ML estimates of the parameter of life (Reliability and hazard rate function given by

$$\hat{S}(t)=1-\hat{F}(t)|_{\beta=\hat{\beta}} \tag{14}$$

$$\hat{h}(t)=h(t)|_{\beta=\hat{\beta}} \tag{15}$$

(ii) When $J=k=r$ the experiment run only at normal conditions and $\hat{\lambda}=0$

(iii) The initial value of iteration method can be obtained from the joint profile log-likelihood function (11).

3.2. Point Bayesian estimation

In this section, A challenging problem in statistical inference is selecting a prior distribution that is suitable for the information about the parameters. In the problem at hand, the conjugate prior distribution does not exist. Due to several distributions, such as the chi-square distribution and the exponential being a special case of the gamma prior. Therefore, we adopt gamma prior distribution of the model parameters and non-informative prior information for the accelerated factor as follows.

$$P_1(\beta)\propto\beta^{a-1}\exp\{-b\beta\},\ a,b>0, \tag{16}$$

and

$$P_2(\lambda)\propto\frac{1}{\lambda}. \tag{17}$$

Therefore, the joint prior information

$$P(\lambda,\beta)\propto\lambda^{-1}\beta^{a-1}\exp\{-b\beta\}. \tag{18}$$

Generality, the joint posterior density function of λ and β compute from

$$\pi(\lambda,\beta|\mathbf{t})=\frac{P_1(\beta)P_2(\beta)L(\lambda,\beta|\mathbf{t})}{\iint P_1(\beta)P_2(\beta)L(\lambda,\beta|\mathbf{t})\,d\lambda d\beta}\propto P_1(\beta)P_2(\beta)L(\lambda,\beta|\mathbf{t}). \tag{19}$$

Also, under squared error loss function the Bayes estimate of any function of λ and β say $g(\lambda,\beta)$ is given by

$$\widehat{g}(\lambda,\beta)=\iint g(\lambda,\beta)\pi(\lambda,\beta|\mathbf{t})d\lambda d\beta. \tag{20}$$

The closed form of posterior distribution (19) and the estimated value in (20) are generally more complicated especially in a parameter vector with a large dimension. Therefore, the approximation method is the natural alternative method. Numerical integration, Lindley's approximation, and MCMC methods can be applied. in this paper, we adopt the important one called the MCMC method as follows.

Bayesian estimation using MCMC:

From (19) the joint posterior distribution by using (18) and (9) reduce to

$$\pi(\lambda,\beta|\mathbf{t}) \propto \left(\frac{\beta^3}{\beta^3+2}\right)^r \beta^{a-1}\lambda^{r-J-1}\left(\prod_{i=1}^{J}(\beta+t_i^2)\right)\left(\prod_{i=J+1}^{r}(\beta+z_i^2)\right)\left(1+\frac{\beta z_r(\beta z_r+2)}{\beta^3+2}\right)^{(n-r)} \times \exp\left\{-b\beta-\beta\sum_{i=1}^{J}t_i-\beta\sum_{i=J+1}^{r}z_i-\beta(n-r)z_r)\right\}. \tag{21}$$

The joint posterior distribution (22) is reduce to two full conditional distributions given by

$$\pi_1(\lambda,|\beta,\mathbf{t}) \propto \lambda^{r-J-1}\left(\prod_{i=J+1}^{r}(\beta+z_i^2)\right)\left(1+\frac{\beta z_r(\beta z_r+2)}{\beta^3+2}\right)^{(n-r)} \times \exp\left\{-\beta\sum_{i=J+1}^{r}(\eta+\lambda(t-\eta)-\beta(n-r)(\eta+\lambda(t-\eta))\right\}, \tag{22}$$

and

$$\pi_2(\beta|\lambda,\mathbf{t}) \propto \left(\frac{\beta^3}{\beta^3+2}\right)^r \beta^{a-1}\left(\prod_{i=1}^{J}(\beta+t_i^2)\right)\left(\prod_{i=J+1}^{r}(\beta+z_i^2)\right)\left(1+\frac{\beta z_r(\beta z_r+2)}{\beta^3+2}\right)^{(n-r)} \times \exp\left\{-b\beta-\beta\sum_{i=1}^{J}t_i-\beta\sum_{i=J+1}^{r}z_i-\beta(n-r)z_r)\right\}. \tag{23}$$

The full conditional distributions (22) and (23) have shown that Metropolis-within-Gibbs samplers are more suitable algorithms. The flowering algorithm is used to generate from the posterior distribution (21) and hence obtain the empirical posterior distribution.

Algorithm 1 (MCMC algorithm).

(i) Begin with initial guess value $\lambda^{(0)}=\widehat{\lambda}$ and $\beta^{(0)}=\hat{\beta}$.

(ii) Put $s=1$.

(iii) Generate $\lambda_1^{(s)}$ from (22) using (MH) algorithm.

(iv) Generate $\beta_1^{(s)}$ from (23) using (MH) algorithm.

- Generate candidate sample points from normal distribution.

- Compute the acceptance probability from P_l =min $\left(1, \frac{\pi_l(.|\lambda,\mathbf{t})}{\pi_l(.|\lambda,\mathbf{t})}\right)$, l =1,2.
- Generate from uniform 0 and 1, U_l.
- Accept the candidate sample points if $U_l < P_l$. Otherwise, reject the candidate point and repeat the last one.

(v) For given t compute $S(t, \beta_1^{(s)})$ and $h(t, \beta_1^{(s)})$

(vi) Put $s = s + 1$.

(vii) Repeat Steps 3 to 4 N times to get ($\lambda^{(1)}$,$\beta^{(1)}$), . . . ,($\lambda^{(N)}$,$\beta^{(N)}$).

(viii) The Bayes estimates of λ and β with respect to the SEL function as are given by

$$\hat{\lambda}_{\mathrm{B}} = \frac{1}{N - N^*} \sum_{i=N^*+1}^{N} \lambda^{(i)}, \tag{24}$$

and

$$\hat{\beta}_{\mathrm{B}} = \frac{1}{N - N^*} \sum_{i=N^*+1}^{N} \lambda^{(i)}, \tag{24}$$

where N^* s the number of iterations to reach the stationary distribution.

(ix) Bayes estimates of the parameter of life (Reliability and hazard rate function) are given by

$$\hat{S}_{\mathrm{B}}(t) = \frac{1}{N - N^*} \sum_{i=N^*+1}^{N} S(t, \beta_1^{(i)}) \tag{25}$$

$$\hat{h}_{\mathrm{B}}(t) = \frac{1}{N - N^*} \sum_{i=N^*+1}^{N} h(t, \beta_1^{(i)}) \tag{26}$$

4. Interval Estimation

In this section, we discuss the interval estimation of the ID parameter and the accelerated factor with three different methods. The first one, confidence intervals depend on the asymptotic property of the ML estimate (Approximate ML confidence intervals), the second is the bootstrap confidence interval. Finally, we consider the probability credible intervals.

4.1. Approximate ML confidence intervals

Interval estimation of the parameters depends on Fisher information matrix, which was defined as the minus expectation of the second partial derivative of the log-likelihood function as

$$\digamma(\lambda, \beta) = E\left(-\frac{\partial^2 \ell(\lambda, \beta|\mathbf{t})}{\partial\theta_i \partial\theta_j}\right), \ i, j = 1, 2, \ \theta_1 = \lambda \text{ and } \theta_2 = \beta,$$

Generally, for models that have several parameters, the problem of computing expectation is more complicated. Therefore, replace the Fisher information matrix with the approximate information matrix is given by

$$\Psi(\lambda,\beta)=\left(-\frac{\partial^2\ell\left(\lambda,\beta|\mathbf{t}\right)}{\partial\theta_i\partial\theta_j}\right)|_{\lambda,\beta=\hat{\lambda},\hat{\beta}},\ i,j=1,2,\ \theta_1=\lambda \text{ and } \theta_2=\beta. \tag{23}$$

From the log-likelihood function (10) the second partial derivative is obtained. Also, under bivariate normal distribution for the values of ML estimate of parameters $\hat{\lambda}$ and $\hat{\beta}$ with mean (λ, β) and variance as the diagonal of $\Psi^{-1}(\hat{\lambda},\hat{\beta})$. Therefore, we say

$$\left(\hat{\lambda},\hat{\beta}\right)\longrightarrow\mathcal{N}\left(\left(\lambda,\beta\right),\Psi^{-1}(\hat{\lambda},\hat{\beta})\right). \tag{24}$$

Hence, the corresponding (1-2α)100% approximate confidence intervals of parameters are given by

$$\hat{\lambda}\mp\xi_\alpha\sqrt{\Psi^{-1}_{1,1}},\ \text{and}\ \hat{\beta}\mp\xi_\alpha\sqrt{\Psi^{-1}_{2,2}}\ , \tag{25}$$

where, $\Psi^{-1}_{i,j}$, $i,j=1,2$ are the element of diagonal for $\Psi^{-1}_{i,j}(\hat{\lambda},\hat{\beta})$ and the value ξ_α is a tabulated standard normal value with confidence level 2α.

4.2. Bootstrap confidence intervals

Bootstrap techniques are commonly used methods in literature to formulate confidenca e interval of the parameters. In the problem at hand, paramteric bootstrap techniques is used to formulate boot-p and boot-t confidence interval, see [25], [26] and [27] as follows

Algorithm 2 (Bootstrap confidence intervals):

Step 1: From a real life population and the prior integer k, m and n as well as $\tau<\eta$ determine the original data set $\mathbf{t}$ =(t_1, t_2, ..., t_J, t_{J+1}, ..., t_r).

Step 2: The ML estimates $\hat{\lambda}$ and $\hat{\beta}$ are computed from the original data set $\mathbf{t}$ =(t_1, t_2, ..., t_J, t_{J+1}, ..., t_r).

Step 3: Put $s=1$.

Step 4: with the same prior integer k, m and n as well as $\tau<\eta$ generate a type-I GHC random from ID with parameters $\hat{\beta}$.

Step 5: Applied the transformation (4) for all data larger than η to get the accelerated type-I GHC bootstrap sample data $\mathbf{t}^*$ =(t_1^*, t_2^*, ..., t_J^*, $t^*{}_{J+1}$, ..., t_r^*).

Step 6: Compute The ML estimates $\hat{\lambda}^*$ and $\hat{\beta}^*$ from bootstrap sample data $\mathbf{t}^*$ =(t_1^*, t_2^*, ..., t_J^*, $t^*{}_{J+1}$, ..., t_r^*).

Step 7: Put $s=s+1$.

Step 8: Repeat the steps from 3 to 7 NB-times, we obtain the bootstrap sample estimate

$$\theta_l^{*(1)}, \theta_l^{*(2)}, ..., \theta_l^{*(NB)},\ l = 1{,}2,\ \theta_1^* = \lambda^* \text{ and } \theta_2^* = \beta^* \tag{26}$$

Boot-p confidence intervals (PBCI**)**

The ordered value of (26) is given by

$$\theta_{l(1)}^*, \theta_{l(2)}^*, ..., \theta_{l(NB)}^*,$$

Suppose $\Gamma(x)$ the emperical distribution of the ordered bootstrap sample estimated (27) then, $F(x) = P(\theta_l \leqslant x)$, i =1, 2 be a CDF of θ_l and the corresponding $(1-2\alpha)100\%$ PBCIs are given by

$$\left(\theta_{l(\alpha\mathbf{NB})}^*,\ \theta_{l((1\text{-}\alpha)\mathbf{NB})}^*\right), \tag{28}$$

where of $\theta_{l(.)}^* = \Gamma^{-1}(x)$.

Boot-t confidence interval (BTCI**)**

From the ordered sample $\theta_{l(1)}^*$, $\theta_{l(2)}^*$, ..., $\theta_{l(\mathbf{NB})}^*$, we difene the statistics $\Phi_{l(1)}^* < \Phi_{l(2)}^* < ... < \Phi_{l(\mathbf{MB})}^*$ by

$$\Phi_{l(j)}^* = \sqrt{r}\frac{\theta_{l(j)}^* - \hat{\theta}_l}{\sqrt{\text{var}\left(\theta_{l(j)}^*\right)}},\ l = 1, 2\ . \tag{29}$$

Let $\Gamma(x) = P(\Phi_i^* \leqslant x)$ be the CDF of Φ_i^*. Therefore for given x, we define

$$\theta_{l\text{boot-t}}^* = \hat{\theta}_l + \sqrt{r\text{Var}(\hat{\theta}_l)}\Gamma^{-1}(x), \tag{30}$$

The corresponding $(1-2\alpha)100\%$ PTCIs are given by

$$\left(\theta_{l\text{boot-t}(\alpha)}^*, \theta_{l\text{boot-t}(1\text{-}\alpha)}^*\right) \tag{31}$$

4.3. Bayesian credible interval

From the Bayes estimate sample generated from MCMC method

$$\hat{\theta}_{\text{B}l}^{(1)},\ \hat{\theta}_{\text{B}l}^{(2)},\ ...,\ \hat{\theta}_{\text{B}l}^{(N)}. \tag{32}$$

The ordred value

$$\hat{\theta}_{\text{B}l(1)},\ \hat{\theta}_{\text{B}l(2)},\ ...,\ \hat{\theta}_{\text{B}l(N)},\ i = 1, 2. \tag{33}$$

The $(1-2\alpha)100\%$ equal two side credible intervals of the model parameters is defined by

$$\left(\hat{\theta}_{\text{B}l\left(\frac{\alpha}{2}(N-N^*)\right)},\ \hat{\theta}_{\text{B}l\left((1-\frac{\alpha}{2})(N-N^*)\right)}\right) \tag{34}$$

5. Numerical Results

In this section, we adopt numerical computation in the form real data analysis and simulation study. For data analysis consider the data reported by Fuller et al. [28] of glass of the aircraft window and Bader and Priest [29] about the strength of single-carbon fibres in GPA.

5.1. Data analysis

In this subsection, we consider two real data sets and the goodness of fit of these data to ID has been done by Shanker and Shukla [3].

Example 1: The following data describe strength data reported by Fuller et al. [28] of glass of the aircraft window

Table 2: The orginal and the corresponding accelerated type-I GHC data sets Fuller et al. [28].

Orginal	18.83	20.800	21.657	23.030	23.230	24.050	24.321	25.500	25.520
data	25.800	26.69	26.770	26.780	27.050	27.670	29.900	31.110	33.200
	33.730	33.760	33.890	34.760	35.750	35.910	36.980	37.080	37.090
	39.580	44.045	45.290	45.381					
accelerated	18.830	20.800	21.657	23.030	23.230	24.050	24.321	25.500	25.520
type-I GHC	25.800	26.690	26.770	26.780	27.017	27.223	27.967	28.370	29.067
data	29.243	29.253	29.297	29.587					

Under consideration, the original data of size n =31 Fuller et al. [28], suppose m =25 k =15, τ =35.0 and the stress change time η =27 , the accelerated type-I GHC data can be obtained from the original data in Table 1 by considered λ =3.0. From accelerated type-I GHC data the integer numbers r =22 and J =13. The prior information about ID parameter is taken to be non-informative prior. With respect to MCMC method, chen iteration is running 11000 iterations and delete the first 1000 iterations as burn-in. Figures 4 and 5 show the empirical posterior distribution and its convergence under the Bayesian approach. The point estimate of the model parameters and the parameters of life when t =1.5 are reported in Table 3. The 95% interval estimation of the model parameters is reported in Table 3

Table 3: The results of point and interval estimate under 5% confidence level.

Pa.	(.)ML	(.)B-MCMC	95% ACI	95% PBCI	95% BTCI	95% CI
β	0.078268	0.078090	(0.0575, 0.0990)	(0.0488, 0.0997)	(0.0542, 0.0991)	(1.1596, 4.6482)
λ	2.524430	2.599440	(0.7772, 4.2716)	(0.6542, 4.2744)	(0.7723, 4.2722)	(1.1596, 4.6482)
$S(.)$	0.999726	0.999712				
$h(.)$	0.000496	0.000520				

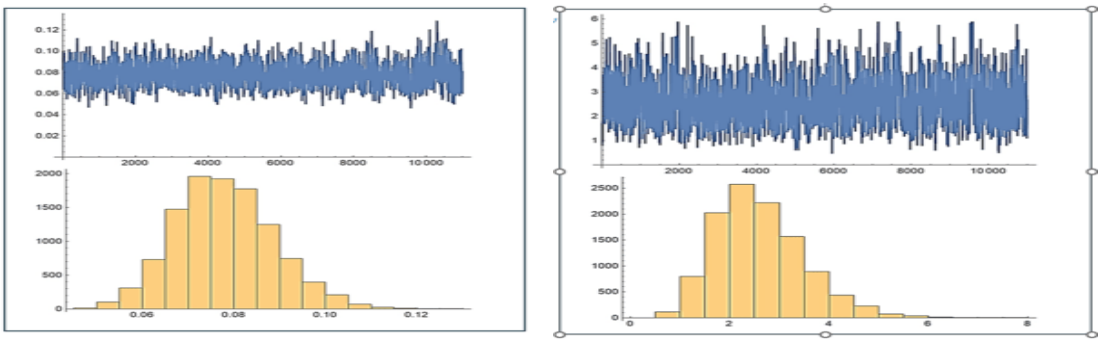

Figure 4: Simulation number and Histogram of

β and λ generated by MCMC method, respectively.

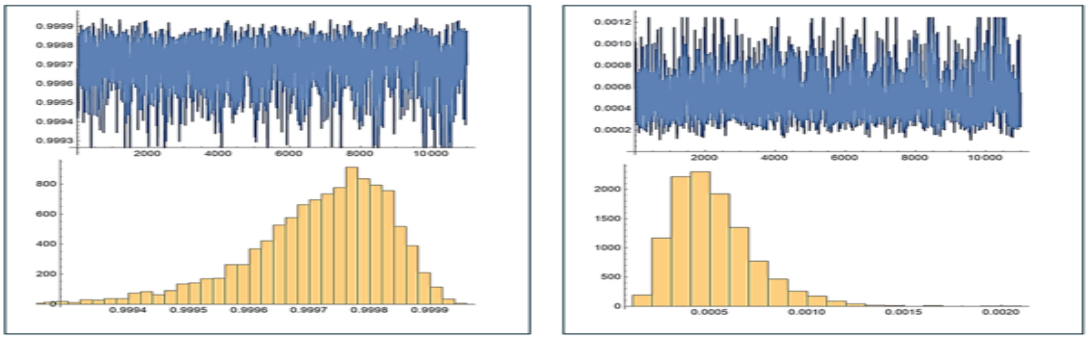

Figure 5: Simulation number and Histogram of

S and h generated by MCMC method, respectively.

Example 2: The second data is the tensile strength, measured in GPa, of 69 carbon fibers represent tested under tension at gauge lengths of 20mm, Bader and Priest [29]

The original Bader and Priest [29] data of size n =69 , suppose m =50 k =25, τ =2.0 and the stress change time η =1.0 , the accelerated type-I GHC data can be obtained from the original data in Table 2 by considered λ =3.0. The integer numbers r =25 and J =0 from accelerated type-I GHC data. With 11000 iterations and delete the first 1000 iterations as burn-in MCMC method is running . Figure 6 and 7 show the empirical posterior distribution and its convergence under Bayesian approach. The point estimate of the model parameters and the parameters of life when t =1.5 are reported in Table 5. The 95% interval estimation of the model parameters is reported in Table 5.

5.2. Simulation study

In this section, the developed estimation results are discussed through a Monte Carlo simulation study to assess and compare the results. Through this study, we discuss the effect of changing the ideal test time and the stress change time. Also, we test the results for the change in the values of the parameters and sample sizes. For the point estimate, check the ML estimate with the Bayes estimate for different prior information. In cases of interval estimation, compare the approximate confidence intervals, the bootstrap confidence

Table 4: The orginal and the corresponding accelerated type-I GHC data sets Bader and Priest [28].

Orginal	1.312	1.314	1.479	1.552	1.700	1.803	1.861	1.865	1.944	1.958
data	1.966	1.997	2.006	2.021	2.027	2.055	2.063	2.098	2.140	2.179
	2.224	2.240	2.253	2.270	2.272	2.274	2.301	2.301	2.359	2.382
	2.382	2.426	2.434	2.435	2.478	2.490	2.511	2.514	2.535	2.554
	2.566	2.570	2.586	2.629	2.633	2.642	2.648	2.684	2.697	2.726
	2.770	2.773	2.800	2.809	2.818	2.821	2.848	2.880	2.954	3.012
	3.067	3.084	3.090	3.096	3.128	3.233	3.433	3.585	3.585	
accelerated	1.104	1.105	1.160	1.184	1.233	1.268	1.287	1.288	1.315	1.319
type-I GHC	1.322	1.332	1.335	1.340	1.342	1.352	1.354	1.366	1.380	1.393
data	1.408	1.413	1.418	1.423	1.424					

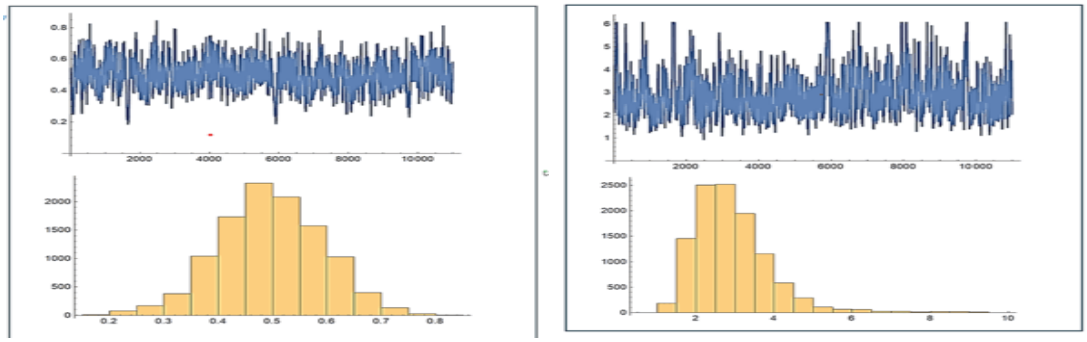

Figure 6: Simulation number and Histogram of

β and λ generated by MCMC method, respectively.

interval, and the Bayesian credible intervals. The simulation results were formulated for 1000 different samples generated from the proposed model. The point estimate is tested under the mean estimate (ME) and mean squared error (MSE). In an interval estimate, we compute the mean interval length (MIL) and coverage percentage (CP). The following algorithms describe what happens in the simulation results

Algorithm 3: (Monte Carlo simulation studying)

Step 1: From ID with parameter β generate a random sample of size n .

Step 2: The generate type-I GHC data is reported respected to k, m and τ.

Step 3: Transform the type-I GHC data to accelerated type-I GHC data based on transfor-

Table 5: The results of point and interval estimate under 5% confidence level.

Pa.	$(.)_{\text{ML}}$	$(.)_{\text{B-MCMC}}$	95% ACI	95% PBCI	95% BTCI	95% CI
β	0.51064	0.49666	(0.3178, 0.7035)	(0.3001, 0.7457)	(0.3124, 0.7011)	(1.5527, 5.2164)
λ	2.65934	2.91626	(1.0371, 4.2816)	(1.0422, 4.2987)	(1.0341, 4.2807)	(1.5527, 5.2164)
$S(.)$	0.92661	0.92657				
$h(.)$	0.08645	0.08529				

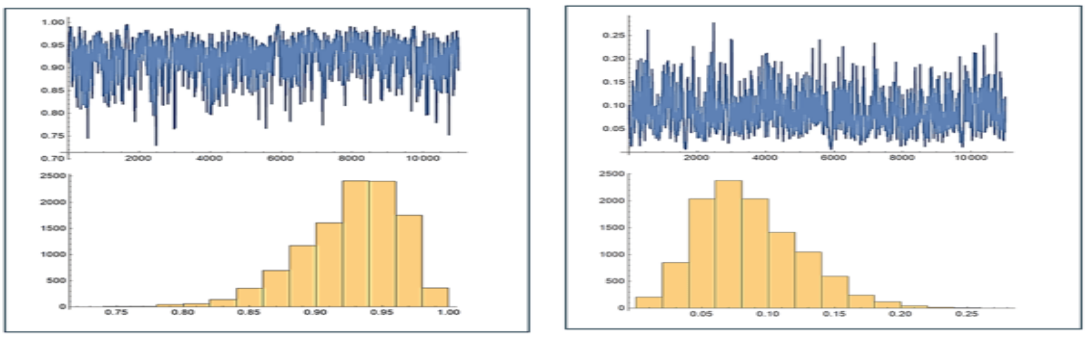

Figure 7: Simulation number and Histogram of

S and h generated by MCMC method, respectively.

mation (4) and accelerated factor λ.

Step 4: Compute the point ML and Bayes estimate of λ and β.

Step 5: Compute the interval ML confidence interval, bootstrap confidence intervals and Bayesian credible interval.

Step 5: Repeated steps from 1 to 5 1000 times.

Step 6: The values of MEs, MSEs, MILs and PCs are computed and results record in Tables (6) to (9).

Table 6. Average ME and the corresponding MSEs when β =0.5 and λ =2.5.

(k, m, n)	(η, τ)		MLE		Bayes P0		Bayes P1	
			β	λ	β	λ	β	λ
(15,20,40)	(2.5,6.5)	ME	0.742	2.853	0.727	2.847	0.687	2.744
		MSE	0.147	0.336	0.139	0.342	0.103	0.267
(15,20,40)	(5,6.5)	ME	0.719	2.871	0.705	2.869	0.661	2.761
		MSE	0.136	0.341	0.125	0.359	0.099	0.271
(15,30,40)	(2.5,6.5)	ME	0.641	2.761	0.615	2.739	0.601	2.630
		MSE	0.118	0.305	0.114	0.318	0.082	0.231
(15,30,30)	(5,6.5)	ME	0.630	2.782	0.672	2.800	0.598	2.670
		MSE	0.112	0.315	0.101	0.331	0.075	0.242
(25,35,70)	(2.5,6.5)	ME	0.611	2.759	0.647	2.771	0.569	2.645
		MSE	0.099	0.302	0.088	0.311	0.063	0.229
(25,35,70)	(5,6.5)	ME	0.601	2.769	0.631	2.789	0.551	2.653
		MSE	0.092	0.308	0.081	0.313	0.060	0.234
(25,50,70)	(2.5,6.5)	ME	0.589	2.601	0.577	2.593	0.544	2.587
		MSE	0.076	0.234	0.069	0.227	0.041	0.175
(25,50,70)	(5,6.5)	ME	0.565	2.612	0.564	2.597	0.528	2.591
		MSE	0.071	0.239	0.063	0.232	0.035	0.181

Table 7. Average MIL and the corresponding PC when β =0.5 and λ =2.5.

(k, m, n)	(η, τ)		MLE		Boot-p		Boot-t		Bayes P1	
			β	λ	β	λ	β	λ	β	λ
(15,20,40)	(2.5,6.5)	MIL	0.642	4.245	0.666	4.542	0.626	4.011	0.603	3.897
		PC	0.89	0.90	0.91	0.89	0.91	0.92	0.92	0.92
(15,20,40)	(5.0,6.5)	MIL	0.631	4.253	0.649	4.551	0.615	4.017	0.598	3.901
		PC	0.91	0.90	0.92	0.90	0.91	0.93	0.94	0.91
(15,30,40)	(2.5,6.5)	MIL	0.601	4.225	0.618	4.527	0.589	4.001	0.579	3.891
		PC	0.90	0.93	0.91	0.93	0.94	0.96	0.93	0.95
(15,30,30)	(5.0,6.5)	MIL	0.589	4.233	0.609	4.534	0.580	4.014	0.561	3.898
		PC	0.91	0.93	0.93	0.94	0.95	0.92	0.96	0.94
(25,35,70)	(2.5,6.5)	MIL	0.566	4.211	0.575	4.508	0.550	3.980	0.542	3.859
		PC	0.94	0.92	0.93	0.90	0.94	0.94	0.96	0.92
(25,35,70)	(5.0,6.5)	MIL	0.551	4.219	0.562	4.517	0.535	3.991	0.527	3.861
		PC	0.93	0.93	0.93	0.96	0.92	0.94	0.95	0.95
(25,50,70)	(2.5,6.5)	MIL	0.515	4.170	0.531	4.466	0.505	3.938	0.511	3.815
		PC	0.93	0.95	0.93	0.92	0.93	0.94	0.93	0.94
(25,50,70)	(5.0,6.5)	MIL	0.502	4.176	0.519	4.471	0.501	3.943	0.495	3.822
		PC	0.92	0.92	0.96	0.94	0.93	0.95	0.95	0.91

Table 8. Average ME and the corresponding MSEs when β =1.2 and λ =1.5.

(k, m, n)	(η, τ)		MLE		Bayes P0		Bayes P1	
			β	λ	β	λ	β	λ
(15,20,40)	(0.8,2.0)	ME	1.454	2.147	1.404	2.103	1.389	2.001
		MSE	0.284	0.314	0.275	0.303	0.197	0.287
(15,20,40)	(1.2,2.0)	ME	1.445	2.151	1.389	2.111	1.365	2.008
		MSE	0.255	0.322	0.242	0.312	0.162	0.291
(15,30,40)	(0.8,2.0)	ME	1.392	2.118	1.379	1.893	1.314	1.872
		MSE	0.225	0.289	0.222	0.275	0.135	0.214
(15,30,30)	(1.2,2.0)	ME	1.362	2.123	1.355	1.914	1.289	1.885
		MSE	0.195	0.294	0.187	0.282	0.114	0.219
(25,35,70)	(0.8,2.0)	ME	1.278	1.874	1.282	1.971	1.244	1.754
		MSE	0.135	0.166	0.127	0.159	0.094	0.114
(25,35,70)	(1.2,2.0)	ME	1.254	1.881	1.259	1.977	1.232	1.757
		MSE	0.122	0.169	0.124	0.161	0.085	0.099
(25,50,70)	(0.8,2.0)	ME	1.247	1.866	1.244	1.974	1.225	1.761
		MSE	0.115	0.151	0.127	0.153	0.079	0.084
(25,50,70)	(1.2,2.0)	ME	1.232	1.869	1.225	1.975	1.212	1.763
		MSE	0.098	0.153	0.115	0.157	0.070	0.079

Table 9. Average MIL and the corresponding PC when β =1.2 and λ =1.5.

(k, m, n)	(η, τ)		MLE		Boot-p		Boot-t		Bayes P1	
			β	λ	β	λ	β	λ	β	λ
(15,20,40)	(0.8,2.0)	MIL	2.147	2.894	2.165	2.899	2.132	2.855	2.102	2.812
		PC	0.90	0.90	0.91	0.90	0.91	0.92	0.92	0.90
(15,20,40)	(1.2,2.0)	MIL	2.122	2.899	2.142	2.915	2.112	2.866	2.055	2.817
		PC	0.91	0.91	0.92	0.91	0.91	0.91	0.93	0.92
(15,30,40)	(0.8,2.0)	MIL	2.102	2.842	2.122	2.877	2.055	2.814	2.001	2.750
		PC	0.93	0.93	0.90	0.93	0.96	0.93	0.94	0.94
(15,30,30)	(1.2,2.0)	MIL	2.072	2.851	2.100	2.884	2.033	2.821	1.982	2.754
		PC	0.92	0.92	0.93	0.91	0.92	0.91	0.93	0.95
(25,35,70)	(0.8,2.0)	MIL	2.025	2.795	2.027	2.825	2.001	2.785	1.954	2.689
		PC	0.93	0.90	0.94	0.94	0.96	0.93	0.94	0.93
(25,35,70)	(1.2,2.0)	MIL	2.002	2.797	2.001	2.831	1.975	2.789	1.932	2.691
		PC	0.91	0.93	0.93	0.92	0.92	0.92	0.97	0.91
(25,50,70)	(0.8,2.0)	MIL	1.952	2.735	1.945	2.715	1.912	2.704	1.854	2.625
		PC	0.94	0.92	0.92	0.94	0.94	0.95	0.93	0.94
(25,50,70)	(1.2,2.0)	MIL	1.924	2.741	1.931	2.722	1.900	2.713	1.832	2.631
		PC	0.92	0.93	0.96	0.93	0.94	0.92	0.95	0.96

5.3. Numerical discussion

From the numerical results presented in the data analysis and the Monte Carlo simulation study observe some points are reported as follows

1- In two real example, the results are formulated under a non-informative prior distribution. Therefore, the results of ML and Bayes estimates are closed.

2- Figures from (4) to (7) show the convergence happening in MCMC.

3- All results of point and interval estimation serve well for the larger ideal test time and larger sample size.

4- The results for the boot-t confidence interval estimate is best than ML and boot-p confidence interval.

5- The values of MSEs are reducing for increasing value of sample size and effective sample size.

6- For point estimate informative Bayes estimate is better than ML and the non-informative Bayes estimate.

7- For interval estimate, informative Bayes and bootstrap-t serve well than ML and boot-p.

The results more suitable for large value of τ.

6. Conclusions

In life-testing experiment and reliability studying under cost and time consideration may be need form some not complete information on failure times. In this paper, we consider ID which has several application in biomedical science and engineering lifetime data. Type-I GHCS with Partially step-stress accelerated life tests model was considered. Also, the conventional of type-I generalized hybrid censoring scheme for saving times and cost inducing by units. We examined the results under real data analysis and simulation study. The results obtained in this paper can be developed for any model extension of competing risks model from this model

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