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Modified Optimal Homotopy Asymptotic Method for Singular Two-Point Boundary Value Problems

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Abstract. This work applies the Modified Optimal Homotopy Asymptotic Method (MOHAM) to two benchmark singular two–point boundary value problems that were previously analyzed using the standard OHAM. The modification introduces a refined homotopy framework in which nonlinear and singular terms are systematically distributed across successive embedding orders, while the auxiliary function is optimally tuned through residual minimization. With this enhancement, MOHAM delivers first–order analytical approximations that are nearly identical to the exact solutions, achieving higher accuracy, faster convergence, and greater computational efficiency than the conventional OHAM even when the latter is extended to three terms of approximation.

2020 Mathematics Subject Classifications: 26A33, 34A08, 65L10

Key Words and Phrases: MOHAM, singular boundary value problems, residual minimization, analytical approximation, convergence efficiency

1. Introduction

Differential equations constitute a central pillar in applied mathematics, as they provide a powerful framework to explain and predict various real-world phenomena. They are widely used in modeling processes in physics, engineering, biology, chemistry, and economics. Two-point boundary value problems (BVPs) are particularly important in this

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regard, as they occur when conditions are imposed at more than one boundary point. Singular two-point BVPs, in which the governing equations contain singularities at the end points of the domain, frequently arise in many physical situations, such as thin film flows, boundary layer problems in fluid mechanics, reaction–diffusion processes, chemical kinetics, and optimal control theory [1]. Singular boundary value problems (BVPs) appear in fluid mechanics, reaction–diffusion, chemical kinetics, and optimal control. Representative forms include

$$\frac{1}{p}u''(x) + \frac{1}{q(x)}u'(x) + \frac{1}{r(x)}u(x) = g(x), \qquad 0 \leq x \leq 1, \tag{1}$$

with boundary conditions such as $u(0) = \alpha_1$, $u(1) = \beta$ or $u'(0) = \alpha_2$, $u(1) = \beta$. In these settings, differential equations provide not only a descriptive mechanism but also an explanatory tool for the underlying dynamics. However, because non-linear and singular models rarely admit closed-form solutions, the development of accurate and efficient approximate or numerical methods becomes essential. Over the past few decades, various approaches have been introduced to deal with the difficulties posed by singular two-point BVPs. Abu Arqub et al. [2] proposed a continuous genetic algorithm that effectively approximates the solutions of such problems, showing that evolutionary computation techniques can achieve smooth and stable results. Bakodah et al. [3] developed an efficient decomposition shooting method that combined decomposition strategies with shooting techniques, enhancing stability and accuracy. Another direction was pursued by Cui and Geng [4], who presented a kernel space reproducing approach and rigorously established convergence properties for singular BVPs. Other notable contributions include numerical schemes proposed by Anakira et al. [5], spline-based approximations [6], the differential transformation method [7, 8], Adomian decomposition methods [9], and the variational iteration method [10] and others [11, 12] . Together, these works reflect the importance of finding effective strategies that can handle the singular nature of boundary value models while providing accurate solutions.

Although these approaches have enriched the literature, they often involve complex iterative computations, high computational cost, or difficulties in handling nonlinearities. To overcome such limitations, researchers have increasingly turned to homotopy-based techniques, which construct a deformation (homotopy) that continuously transforms a simple solvable problem into the original nonlinear model. Among these, the OHAM introduced by Marinca et al. [13], has received particular attention. Unlike perturbation methods that require a small or large parameter, OHAM avoids this restriction and instead introduces auxiliary convergence control parameters that are optimally determined to accelerate convergence. Marinca et al. [13] successfully applied OHAM to thin-film flow, demonstrating its efficiency, and since then it has been extended to a wide variety of linear and non-linear problems in applied sciences. For example, Anakira [14] applied OHAM to systems of ordinary differential equations, Agarwal et al. [15] solved systems of Volterra integro-differential equations, and Jameel et al. [16] addressed fuzzy differential equations. These studies confirm the adaptability and precision of OHAM in many applied contexts.

The strength of the OHAM algorithm lies in its ability to produce rapidly convergent approximate solutions using relatively few terms, typically three or four, while maintaining excellent agreement with exact or numerical results. Several studies [17–21] have shown that OHAM can outperform traditional analytical and iterative techniques in terms of the number of terms required to achieve the same level of accuracy. Unlike other methods that may require more than four terms, OHAM achieves comparable results with fewer terms, thereby reducing computational time, effort, and cost. To further improve efficiency and accuracy, improved approaches have been developed to provide more precise or near-exact solutions through several previous studies conducted by many researchers [22–29]. This has led to efforts and initiatives to expand the scope of significant developments within the OHAM framework, either by improving the auxiliary function or by restructuring the symmetry equation to accelerate convergence and simplify the computational process. One of the key advantages of the MOHAM algorithm is its ability to construct highly accurate solutions using only a single-term (monomial) approximation, which significantly reduces computational complexity while maintaining accuracy.

In summary, singular two-point boundary value problems are of great significance in applied mathematics, as they describe fundamental processes in fluid mechanics, reaction–diffusion systems, chemical kinetics, optimal control, and many other fields. Differential equations, as the language of these phenomena, require effective approximate techniques when exact solutions are not attainable. A wide range of methods have been proposed, but the OHAM by Marinca et al. [13] and its subsequent modifications represent particularly powerful approaches. The MOHAM, in particular, is capable of producing accurate solutions even with a single approximation term, highlighting its potential as an efficient and reliable method for addressing nonlinear singular boundary value problems. The OHAM, introduced by Marinca and collaborators[13], provides a convenient convergence control via auxiliary constants chosen by minimizing the residual. OHAM has been successfully applied to singular BVPs [5]. This work adopts a MOHAM a refined homotopy in which the nonlinear/singular operator is uniquely represented across powers of the embedding parameter, allowing faster convergence with fewer series terms. We follow the structure in [5] and re–solve the same examples, comparing MOHAM with OHAM.

2. Modified Optimal Homotopy Asymptotic Method

We consider a general nonlinear boundary value problem

$$\mathcal{L}(u(x)) + g(x) + \mathcal{N}(u(x)) = 0, \qquad \mathcal{B}(u, \tfrac{du}{dx}) = 0, \tag{2}$$

where $\mathcal{L}$ is a linear operator, $\mathcal{N}$ is a nonlinear operator, $g(x)$ is a known function, and $\mathcal{B}$ denotes the boundary conditions.

According to the modified OHAM, we construct the homotopy

$$(1-p)\big[\mathcal{L}(v(x,p)) - \mathcal{L}(u_0(x))\big] = H(x,p)\,\big[\mathcal{L}(v(x,p)) + g(x) + \mathcal{N}(v(x,p))\big], \tag{3}$$

with the boundary condition

$$\mathcal{B}\Big(v(x,p), \tfrac{\partial v(x,p)}{\partial x}\Big) = 0, \tag{4}$$

where $p \in [0,1]$ is the embedding parameter, $u_0(x)$ is the initial guess satisfying

$$\mathcal{L}(u_0(x)) = 0, \qquad \mathcal{B}(u_0, \tfrac{du_0}{dx}) = 0. \tag{5}$$

Unlike standard OHAM where $H(p)$ is a scalar function, here we generalize to

$$H(x,p) = \sum_{n=1}^{N} \sum_{m=0}^{M} c_{nm}\, x^m p^n, \tag{6}$$

where c_{nm} are unknown parameters to be optimally determined.

Expanding $v(x,p)$ in a Taylor series about $p = 0$ gives

$$v(x,p) = u_0(x) + \sum_{k=1}^{\infty} u_k(x)\, p^k. \tag{7}$$

Substituting the expansion of $v(x,p)$ and $H(x,p)$ into the homotopy equation and equating like powers of p yields a hierarchy of linear subproblems:

- **Zeroth-order:**
$$\mathcal{L}(u_0(x)) = 0, \qquad \mathcal{B}(u_0) = 0. \tag{8}$$

- **First-order:**
$$\mathcal{L}(u_1(x)) + g(x) = \sum_{m=0}^{M} c_{1m} x^m\, \mathcal{N}_0(u_0(x)), \qquad \mathcal{B}(u_1) = 0. \tag{9}$$

- **Second-order:**
$$\begin{aligned}\mathcal{L}(u_2(x)) - \mathcal{L}(u_1(x)) &= \sum_{m=0}^{M} c_{2m} x^m\, \mathcal{N}_0(u_0(x)) \\ &\quad + \sum_{m=0}^{M} c_{1m} x^m \Big(\mathcal{L}(u_1(x)) + \mathcal{N}_1(u_0, u_1)\Big),\end{aligned} \tag{10}$$

  with $\mathcal{B}(u_2) = 0$.

- **k-th order ($k \geq 2$):**
$$\begin{aligned}\mathcal{L}(u_k(x)) - \mathcal{L}(u_{k-1}(x)) &= \sum_{m=0}^{M} c_{km} x^m\, \mathcal{N}_0(u_0(x)) \\ &\quad + \sum_{i=1}^{k-1} \sum_{m=0}^{M} c_{im} x^m \Big(\mathcal{L}(u_{k-i}(x)) + \mathcal{N}_{k-i}(u_0, \ldots, u_{k-i})\Big),\end{aligned} \tag{11}$$

  with $\mathcal{B}(u_k) = 0$.

Truncating at order m gives the m-th order approximate solution

$$\tilde{u}(x;\{c_{nm}\}) = u_0(x) + \sum_{k=1}^{m} u_k(x). \tag{12}$$

The residual is defined by

$$R(x;\{c_{nm}\}) = \mathcal{L}(\tilde{u}(x)) + g(x) + \mathcal{N}(\tilde{u}(x)), \tag{13}$$

and the optimal coefficients $\{c_{nm}\}$ are determined by minimizing

$$J(\{c_{nm}\}) = \int_a^b R(x;\{c_{nm}\})^2\,dx, \qquad \frac{\partial J}{\partial c_{nm}} = 0. \tag{14}$$

This yields the optimal auxiliary function $H(x,p)$ and the final MOHAM solution.

3. Numerical Examples

In this part, we apply the proposed method to two examples of singular two-point boundary value problems to assess its efficiency and capability, compared with the standard OHAM results of three order, in producing near-exact solutions using only a first-order approximation.

Example 1

As a first example, we examine the singular two-point boundary value problem considered by Kanth, A. R., & Reddy, Y. N. (2005)[30].

$$y''(x) + \frac{2}{x}y'(x) - 4y(x) \;=\; -2, \quad 0 < x \leq 1, \tag{15}$$

$$y'(0) \;=\; 0, \quad y(1) = 5.5. \tag{16}$$

Following the MOHAM procedure outlined in the previous section and based on the flexibility of standard OHAM in selecting the initial guess, we define the linear operator the nonlinear operator as follows

$$L[v(x,p)] \;=\; x\frac{d^2v(x,p)}{dx^2} + 2x \tag{17}$$

$$N[v(x,p)] \;=\; x\frac{d^2v(x,p)}{dx^2} + \frac{dv(x,p)}{dx} - 4xv(x,p) + 2x. \tag{18}$$

Then, we construct the following family of homtopy equations $h(u(x;p) : R- > [0,1]$

$$(1-q)\,L(v(x;q) - y_0(x)) = H(q)\left(L[v(x;q)] + N[v(x;q)]\right), \tag{19}$$

with boundary conditions

$$(1-q)\Big(v'(0;q)-y_0'(0)\Big)=H(q)\Big(v'(0;q)-0\Big), \tag{20}$$

$$(1-q)\Big(v(1;q)-y_0(1)\Big)=H(q)\Big(v(1;q)-5.5\Big). \tag{21}$$

Thus, as p varies from 0 to 1, the solution approach from $u_0(x)$ to $u(x)$, where $u_0(x)$ is the zeroth -order problem that can be obtained

$$u_0''(x)=-2, \quad u_0'(0)=0, \quad u(1)=5.5. \tag{22}$$

Then, by expanding $u(x;p;c_i), i=1$ in the Taylor series on p, we have

$$u(x;p;c_i)=\sum_{k=0}^{m} u_k(x,C_1,C_2,\ldots,C_7)\,p^k. \tag{23}$$

Substituting Eq.(23) into Eq. (19) and by equating the coefficient of similar power p, yields

The first-order deformation problem

$$\begin{aligned}u_1'(x)&=1.33333C_7x^{10}+1.33333C_6x^9+1.33333C_5x^8-4.C_7x^8+1.33333C_4x^7-4.C_6x^7\\&-21.3333C_7x^7+1.33333C_3x^6-4.C_5x^6-21.3333C_6x^6+1.33333C_2x^5-4.C_4x^5\\&-21.3333C_5x^5+1.33333C_1x^4-4.C_3x^4-21.3333C_4x^4+1.33333C_0x^3-4.C_2x^3\\&-21.3333C_3x^3-4.C_1x^2-21.3333C_2x^2-4.C_0x-21.3333C_1x-21.3333C_0+2\end{aligned} \tag{24}$$

By solving Eqs. (22) and (24) and substituting them into Eq.(23), the first-order approximate solution by MOHAM becomes:

$$\begin{aligned}u_1(x)&=0.333333\Big(17.5\;-1.x^3\Big)+0.010101c_7x^{12}+0.0121212c_6x^{11}+0.0148148c_5x^{10}\\&-0.0444444c_7x^{10}+0.0185185c_4x^9-0.0555556c_6x^9-0.296296c_7x^9+0.0238095c_3x^8\\&-0.0714286c_5x^8-0.380952c_6x^8+0.031746c_2x^7-0.0952381c_4x^7-0.507937c_5x^7\\&-0.133333c_3x^6-0.711111c_4x^6-0.2c_2x^5-1.06667c_3x^5-1.77778c_2x^4\\&+c_1\Big(0.0444444x^6-0.333333x^4-3.55556x^3+3.84444\Big)\\&+c_0\Big(0.0666667x^5-0.666667x^3-10.6667x^2+11.2667\Big)+1.94603c_2\\&+1.17619c_3+0.787831c_4+0.56455c_5+0.424387c_6+0.33064c_7+1.x^2-1.\end{aligned} \tag{25}$$

By using the proposed method of section 2 on $[0,1]$, we use the residual error,

$$\begin{aligned}R &= x\widetilde{u}''(x,C_1,\cdots,C_7)+2\widetilde{u}'(x,C_1,\cdots,C_7)\\&\quad -4x\widetilde{u}(x,C_1,\cdots,C_7)+2x.\end{aligned} \tag{26}$$

The Less Square error can be formed as

$$J(\partial J(C_1,\cdots,C_7)) = \int_0^1 R^2 dx, \tag{27}$$

and

$$\frac{\partial J(C_1,\cdots,C_7)}{\partial C_1}=\frac{\partial J(C_1,\cdots,C_7)}{\partial C_2}=\cdots=\frac{\partial J(C_1,\cdots,C_7)}{\partial C_7}=0. \tag{28}$$

Thus, the following optimal values of C_i's are obtained:

$$C_0=-0.078574,\quad C_1=-0.0790727,\quad C_2=-0.191309\quad C_3=0.0274625.$$

$$C_4=-0.0495492,\quad C_5=-0.0175115,\quad C_6=0.0112912\quad C_7=-0.00830839.$$

Upon substituting those values into equation (25), the first-order approximate solution is obtained.

$$\begin{aligned}\widetilde{u}(x) &= 3.25721+1.83812x^2+0.000196623x^3+0.366462x^4+0.0037302x^5+0.028059x^6\\ &+0.00754041x^7-0.00239673x^8+0.000916876x^9+0.000109832x^{10}\\ &+0.000136863x^11-0.0000839231x^{12}\end{aligned} \tag{29}$$

The numerical comparison presented in Table 1 clearly demonstrates the superiority of the modified OHAM (MOHAM) with only one order of approximation over the standard OHAM using three terms. At $x=0.2$, the OHAM absolute error is of order 10^{-4}, whereas MOHAM reduces the error dramatically to the order of 10^{-9}, showing an improvement of nearly five orders of magnitude. A similar trend is observed at $x=0.4, 0.6$, and 0.8, where OHAM produces errors between 10^{-4} and 10^{-5}, while MOHAM consistently achieves errors in the range of 10^{-9}–10^{-10}. At the endpoint $x=1$, both methods attain machine-precision accuracy, but overall the MOHAM approach achieves substantially higher accuracy with fewer approximation terms, confirming its efficiency and effectiveness.

Table 1: Numerical results resulted from example 1.

x	Exact Solution	OHAM Absolute Error	MOHAM Absolute Error
0.2	3.331321581291895	6.90×10^{-4}	2.15×10^{-9}
0.4	3.560863537324634	5.60×10^{-5}	3.7×10^{-9}
0.6	3.9682461451285476	1.50×10^{-4}	2.59×10^{-9}
0.8	4.593705860688229	4.74×10^{-4}	5.55×10^{-10}
1.0	5.500000000000000	8.88×10^{-16}	8.86×10^{-16}

Example 2

As a second example, we examine the singular two-point boundary value problem considered by Anakira et al. (2013).

$$u''+\frac{1}{x}u'+u=g(x),\quad 0\le x\le 1,\qquad u'(0)=0,\ u(1)=\tfrac{17}{16}, \tag{30}$$

$$g(x) = \frac{5}{4} + \frac{x^2}{16}, \qquad u_{\text{exact}}(x) = 1 + \frac{x^2}{16}. \tag{31}$$

By applying the same procedure as in the previous example, we determine the corresponding optimal values of C_i.

$$C_0 = -0.666669, \quad C_1 = 0.0000909446, \quad C_2 = 0.221205 \quad C_3 = 0.00521847.$$

$$C_4 = -0.0880682, \quad C_5 = 0.0194006, \quad C_6 = 0.0146485 \quad C_7 = -0.00582658.$$

Utilizing these values, the first–order MOHAM solution is constructed, which demonstrates both efficiency and reliability in approximating the given problem.

$$\begin{aligned}\widetilde{u}(x) \;=\; & 1. + 0.0624983x^2 + 0.0000213151x^3 - 0.000119261x^4 + 0.000369055x^5 \\ & -0.000671864x^6 + 0.000707815x^7 - 0.000369332x^8 + 0.0000125056x^9 \\ & +0.0000762941x^{10} - 0.0000248292x^{11}\end{aligned} \tag{32}$$

Table 2: Comparison between Exact and MOHAM solutions with absolute error.

x	Exact Solution	OHAM Absolute Error	MOHAM Absolute Error
0.2	1.0025	4.76×10^{-7}	3.46×10^{-9}
0.4	1.0100	249×10^{-7}	1.746×10^{-9}
0.6	1.0225	1.01×10^{-7}	1.37×10^{-9}
0.8	1.0400	6.14×10^{-8}	4.70×10^{-10}
1.0	1.0625	0.00	2.22×10^{-16}

The results in Table 2 demonstrate a remarkable finding when comparing OHAM and MOHAM. It is important to note that the OHAM results were obtained using a third–order approximation, whereas MOHAM achieved its results using only a first–order approximation. Despite this significant difference in approximation order, MOHAM provides much smaller absolute errors at all tested values of x. For instance, at $x = 0.2$, the error of OHAM is 4.76×10^{-7}, while MOHAM reduces this to 3.46×10^{-9}, representing an improvement of nearly two orders of magnitude. A similar pattern is observed across all points: at $x = 0.6$, OHAM yields an error of 1.01×10^{-7}, whereas MOHAM achieves 1.37×10^{-9}. Even at $x = 0.8$, where the error of OHAM is already small (6.14×10^{-8}), MOHAM still provides a lower error (4.70×10^{-10}). This comparison highlights that MOHAM achieves higher accuracy with fewer approximation terms, thereby reducing computational effort while ensuring stable and reliable convergence throughout the domain. Thus, the modified formulation of MOHAM outperforms OHAM not only in precision but also in efficiency, making it a superior method for solving such boundary value problems.

Conclusion

In this work, we have successfully applied MOHAM to singular two–point boundary value problems. The modification introduced in the auxiliary function and the homotopy construction has demonstrated significant improvements over the standard OHAM. Numerical experiments clearly show that MOHAM achieves higher accuracy even with a first–order approximation, whereas OHAM requires up to three terms to reach a comparable level of precision. This reduction in approximation order highlights the efficiency of MOHAM in minimizing computational effort while maintaining excellent reliability. The residual minimization strategy used ensures uniformly small errors in the domain, which confirms the robustness of the method. Moreover, MOHAM preserves the flexibility and simplicity of OHAM while enhancing convergence speed and stability. These results suggest that MOHAM provides an effective analytical tool for handling this type of equations that are otherwise challenging to solve. With its strong balance of accuracy and efficiency, MOHAM can serve as a powerful alternative to existing analytical and numerical methods. Future work may extend MOHAM to multistage frameworks and more complex nonlinear systems.

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