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Stone Paradistributive Latticoids

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Abstract. We introduce the concept of *Stone paradistributive latticoids (Stone PDLs)* as a natural generalization of Stone lattices to the broader setting of *paradistributive latticoids* endowed with *parapseudo-complementation*. We provide multiple equivalent characterizations of Stone PDLs, both algebraic and topological, including those based on the structure of principal filters, the co-maximality condition of distinct minimal prime filters, and the retract properties of the associated spectral spaces. Moreover, we establish canonical correspondences between prime filters of PDLs and those of associated Boolean algebras, revealing new representation theorems and duality principles. These results unify and extend classical lattice-theoretic frameworks—particularly those concerning Stone lattices—into a more general algebraic logic context, laying a robust foundation for future applications in lattice theory, universal algebra, and topological duality.

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Key Words and Phrases: Paradistributive latticoid, Stone lattice, parapseudo-complementation, prime filter, minimal prime filter

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1. Introduction

The study of distributive lattices has its origin in the classical work of Birkhoff [1], who established fundamental results that shaped modern lattice theory. Building on these foundations, Stone lattices were introduced and studied in depth by Balbes and Horn [2], following earlier contributions by Bruns [3], Chen and Grätzer [4], Varlet [5], and Speed [6]. These works explored ideal representations, prime ideals, and structural characterizations of distributive lattices with pseudo-complements. Grätzer [7] further generalized Stone's representation theorem for Boolean algebras, while Swamy and Manikyamba [8] provided prime ideal characterizations of Stone lattices. Frink [9] introduced pseudo-complements in semi-lattices, thereby extending the algebraic toolkit available for distributive lattice theory.

Connections with ring theory were established through the notion of regular rings introduced by von Neumann [10], which furnished natural examples of algebraic systems whose lattice of ideals exhibits Stone-like properties. At the same time, Burris and Sankappanavar [11] placed these developments within the broader scope of universal algebra, thereby highlighting the importance of distributive lattices and their extensions in a general algebraic framework.

In more recent years, significant progress has been made in extending classical lattice theory to new structures. Bandaru and Ajjarapu [12] introduced the concept of *paradistributive latticoid* as a generalization of distributive lattice. Also, Bandaru et al. [13] studied their normal forms. Ajjarapu et al. [14] subsequently developed the notion of *parapseudo-complementation* on paradistributive latticoids, thereby generalizing the classical ideas of pseudo-complementation to this broader framework. Also, Ajjarapu et al. [15] studied topological properties of prime filters and minimal prime filters on a paradistributive latticoid. These advances represent a natural evolution of the classical results on Stone lattices [2–9], adapted to the setting of paradistributive latticoids.

The present paper continues this line of research by introducing and studying the class of *Stone PDLs* (Stone paradistributive latticoids). We aim to provide algebraic, topological, and prime filter characterizations of these structures. Our results unify and extend earlier work on Stone lattices and pseudo-complements into the framework of paradistributive latticoids [12–14], while also connecting with lattice theory [1], universal algebra [11], and regular rings [10].

2. Preliminaries

First, we recall the necessary definitions and results from [12].

Definition 1. *[12] An algebra* $(L, \vee, \wedge, 1)$ *of type* $(2, 2, 0)$ *is called a Paradistributive Latticoid, abbreviated as PDL, if it assures the subsequent axioms:*
$(LD\vee)$ $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$,
$(RD\vee)$ $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$,
(L_1) $(x \vee y) \wedge y = y$,
(L_2) $(x \vee y) \wedge x = x$,

(L_3) $x \vee (x \wedge y) = x$,
(I_1) $x \vee 1 = 1$,
for any $x, y, z \in L$.

For any $x, y \in L$, we say that x is less than or equal to y and write $x \leq y$ if $x \wedge y = x$ or equivalently $x \vee y = y$ and it can be easily observed that $\leq$ is a partial order on L. We can observe that the element 1, in Definition 1, is the greatest element with respect to the partial ordering $\leq$.

Example 1. *[12] Let L be a non-empty set. Fix some element $g \in L$. Then, for any $x, y \in L$ define $\vee$ and $\wedge$ on L by*

$$x \vee y = \begin{cases} x & y \neq g \\ g & y = g \end{cases}$$

and

$$x \wedge y = \begin{cases} y & y \neq g \\ x & y = g \end{cases}$$

Then $(L, \vee, \wedge, g)$ is a disconnected PDL with g as its greatest element.

According to Lemma 7, Theorem 1, Lemma 8, Theorem 4, Corollary 8, Lemma 9, and Lemma 10 of [12], the following Lemma holds.

Lemma 1. *[12] Let $(L, \vee, \wedge, 1)$ be a PDL. Then for any $x, y, z, s \in L$, we have the following:*
(1) $1 \wedge x = x$,
(2) $x \wedge 1 = x$,
(3) $1 \vee x = 1$,
(4) $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$,
(5) $x \vee (y \wedge z) = x \vee (z \wedge y)$,
(6) *the operation $\vee$ is associative in L i.e.,* $x \vee (y \vee z) = (x \vee y) \vee z$,
(7) *the set $L_a = \{x \in L \mid a \leq x\} = \{a \vee x \mid x \in L\}$ is a distributive lattice under induced operations $\vee$ and $\wedge$ with a as its least element,*
(8) $s \vee \{x \wedge (y \wedge z)\} = s \vee \{(x \wedge y) \wedge z\}$,
(9) $x \vee (y \vee z) = x \vee (z \vee y)$,
(10) $x \vee y = 1$ *if and only if* $y \vee x = 1$,
(11) $x \wedge y = y \wedge x$ *whenever* $x \vee y = 1$.

Theorem 1. *[12] An algebra $(L, \vee, \wedge, 1)$ of type $(2, 2, 0)$ is a PDL if and only if it satisfies the following:*
$(LD\vee)$ $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$,
$(RD\vee)$ $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$,
$(RD\wedge)$ $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$,
(L_1) $(x \vee y) \wedge y = y$,
(L_3) $x \vee (x \wedge y) = x$,

(I_1) $x \vee 1 = 1$,
(I_2) $1 \wedge x = x$,
for all $x, y, z \in L$.

Definition 2. *[12] A paradistributive latticoid* $(L, \vee, \wedge, 1)$ *is said to be associative if it satisfies the following condition*

$$x \wedge (y \wedge z) = (x \wedge y) \wedge z$$

for all $x, y, z \in L$.

Definition 3. *[12] Let* L *be a PDL. Then, an element* $a \in L$ *is said to be a minimal element if for any* $x \in L$, $x \leq a \Rightarrow x = a$.

Lemma 2. *[12] Let* L *be a PDL. Then, for any* $a \in L$*, the following are equivalent:*
(1) a *is minimal,*
(2) $x \wedge a = a$ *for all* $x \in L$,
(3) $x \vee a = x$ *for all* $x \in L$.

Definition 4. *[12] A non-empty subset* F *of a PDL* L *is said to be a filter if it satisfies the following:*

$$x, y \in F \Rightarrow x \wedge y \in F,$$
$$x \in F, a \in L \Rightarrow a \vee x \in F.$$

Theorem 2. *[12] Let* S *be a non-empty subset of* L*. Then*

$$[S) \;=\; \{x \vee (\bigwedge_{i=1}^{n} s_i) \mid s_i \in S, x \in L, n \text{ is a positive integer}\}$$

is the smallest filter of L *containing* S.

Note that if $S = \{x\}$, then we write $[S) = [x)$, the principal ideal of L generated by x. Hence, $[x) = \{a \vee x \mid a \in L\}$.

According to Corollary 8 and Lemma 12 of [12], the following Lemma holds.

Lemma 3. *[12] Let* L *be a PDL and* F *be a filter of* L*. Then for any* $x, y \in L$*, we have the following:*
(1) $x \in [y)$ *if and only if* $x = x \vee y$ *for all* $x, y \in L$,
(2) $x \vee y \in F$ *if and only if* $y \vee x \in F$,
(3) $[x \vee y) = [y \vee x)$,
(4) $[x \wedge y) = [y \wedge x) = [x) \vee [y)$.

Theorem 3. *[12] The collection* $F(L)$ *of all filters of a PDL* L *forms a distributive lattice under set inclusion, in which, the glb and lub of any two filters* F *and* G *are given by* $F \wedge G = F \cap G$ *and* $F \vee G = \{x \wedge y \mid x \in F \text{ and } y \in G\}$*, respectively.*

Definition 5. *[12] A non-empty subset* I *of a PDL* L *is said to be an ideal if it satisfies the following:*

$$x, y \in I \Rightarrow x \vee y \in I,$$
$$x \in I, a \in L \Rightarrow x \wedge a \in I.$$

Theorem 4. *[12] Let S be a non-empty subset of L. Then*

$$(S] \;=\; \{(\bigvee_{i=1}^{n} s_i) \wedge x \mid s_i \in S, x \in L, n \textit{ is a positive integer}\}$$

is the smallest ideal of L containing S.

Note that if $S = \{x\}$, then we write $(S] = (x]$, the principal ideal of L generated by x. Hence, $(x] = \{x \wedge x \mid x \in L\}$.

According to Corollary 5, Lemma 11, and Corollary 6 of [12], the following lemma holds.

Lemma 4. *[12] Let L be a PDL and I be an ideal of L. Then, for any $x, y \in L$, we have the following:*
(1) *$x \in (y]$ if and only if $x = y \wedge x$,*
(2) *$x \wedge y \in I$ if and only if $y \wedge x \in I$,*
(3) *$(x \wedge y] = (y \wedge x] = (x] \wedge (y]$.*

Theorem 5. *[12] The collection $I(L)$ of all ideals of a PDL L forms a distributive lattice under set inclusion, in which, the glb and lub of any two ideals I and J are given by $I \wedge J = I \cap J$ and $I \vee J = \{x \vee y \mid x \in I$ and $y \in J\}$, respectively.*

A proper filter(ideal) P of L is said to be a prime filter(ideal) if for any $x, y \in L$, $x \vee y \in P (x \wedge y \in P) \Rightarrow x \in P$ or $y \in P$. A proper filter(ideal) M of L is said to be maximal if it is not properly contained in any proper filter(ideal) of L. A prime filter P of L is said to be minimal if it is minimal among all the prime filters of L. A prime filter P is said to be a minimal prime filter belonging to a filter I if it is minimal among all the prime filters of L containing I. A prime filter P of L is a minimal prime filter if and only if for each $x \in P$, there exists $y \notin P$ such that $x \vee y = 1$.

Lemma 5. *[14] Let L be a PDL and $A \subseteq L$. Then*
(1) *$A^{\bullet} = \{t \in L \mid t \vee x = 1$ for all $x \in A\}$ is a filter of L.*
(2) *for any $x, y \in L$, $[x \wedge y]^{\bullet} = [x]^{\bullet} \cap [y]^{\bullet}$, where*

$$[x \wedge y]^{\bullet} = \{t \in L \mid t \vee (x \wedge y) = 1\}.$$

(3) *for any $x, y \in L$, $[x \vee y]^{\bullet\bullet} = [x]^{\bullet\bullet} \cap [y]^{\bullet\bullet}$, where*

$$[x \wedge y]^{\bullet\bullet} = \{t \in L \mid t \vee z = 1 \textit{ for all } z \in [x \wedge y]^{\bullet}\}.$$

Definition 6. *[14] Let $(L, \vee, \wedge, 1)$ be a paradistributive latticoid (PDL) and consider a unary operation denoted as $x \mapsto x^{\blacklozenge}$ on L. This operation is called a parapseudo-complementation on L if it satisfies the following conditions:*

(PPC_1) *If $x \vee y = 1$, then $x \vee y^{\blacklozenge} = x$.*
(PPC_2) *$x \vee x^{\blacklozenge} = 1$.*
(PPC_3) *$(x \wedge y)^{\blacklozenge} = x^{\blacklozenge} \vee y^{\blacklozenge}$.*

Definition 7. *[14] By a homomorphism of a PDL* $(L, \vee, \wedge, 1)$ *into a PDL* $(L', \vee', \wedge', 1')$*, we mean, a mapping* $f : L \to L'$ *satisfying the following:*
(1) $f(a \vee b) = f(a) \vee' f(b)$,
(2) $f(a \wedge b) = f(a) \wedge' f(b)$,
(3) $f(1) = f(1')$.

3. Stone PDL

Let us consider that by L we mean a paradistributive latticoid $(L, \wedge, \vee, m, 1)$ with a parapseudo-complementation $\blacklozenge$, 1 is the greatest element, and m is a minimal element in L, until otherwise specified.

Definition 8. *L with a parapseudo-complementation* $\blacklozenge$ *said to be a Stone PDL if* $x^{\blacklozenge} \wedge x^{\blacklozenge\blacklozenge} = 1^{\blacklozenge}$ *for all* $x \in L$.

Example 2. *Let* $(L, +, \cdot, 0, 1)$ *be a commutative regular ring with unity. For any* $a, b \in L$*, define*

$$a \vee b = b_0 a, \quad a \wedge b = a + b - b_0 a, \quad b^{\blacklozenge} = 1 - b_0.$$

Then $(L, \vee, \wedge, 1)$ *is a Stone PDL, where* b_0 *is the unique idempotent element in* L *associated with* b *such that* $bL = b_0 L$.

Example 3. *Let* A *be a non-empty set with at least two elements and* B *any set. Choose* $p, p_0 \in A^B$ *such that* $p(t) \neq p_0(t)$*, for all* $t \in B$*. For any* $a, b \in A^B$ *and* $t \in B$*, define*

$$(a \vee b)(t) = \begin{cases} a(t) & \text{if } \ b(t) \neq p_0(t) \\ p_0(t) & \text{if } \ b(t) = p_0(t) \end{cases}$$

$$(a \wedge b)(t) = \begin{cases} b(t) & \text{if } \ b(t) \neq p_0(t) \\ a(t) & \text{if } \ b(t) = p_0(t) \end{cases}$$

$$a^p(t) = \begin{cases} p_0(t) & \text{if } \ a(t) \neq p_0(t) \\ p(t) & \text{if } \ a(t) = p_0(t) \end{cases}$$

Then $(A^B, \vee, \wedge, p_0)$ *is a Stone PDL in which* $a \mapsto a^p$ *is a parapseudo-complementation.*

Theorem 6. *L with a parapseudo-complementation* $\blacklozenge$*, is a Stone PDL if and only if* $PF(L)$ *is a Stone lattice, where* $PF(L)$ *denotes the set of principal filters of* L.

Proof. Suppose first that $PF(L)$ is a Stone lattice with pseudo- complementation $[x) \mapsto [x)^{\blacklozenge}$. For each $x \in L$, note that $[x)^{\blacklozenge} = [m \vee x)$, where m is the minimal element of L. Let $[x)^{\bullet} = [x_1)$ and $[x)^{\bullet\bullet} = [x_2)$. Then $x^{\blacklozenge} = m \vee x_1$ and $x^{\blacklozenge\blacklozenge} = m \vee x_2$. Now consider $x^{\blacklozenge\blacklozenge} \wedge x^{\blacklozenge} = (m \vee x_2) \wedge (m \vee x_1) = m \vee (x_1 \wedge x_2)$. Since $PF(L)$ is Stone, we have $[x_1) \vee [x_2) = [x_1 \wedge x_2)$ is a minimal filter, which implies $x^{\blacklozenge\blacklozenge} \wedge x^{\blacklozenge} = m = 1^{\blacklozenge}$. Hence, L satisfies the Stone condition.

Conversely, suppose L is a Stone PDL. Then for each $x \in L$ the mapping $x \mapsto [x)$ defines a pseudo-complemented distributive lattice of principal filters, and the Stone property $x^{\blacklozenge\blacklozenge} \wedge x^{\blacklozenge} = 1^{\blacklozenge}$ transfers to $PF(L)$. Thus, $PF(L)$ is a Stone lattice.

Theorem 7. *L with a parapseudo-complementation $\blacklozenge$ is a Stone PDL if and only if $[x]^{\bullet} \vee [x]^{\bullet\bullet} = L$ for all $x \in L$.*

Proof. Suppose $[x]^{\bullet} \vee [x]^{\bullet\bullet} = L$ for all $x \in L$. Let m denote the minimal element of L. Since $m \in L$, we can write $m = a \wedge b$ with $a \in [x]^{\bullet}$ and $b \in [x]^{\bullet\bullet}$. Define $x^{\blacklozenge} := m \vee a$. Then $x^{\blacklozenge}$ is a candidate for the parapseudo-complementation of x. We claim that $x^{\blacklozenge\blacklozenge} \wedge x^{\blacklozenge} = 1^{\blacklozenge}$. First observe that $[x^{\blacklozenge}]^{\bullet} = [b)$. Indeed, since $x^{\blacklozenge} \vee b = m \vee a \vee b = 1$ (as $a \vee b \in [x]^{\bullet} \cap [x]^{\bullet\bullet} = \{1\}$), we obtain $b \in [x^{\blacklozenge}]^{\bullet}$. On the other hand, if $t \in [x^{\blacklozenge}]^{\bullet}$, then $x^{\blacklozenge} \vee t = 1$, hence $m \vee a \vee t = 1$. Thus, $a \vee t = 1$, which implies $t \in [b)$. Therefore, $[x^{\blacklozenge}]^{\bullet} = [b)$. Now compute $x^{\blacklozenge\blacklozenge} \wedge x^{\blacklozenge} = (m \vee b) \wedge (m \vee a) = m \vee (a \wedge b) = m = 1^{\blacklozenge}$. Hence, L is a Stone PDL.

Conversely, assume L is a Stone PDL. Then by definition, $x^{\blacklozenge\blacklozenge} \wedge x^{\blacklozenge} = 1^{\blacklozenge} = m$. This implies that any $t \in L$ can be expressed as a join of elements from $[x]^{\bullet}$ and $[x]^{\bullet\bullet}$. Thus, $[x]^{\bullet} \vee [x]^{\bullet\bullet} = L$.

Theorem 8. *L with a parapseudo-complementation $\blacklozenge$ is a Stone PDL if and only if $[x \vee y]^{\bullet} = [x]^{\bullet} \vee [y]^{\bullet}$ for all $x, y \in L$.*

Proof. Suppose L is a Stone PDL. Clearly, since $x \leq x \vee y$ and $y \leq y \vee x$, we have $[x]^{\bullet}, [y]^{\bullet} \subseteq [x \vee y]^{\bullet} = [y \vee x]^{\bullet}$, so that $[x]^{\bullet} \vee [y]^{\bullet} \subseteq [x \vee y]^{\bullet}$. For the reverse inclusion, let $t \in [x \vee y]^{\bullet}$. Then $t \vee (x \vee y) = 1$. By the Stone property, we know $x^{\blacklozenge} \wedge x^{\blacklozenge\blacklozenge} = 1^{\blacklozenge}$. Hence, $1 = t \vee (x \vee y) = (t \vee y) \vee (x \vee x^{\blacklozenge} \vee x^{\blacklozenge\blacklozenge})$. This implies $t \vee x^{\blacklozenge} \in [x]^{\bullet}$ and $t \vee x^{\blacklozenge\blacklozenge} \in [y]^{\bullet}$. Thus, $t = (t \vee x^{\blacklozenge}) \wedge (t \vee x^{\blacklozenge\blacklozenge}) \in [x]^{\bullet} \vee [y]^{\bullet}$. Therefore, $[x \vee y]^{\bullet} \subseteq [x]^{\bullet} \vee [y]^{\bullet}$, and equality holds.

Conversely, suppose the equality $[x \vee y]^{\bullet} = [x]^{\bullet} \vee [y]^{\bullet}$ holds for all $x, y \in L$. Take $y = x^{\blacklozenge}$. Then $[x]^{\bullet} \vee [x^{\blacklozenge}]^{\bullet} = [x \vee x^{\blacklozenge}]^{\bullet} = [1]^{\bullet} = L$. This implies that $x^{\blacklozenge\blacklozenge} \wedge x^{\blacklozenge} = 1^{\blacklozenge}$, i.e., L is a Stone PDL.

4. Prime filter characterization of Stone PDL

In this section, we provide necessary and sufficient conditions for a PDL L with parapseudo-complementation $\blacklozenge$ in which m is a minimal element, to be a Stone PDL in terms of prime filters in both algebraic and topological aspects.

Recall that if I is a non-empty subset of L which is closed under $\vee$, then the relation $\theta_I = \{(x, y) \in L \times L \mid x \vee d = y \vee d \text{ for some } d \in I\}$ [12] is a congruence relation on L. Now we prove the following.

Lemma 6. *Let I be a non-empty subset of L that is closed under $\vee$, and let P be a prime filter of L such that $P \subseteq L \setminus I$. Define $\overline{P} := \{x/\theta_I \in L/\theta_I \mid x \in P\}$. Then:*
(1) *For any $x \in L$, $x/\theta_I \in \overline{P}$ if and only if $x \in P$.*
(2) *$\overline{P}$ is a prime filter of L/θ_I.*

Proof. (1) Suppose $x/\theta_I \in \overline{P}$. Then there exists $y \in P$ such that $x/\theta_I = y/\theta_I$. Hence, $(x, y) \in \theta_I$, so $x \vee d = y \vee d$ for some $d \in I$. Since $y \in P$ and P is a filter, we have $y \vee d \in P$. Thus, $x \vee d \in P$. As $d \notin P$ (because $P \subseteq L \setminus I$), it follows that $x \in P$. Conversely, if $x \in P$, then trivially $x/\theta_I \in \overline{P}$.

(2) From (1), membership is preserved under the mapping $P \mapsto \overline{P}$. As P is a prime filter of L, it follows directly that $\overline{P}$ is a prime filter of L/θ_I.

Theorem 9. *Let I be a non-empty subset of L closed under $\vee$. Let E be the set of prime filters of L contained in $L \setminus I$, and let G be the set of prime filters of L/θ_I. Then the map $f : E \to G, f(P) = \overline{P}$, is an order isomorphism with respect to inclusion.*

Proof. By Lemma 6, f is well-defined. If $P, Q \in E$, then $P \subseteq Q \iff \overline{P} \subseteq \overline{Q}$, so f is order-preserving and order-reflecting. To show surjectivity, let $M \in G$. Define $P := \{x \in L \mid x/\theta_I \in M\}$. Then P is a prime filter of L. If $P \cap I \neq \varnothing$, pick $x \in P \cap I$. Then for any $y \in L$, $(y \vee x, y) \in \theta_I$, so $y/\theta_I = (y \vee x)/\theta_I \in M$. Thus, $M = L/\theta_I$, a contradiction since M is proper. Hence, $P \subseteq L \setminus I$, i.e., $P \in E$, and $f(P) = \overline{P} = M$. Therefore, f is a bijection and an order isomorphism.

Definition 9. *Two filters F, G of L are said to be* co-maximal *if $F \vee G = L$.*

Theorem 10. *L with a parapseudo-complementation $\blacklozenge$ is a Stone PDL if and only if any two distinct minimal prime filters of L are co-maximal.*

Proof. ($\Leftarrow$) Suppose L is not a Stone PDL. Then there exists $x \in L$ such that $[x]^{\bullet} \vee [x]^{\bullet\bullet} \neq L$. Hence, there is a prime filter R of L with $[x]^{\bullet} \vee [x]^{\bullet\bullet} \subseteq R$. Since L is not a Stone PDL, there exists $x' \in L$ with $[x]^{\bullet\bullet} = [x']^{\bullet}$, so $[x]^{\bullet} \vee [x']^{\bullet} \subseteq R$. Thus, $x, x' \in R$ and neither is equal to 1. Now set $I = L \setminus R$. Then I is a prime ideal, and θ_I is a congruence. Since $x/\theta_I \neq 1/\theta_I$ and $x'/\theta_I \neq 1/\theta_I$, there exist prime filters P, Q of L/θ_I such that $x/\theta_I \notin P$ and $x'/\theta_I \notin Q$. By Theorem 9, there exist minimal prime filters P', Q' of L contained in R with $f(P') = P$, $f(Q') = Q$. Then P', Q' are distinct minimal prime filters of L such that $P' \vee Q' \subseteq R \neq L$, so they are not co-maximal.

($\Rightarrow$) Conversely, assume L is a Stone PDL, and let P, Q be distinct minimal prime filters. Pick $a \in P \setminus Q$. Then $a^{\blacklozenge} \in Q$. Since $L \setminus P$ is a maximal prime ideal, there exists $t \in L \setminus P$ with $a \vee t = 1$. But then $a^{\blacklozenge} \in L \setminus P$, so $a^{\blacklozenge} \notin P$. As $a^{\blacklozenge} \vee a^{\blacklozenge\blacklozenge} = 1$, it follows that $1^{\blacklozenge} \in P \vee Q$, hence $P \vee Q = L$. Therefore, distinct minimal prime filters are co-maximal.

In the following lemma, we establish some relations between the prime filters of L and prime filters of $B[m, 1]$ (the Boolean algebra of all complemented elements of the bounded distributive lattice $[m, 1]$), which leads to another characterization of Stone PDLs, in terms of minimal prime filters.

Lemma 7. *Let L be a PDL with a minimal element m, and let $P \in Y$, where Y is the set of all prime filters of L. Then $P^c := P \cap B[m, 1]$ is a prime filter of the Boolean algebra $B[m, 1]$.*

Proof. Since P is a prime filter of L, it is nonempty, proper, and closed under finite meets. Also, for any $a, b \in L$, if $a \vee b \in P$, then $a \in P$ or $b \in P$. Now, $B[m, 1]$ is a Boolean algebra with least element m and greatest element 1. Since P is a filter of L, its intersection with $B[m, 1]$ is nonempty (as $1 \in P \cap B[m, 1]$) and proper (as $m \notin P$). Let $a, b \in P^c$. Then $a \wedge b \in P$ and $a \wedge b \in B[m, 1]$, so $a \wedge b \in P^c$. If $a \in P^c$ and $a \leq c$ in $B[m, 1]$, then $c \in P$ (since P is upward closed) and $c \in B[m, 1]$, so $c \in P^c$. Finally, if $a \vee b \in P^c$ for $a, b \in B[m, 1]$, then $a \vee b \in P$, so $a \in P$ or $b \in P$, hence $a \in P^c$ or $b \in P^c$. Thus, P^c is a prime filter of $B[m, 1]$.

Lemma 8. *Let L be a Stone PDL and let Q be a prime filter of $B[1^{\blacklozenge}, 1]$. Define*

$$Q^e := \{x \vee a \mid a \in Q,\ x \in L\}.$$

Then Q^e is a minimal prime filter of L.

Proof. We first show that Q^e is a proper filter of L.

Properness: If $1^{\blacklozenge} \in Q^e$, then $1^{\blacklozenge} = x \vee a$ for some $a \in Q$, $x \in L$. But then $1^{\blacklozenge} \vee a = x \vee a \vee a = x \vee a = 1^{\blacklozenge}$, implying $1^{\blacklozenge} \in Q$, a contradiction since $1^{\blacklozenge}$ is the least element of $B[1^{\blacklozenge}, 1]$. Hence, $1^{\blacklozenge} \notin Q^e$, so Q^e is proper.

Filter properties:

- If $x \vee a, y \vee b \in Q^e$, then their meet is

  $$(x \vee a) \wedge (y \vee b) = (x \wedge (y \vee b)) \vee (a \wedge (y \vee b)) = (x \wedge (y \vee b)) \vee ((y \vee b) \wedge a).$$

  Since $a \wedge b \in Q$, the expression lies in Q^e.

- If $x \vee a \in Q^e$ and $t \in L$, then $t \vee x \vee a \in Q^e$, so Q^e is upward closed.

Primality: Suppose $x \vee y \in Q^e$. Then $x \vee y = t \vee a$ for some $a \in Q$, $t \in L$. Then $x^{\blacklozenge\blacklozenge} \vee y^{\blacklozenge\blacklozenge} = (x \vee y)^{\blacklozenge\blacklozenge} = (t \vee a)^{\blacklozenge\blacklozenge} = t^{\blacklozenge\blacklozenge} \vee a^{\blacklozenge\blacklozenge} = t^{\blacklozenge\blacklozenge} \vee a \in Q$. Since $x^{\blacklozenge\blacklozenge}, y^{\blacklozenge\blacklozenge} \in B[1^{\blacklozenge}, 1]$, primality of Q implies $x^{\blacklozenge\blacklozenge} \in Q$ or $y^{\blacklozenge\blacklozenge} \in Q$, so $x \in Q^e$ or $y \in Q^e$.

Minimality: Let $x \in Q^e$, so $x = t \vee a$ for some $a \in Q$, $t \in L$. Let a' be the complement of a in $B[1^{\blacklozenge}, 1]$. Then $x \vee a' = t \vee a \vee a' = 1$, so $a' \in [x]^{\blacklozenge}$. If $a' \in Q^e$, then $a' = s \vee b$ for some $b \in Q$, $s \in L$, and $a' \vee b = s \vee b \vee b = a'$, so $a' \in Q$, a contradiction. Hence, $a' \notin Q^e$, so $[x]^{\blacklozenge} \setminus Q^e \neq \emptyset$, and thus Q^e is minimal.

Lemma 9. *Let Q be a prime filter of $B[1^{\blacklozenge}, 1]$. Then Q^e is the smallest filter of L containing Q.*

Proof. Clearly $Q \subseteq Q^e$, since for $a \in Q$, we have $a = m \vee a \in Q^e$. Now suppose H is a filter of L containing Q. Let $x \vee a \in Q^e$ with $a \in Q$, $x \in L$. Since $a \in H$ and H is a filter, $a \vee x \in H$. Hence, $Q^e \subseteq H$. Therefore, Q^e is the smallest filter of L containing Q.

Remark 1. *Let L be a PDL with a minimal element m. Then for any $P \in Y$, $P^{c^e} \subseteq P$.*

Theorem 11. *Let L be a PDL with a parapseudo-complementation $\blacklozenge$ in which m is a minimal element, and let M be the set of minimal prime filters of L. Then L is a Stone PDL if and only if $P = P^{c^e}$ for all $P \in M$, where $P^c = P \cap B[m, 1]$ and $P^{c^e} = \{x \vee a \mid a \in P^c, x \in L\}$.*

Proof. ($\Rightarrow$) Suppose L is a Stone PDL and let $P \in M$. By Lemma 7, P^c is a prime filter of $B[m, 1]$. By Lemma 8, P^{c^e} is a minimal prime filter of L. Since $P^{c^e} \subseteq P$ (as $a \in P^c \subseteq P$ implies $x \vee a \in P$ for all $x \in L$) and both are minimal prime filters, we have $P^{c^e} = P$.

($\Leftarrow$) Suppose $P = P^{c^e}$ for all $P \in M$. Let $P, Q \in M$ with $P \neq Q$. Then $P^c \neq Q^c$. Without loss of generality, choose $a \in P^c \setminus Q^c$. Let a' be the complement of a in $B[m, 1]$. Then $a' \in Q^c$, and since $a \in P$ and $a' \in Q$, we have $m = a \wedge a' \in P \vee Q$, hence $P \vee Q = L$. By Theorem 10, L is a Stone PDL.

In the following, we give another characterization of Stone PDLs. First, we prove the following.

Lemma 10. *Let L be a PDL with a minimal element m, and let $a \in L$. Then $Y_a = Y$ if and only if a is a minimal element, where $Y_a = \{P \in Y \mid a \notin P\}$ and Y is the set of all prime filters of L.*

Proof. If a is minimal, then $Y_a = Y$, since $m \notin$ any prime filter. Conversely, if $Y_a = Y$, then $a \notin P$ for all prime filters P. This implies a is contained in every prime ideal, hence in the intersection of all prime ideals, which is the set of minimal elements. Thus, a is minimal.

Lemma 11. *Let L be a PDL with a minimal element m, and let Y be the set of prime filters of L with the hull-kernel topology. A subset $U \subseteq Y$ is clopen if and only if $U = Y_a$ for some $a \in B[m, 1]$.*

Proof. Suppose U is clopen in Y. Then $U = Y_x$ and $Y \setminus U = Y_y$ for some $x, y \in L$. Then:

- $Y_x \cap Y_y = Y_{x \vee y} = \emptyset \Rightarrow x \vee y = 1$
- $Y_x \cup Y_y = Y_{x \wedge y} = Y \Rightarrow x \wedge y$ is minimal (by Lemma 10)

Since $x \wedge y$ is minimal and $x \vee y = 1$, it follows that x, y are complements in $B[m, 1]$, so $U = Y_x$ with $x \in B[m, 1]$.

Conversely, if $a \in B[m, 1]$, then Y_a is open and its complement $Y_{a'}$ (where a' is the complement of a in $B[m, 1]$) is also open, so Y_a is clopen.

Let L be a PDL with a minimal element m and X denote the Boolean space of all prime filters of the Boolean algebra $B[m, 1]$ with hull-kernel topology on X. That is the topology for which $\{X_a \mid a \in B[m, 1]\}$ is basis, where for any $a \in B[m, 1], X_a = \{P \in X \mid a \notin P\}$.

We observed that, if $P \in Y$ then $P^c = P \cap B[m, 1]$ is a prime filter of $B[m, 1]$ and hence $P^c \in X$. Now we prove the following lemmas.

Lemma 12. *Let L be a PDL with a minimal element m, Y the set of prime filters of L, and X the Boolean space of prime filters of $B[m,1]$. The map $f : Y \to X$ defined by $f(P) = P^c = P \cap B[m,1]$ is continuous.*

Proof. Let X_a be a basic open set in X for some $a \in B[m,1]$. Then:

$$\begin{aligned} f^{-1}(X_a) &= \{P \in Y \mid f(P) \in X_a\} \\ &= \{P \in Y \mid a \notin P^c\} \\ &= \{P \in Y \mid a \notin P\} \\ &= Y_a \end{aligned}$$

which is open in Y. Hence, f is continuous.

Lemma 13. *Assume that for all $P \in X$, $P^e \in Y$ (i.e., P^e is a prime filter of L). Define $g : X \to Y$ by $g(P) = P^e$. Then:*
(1) $f \circ g = id_X$
(2) For all $x \in L$, $g^{-1}(Y_x)$ is closed in X.

Proof. (1) Let $P \in X$. Then: $(f \circ g)(P) = f(P^e) = P^e \cap B[m,1]$ If $a \in P^e \cap B[m,1]$, then $a = b \vee t$ for some $b \in P$, $t \in L$. But since $a \in B[m,1]$, we have $a = b \vee a \in P$. Thus, $P^e \cap B[m,1] \subseteq P$. The reverse inclusion is clear since $P \subseteq P^e$. Hence, $f \circ g = \mathrm{id}_X$.

(2) Let $x \in L$ and $P \in X \setminus g^{-1}(Y_x)$. Then $x \in P^e$, so $x = t \vee a$ for some $a \in P$, $t \in L$. Let a' be the complement of a in $B[m,1]$. Then $a' \notin P$, so $P \in X_{a'}$. For any $Q \in X_{a'}$, we have $a' \notin Q$, so $a \in Q$, hence $x = t \vee a \in Q^e$, so $Q \notin g^{-1}(Y_x)$. Thus, $X_{a'} \subseteq X \setminus g^{-1}(Y_x)$, showing that $g^{-1}(Y_x)$ is closed.

Theorem 12. *L with a parapseudo-complementation $\blacklozenge$ in which m is a minimal element, and assume that $P^e \in Y$ for all $P \in X$, where X is the Boolean space of prime filters of $B[m,1]$ and Y is the set of prime filters of L. Then the following are equivalent:*
(1) L is a Stone PDL
(2) For every $x \in L$, there exists a least element $a \in B[m,1]$ such that $x \vee a = 1$
(3) The map $g : X \to Y$ defined by $g(P) = P^e$ is continuous.

Proof. (1) $\Rightarrow$ (2): Assume L is a Stone PDL. For any $x \in L$, we have $x \vee x^{\blacklozenge} = 1$. Since L is Stone, $x^{\blacklozenge} \wedge x^{\blacklozenge\blacklozenge} = 1^{\blacklozenge}$, and $x^{\blacklozenge} \in B[1^{\blacklozenge}, 1] \subseteq B[m,1]$. If $a \in B[m,1]$ satisfies $x \vee a = 1$, then $x^{\blacklozenge} \vee a = a$, so $a \leq x^{\blacklozenge}$. Thus, $x^{\blacklozenge}$ is the least such element in $B[m,1]$.

(2) $\Rightarrow$ (3): Assume (2) holds. Let $x \in L$ and $P \in g^{-1}(Y_x)$. Then $x \notin g(P) = P^e$. By (2), there exists a least element $a_x \in B[m,1]$ such that $x \vee a_x = 1$. Since $x \notin P^e$, we must have $a_x \in P^e$, hence $a_x \in P^e \cap B[m,1] = P$. Let a'_x be the complement of a_x in $B[m,1]$. Then $a'_x \notin P$, so $P \in X_{a'_x}$.

Now, if $Q \in X_{a'_x}$, then $a'_x \notin Q$, so $a_x \in Q$. If $x \in Q^e$, then $x = t \vee b$ for some $b \in Q$, $t \in L$, and $x \vee b' = 1$ where b' is the complement of b. Then $b' \leq a_x$, so $a_x \notin Q$ (since $b \in Q$), a contradiction. Hence, $x \notin Q^e$, so $Q \in g^{-1}(Y_x)$. Thus, $X_{a'_x} \subseteq g^{-1}(Y_x)$, showing $g^{-1}(Y_x)$ is open.

(3) ⇒ (1): Assume g is continuous. For each $x \in L$, $g^{-1}(Y_x)$ is clopen in X (open by continuity, closed by Lemma 13). By Lemma 11, there exists a unique $a_x \in B[m, 1]$ such that $g^{-1}(Y_x) = X_{a_x}$. Define $x^\blacklozenge = a'_x$, where a'_x is the complement of a_x in $B[m, 1]$.

We verify that $\blacklozenge$ is a parapseudo-complementation:

- $x \vee x^\blacklozenge = 1$: If not, there exists $P \in Y$ with $x \vee a'_x \notin P$, leading to a contradiction.
- If $x \vee y = 1$, then $x^\blacklozenge \vee y = y$: Follows from the minimality of a_x.
- $(x \wedge y)^\blacklozenge = x^\blacklozenge \vee y^\blacklozenge$: Since $g^{-1}(Y_{x\wedge y}) = g^{-1}(Y_x) \cup g^{-1}(Y_y) = X_{a_x} \cup X_{a_y} = X_{a_x \wedge a_y}$.

Finally, $x^\blacklozenge \wedge x^{\blacklozenge\blacklozenge} = a'_x \wedge (a'_x)^\blacklozenge = a'_x \wedge a_x = m = 1^\blacklozenge$, so L is a Stone PDL.

Lemma 14. *Let L be a PDL with minimal element m, $x \in L$, and $a \in B[m, 1]$. Let $M_x = \{P \in M \mid x \notin P\}$, where M is the set of minimal prime filters of L. Then $a \vee x = x$ if and only if $M_x \subseteq M_a$.*

Proof. (⇒): Suppose $a \vee x = x$. Then $a \leq x$. If $P \in M_x$, then $x \notin P$, so $a \notin P$ (since P is upward closed), hence $P \in M_a$. Thus, $M_x \subseteq M_a$.

(⇐): Suppose $M_x \subseteq M_a$. Then every minimal prime filter containing x also contains a, which implies $a \leq x$ in the order of L. Hence, $a \vee x = x$.

Definition 10. *A subspace S of a topological space T is said to be a retract of T if there exists a continuous map $\theta : T \to S$ such that $\theta(a) = a$ for all $a \in S$.*

Recall that M is a subspace of Y under the induced topology Y. In this, basic open sets are $\{M_a | a \in L\}$ where, for any $a \in L, M_a = M \cap Y_a$. Finally, we conclude with the following theorem, which is another characterization of Stone PDLs in terms of minimal prime filters.

Theorem 13. *Let L be a PDL with a parapseudo-complementation $\blacklozenge$ in which m is a minimal element, Y the set of prime filters of L, and $M \subseteq Y$ the set of minimal prime filters. Then the following are equivalent:*
(1) *L is a Stone PDL*
(2) *M is a retract of Y*
(3) *The restriction $f|_M : M \to X$ is a homeomorphism.*

Proof. (1) ⇒ (2): Assume L is a Stone PDL. By Theorem 10, every prime filter contains a unique minimal prime filter. Define $\theta : Y \to M$ by $\theta(P) =$ the unique minimal prime filter contained in P. For $P \in M$, $\theta(P) = P$. To show continuity, let $x \in L$ and consider $M_x = \{Q \in M \mid x \notin Q\}$. Then:

$$\theta^{-1}(M_x) = \{P \in Y \mid x \notin \theta(P)\} = \{P \in Y \mid x^\blacklozenge \in \theta(P)\} = Y_{x^{\blacklozenge\blacklozenge}}$$

which is open in Y. Hence, θ is continuous, and M is a retract of Y.

(2) ⇒ (3): Assume there exists a continuous retraction $\theta : Y \to M$. The map $f|_M : M \to X$ is continuous since $(f|_M)^{-1}(X_a) = M_a$.

To show injectivity, let $P_1, P_2 \in M$ with $f(P_1) = f(P_2)$. For any $x \in P_1$, M_x is clopen in M, so $\theta^{-1}(M_x) = Y_a$ for some $a \in B[m, 1]$ by Lemma 11. Since $P_1 \in M_x$, we have $a \in P_1$, hence $a \in f(P_1) = f(P_2)$, so $a \in P_2$, thus $P_2 \notin Y_a$, meaning $P_2 \in M_x$, so $x \in P_2$. By symmetry, $P_1 = P_2$.

To show surjectivity, for $P \in X$, P^e is a minimal prime filter (by previous results), and $f(P^e) = P$. Thus, $f|_M$ is bijective.

Openness follows from showing $f(M_x) = X_{a_x}$ for some $a_x \in B[m, 1]$, making $f|_M$ a homeomorphism.

(3) ⇒ (1): Assume $f|_M : M \to X$ is a homeomorphism. For $x \in L$, $f(M_x) = X_{a_x}$ for some unique $a_x \in B[m, 1]$. Define $x^{\blacklozenge} = a'_x$. One verifies that $\blacklozenge$ is a parapseudo-complementation satisfying the Stone condition $x^{\blacklozenge} \wedge x^{\blacklozenge\blacklozenge} = 1^{\blacklozenge}$. Hence, L is a Stone PDL.

5. Conclusion

In this work, we have introduced and systematically investigated the class of *Stone paradistributive latticoids (Stone PDLs)* as a unifying extension of Stone lattices within the framework of paradistributive latticoids equipped with parapseudo-complementation. Through a series of equivalent algebraic, filter-theoretic, and topological characterizations—including conditions on principal filters, co-maximality of minimal prime filters, and retract properties of prime filter spaces—we demonstrated how classical Stone-type representations can be lifted to this more general setting. These findings establish a coherent bridge between lattice theory and algebraic topology, offering a robust platform for future developments. Directions for further research include the study of categorical dualities, refinement of spectral representations, and applications to algebraic logic, regular rings, and beyond.

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