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Nonlocal Fractional Hahn Integral Conditions in Sequential Nonlinear Integro-Difference Equations: Existence and Stability Perspectives

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Abstract. In this paper, we investigate the existence, uniqueness, and stability of solutions for a class of sequential fractional Hahn integro-difference boundary value problems. To facilitate the analysis, several key properties of the fractional Hahn integral are derived and utilized as computational tools. The considered problem involves a combination of three distinct fractional Hahn difference operators together with two fractional Hahn integrals of varying orders, which provides a richer framework than existing studies. By applying both the Banach fixed point theorem and the Schauder fixed point theorem, we establish rigorous conditions ensuring the existence and uniqueness of solutions. Furthermore, we demonstrate the Hyers–Ulam stability of the proposed model, highlighting its robustness under perturbations. An illustrative example is also provided to confirm the effectiveness and applicability of the theoretical results.

2020 Mathematics Subject Classifications: 34K10, 39A10, 39A11, 39A13, 39A70

Key Words and Phrases: Fractional Hahn integral, Riemann–Liouville fractional Hahn difference, boundary value problems, existence, Hyers–Ulam stability

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Introduction

Quantum calculus focuses on calculus without the concept of limits, addressing a class of non-differentiable functions. Quantum operators play a significant role in various mathematical areas, including hypergeometric series, complex analysis, orthogonal polynomials,

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combinatorics, hypergeometric functions, and the calculus of variations. Additionally, quantum calculus has practical applications in fields such as quantum mechanics and particle physics [1]-[12].

The Hahn difference operator was introduced by Wolfgang Hahn in 1949 [13] as a unification of two significant operators in the study of difference calculus: the forward difference operator and the Jackson q-difference operator. Hahn's work aimed to generalize and extend the applicability of these operators, providing a powerful tool for analyzing sequences and functions within the framework of q-calculus and difference equations. This operator has since become a fundamental concept in the study of discrete mathematics and mathematical analysis. The Hahn difference operator is define by

$$D_{q,\omega}f(t) = \frac{f(qt+\omega) - f(t)}{t(q-1)+\omega}, \quad t \neq \omega_0 := \frac{\omega}{1-q}.$$

We point out that

$$D_{q,\omega}f(t) = \Delta_\omega f(t) \text{ whenever } q = 1, \qquad D_{q,\omega}f(t) = D_q f(t) \text{ whenever } \omega = 0$$

$$\text{and} \quad D_{q,\omega}f(t) = f'(t) \text{ whenever } q = 1, \omega \to 0.$$

The Hahn difference operator has been employed in the study of families of orthogonal polynomials and in addressing various approximation problems, see [14]-[16].

In 2009, Aldwoah [17]-[18] proposed the right inverse of the Hahn difference operator. This operator is expressed in terms of the Jackson q-integral, which incorporates the right inverse of D_q [19] and the Nörlund sum, which involves the right inverse of Δ_ω [19].

In 2010, Malinowska and Torres [20]-[21] introduced the Hahn quantum variational calculus. Subsequently, in 2013, Malinowska and Martins [22] presented the generalized transversality conditions for this calculus. Furthermore, Hamza and Ahmed [23]- [24] developed the theory of linear Hahn difference equations. These authors also examined the existence and uniqueness of solutions for initial value problems related to Hahn difference equations by utilizing the method of successive approximations. Additionally, they established Gronwall's and Bernoulli's inequalities within the framework of the Hahn difference operator and explored the mean value theorems associated with this calculus. In 2016, Hamza and Makharesh [25] investigated Leibniz's rule and Fubini's theorem in the context of the Hahn difference operator. In the same year, Sitthiwirattham [26] conducted a study on the nonlocal boundary value problem (BVP) for nonlinear Hahn difference equations.

In 2010, $\check{C}erm\acute{a}k$ and Nechv$\acute{a}$tal [27] introduced the fractional (q,h)-difference operator and the fractional (q,h)-integral for $q > 1$. Subsequently, in 2011, $\check{C}erm\acute{a}k$, Kisela and Nechv$\acute{a}$tal [28] investigated linear fractional difference equations involving discrete Mittag-Leffler functions for $q > 1$. During the same period, Rahmat [29]-[30] proposed the (q,h)-Laplace transform along with several (q,h)-analogues of integral inequalities on discrete time scales for $q > 1$. In 2016, Du et al. [31] conducted a study on the monotonicity and convexity for nabla fractional (q,h)-difference for $q > 0,\ q \neq 1$. It is worth noting that since fractional Hahn operators requires the condition $0 < q < 1$, the aforementioned opertors are not classified as fractional Hahn operators. Recently, the fractional Hahn operators have been introduced by Brikshavana and Sitthiwirattham [32]. Several research papers focus on boundary value problems (BVPs) for fractional Hahn difference equations, such as [33]-[38].

In this paper, we aim to deepen our understanding of fractional Hahn operators by examining the BVP associated with fractional Hahn difference equations. Specifically, we focus on the Riemann-Liouville fractional Hahn integral boundary condition for a sequential fractional Hahn integro difference equation of the form

$$\begin{aligned} &D^{\alpha}_{q,\omega}D^{\beta}_{q,\omega}u(t) = F\left[t, u(t), \Psi^{\gamma}_{q,\omega}u(t), D^{\nu}_{q,\omega}u(t)\right], \quad t \in I^{T}_{q,\omega}, \\ &u(\omega_0) = u(T) \\ &\mathcal{I}^{\theta}_{q,\omega}g(\eta)u(\eta) = \varphi(u), \quad \eta \in I^{T}_{q,\omega} - \{\omega_0, T\} \end{aligned} \tag{0.1}$$

where $[\omega_0, T]_{q,\omega} := \{q^k T + \omega[k]_q : k \in \mathbb{N}_0\} \cup \{\omega_0\}$; $0 < q < 1, \omega > 0$; $\alpha, \beta, \gamma, \nu, \theta \in (0, 1]$; $F \in C\big([\omega_0, T]_{q,\omega} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}\big)$ and $g \in C\big([\omega_0, T]_{q,\omega}, \mathbb{R}^+\big)$ are given functions; $\varphi : C\big([\omega_0, T]_{q,\omega}, \mathbb{R}\big) \to \mathbb{R}$ are given functionals; and for $\phi \in C\big([\omega_0, T]_{q,\omega} \times [\omega_0, T]_{q,\omega}, [0, \infty)\big)$, we define an operator of the (q, ω)-integral of the product of functions ϕ and u as

$$\left(\Psi^{\gamma}_{q,\omega}u\right)(t) := \left(\mathcal{I}^{\gamma}_{q,\omega}\phi\, u\right)(t) = \frac{1}{\Gamma_q(\gamma)}\int_{\omega_0}^{t}\left(t - \sigma_{q,\omega}(s)\right)_{q,\omega}^{\underline{\gamma-1}}\phi(t, s)\, u(s)\, d_{q,\omega}s. \tag{0.2}$$

In Section 2, we present the foundational definitions, properties, and lemmas that serve as the basis for this study. In Section 3, we demonstrate the existence results of problem (0.1) and (0.2), respectively. Specifically, we establish the existence and uniqueness of a solution using the Banach Fixed Point Theorem, and prove the existence of at least one solution through the application of the Schauder Fixed Point Theorem. In Section 4, the stability of our problem is also studied based on Hyers-Ulam stability analysis. Finally, illustrative examples are provided in the concluding section to highlight the applicability of our results.

1. Preliminaries

In this section, we suggest some notations, definitions, and lemmas which are used in the main results. Let $q \in (0, 1)$, $\omega > 0$ and define

$$[n]_q := \frac{1 - q^n}{1 - q} = q^{n-1} + ... + q + 1 \qquad \text{and} \qquad [n]_q! := \prod_{k=1}^{n}\frac{1 - q^k}{1 - q}, \qquad n \in \mathbb{R}.$$

The q-analogue of the power function $(a - b)_q^{\underline{n}}$ with $n \in \mathbb{N}_0 := [0, 1, 2, ...]$ is defined by

$$(a - b)_q^{\underline{0}} := 1, \qquad (a - b)_q^{\underline{n}} := \prod_{k=0}^{n-1}(a - bq^k), \qquad a, b \in \mathbb{R}.$$

The q,ω-analogue of the power function $(a-b)_{q,\omega}^{\underline{n}}$ with $n \in \mathbb{N}_0 := [0,1,2,...]$ is defined by

$$(a-b)_{q,\omega}^{\underline{0}} := 1, \qquad (a-b)_{q,\omega}^{\underline{n}} := \prod_{k=0}^{n-1}\left[a-(bq^k+\omega[k]_q)\right], \qquad a,b \in \mathbb{N}.$$

In general, for $\alpha \in \mathbb{R}$, we define

$$(a-b)_q^{\underline{\alpha}} = a^{\alpha}\prod_{n=0}^{\infty}\frac{1-\left(\frac{b}{a}\right)q^n}{1-\left(\frac{b}{a}\right)q^{\alpha+n}},\ a \neq 0,$$

$$(a-b)_{q,\omega}^{\underline{\alpha}} = (a-\omega_0)^{\alpha}\prod_{n=0}^{\infty}\frac{1-\left(\frac{b-\omega_0}{a-\omega_0}\right)q^n}{1-\left(\frac{b-\omega_0}{a-\omega_0}\right)q^{\alpha+n}} = \Big((a-\omega_0)-(b-\omega_0)\Big)_q^{\underline{\alpha}},\ a \neq \omega_0.$$

We note that, $a_q^{\underline{\alpha}} = a^{\alpha}$ and $(a-\omega_0)_{q,\omega}^{\underline{\alpha}} = (a-\omega_0)^{\alpha}$ and use the notation $(0)_q^{\underline{\alpha}} = (\omega_0)_{q,\omega}^{\underline{\alpha}} = 0$ for $\alpha > 0$. The q-gamma and q-beta functions are defined by

$$\begin{aligned}\Gamma_q(x) &:= \frac{(1-q)_q^{\underline{x-1}}}{(1-q)^{x-1}}, \qquad x \in \mathbb{R}\setminus\{0,-1,-2,...\},\\ B_q(x,s) &:= \int_0^1 t^{x-1}(1-qt)_q^{\underline{s-1}}d_qt = \frac{\Gamma_q(x)\Gamma_q(s)}{\Gamma_q(x+s)}.\end{aligned}$$

Definition 1. *For* $q \in (0,1)$, $\omega > 0$ *and* f *defined on an interval* $I \subseteq \mathbb{R}$ *which containing* $\omega_0 := \frac{\omega}{1-q}$, *the Hahn difference of* f *is defined by*

$$D_{q,\omega}f(t) = \frac{f(qt+\omega)-f(t)}{t(q-1)+\omega} \quad \textit{for} \;\; t \neq \omega_0,$$

and $D_{q,\omega}f(\omega_0) = f'(\omega_0)$. *Providing that* f *is differentiable at* ω_0, *we call* $D_{q,\omega}f$ *the* q,ω-*derivative of* f, *and say that* f *is* q,ω-*differentiable on* I.

Remarks We give some properties for the Hahn difference as follows.

(1) $D_{q,\omega}[f(t)+g(t)] = D_{q,\omega}f(t) + D_{q,\omega}g(t)$

(2) $D_{q,\omega}[\alpha f(t)] = \alpha D_{q,\omega}f(t)$

(3) $D_{q,\omega}[f(t)g(t)] = f(t)D_{q,\omega}g(t) + g(qt+\omega)D_{q,\omega}f(t)$

(4) $D_{q,\omega}\left[\dfrac{f(t)}{g(t)}\right] = \dfrac{g(t)D_{q,\omega}f(t)-f(t)D_{q,\omega}g(t)}{g(t)g(qt+\omega)}$.

Letting $a,b \in I \subseteq \mathbb{R}$ with $a < \omega_0 < b$ and $[k]_q = \frac{1-q^k}{1-q}$, $k \in \mathbb{N}_0 := \mathbb{N}\cup\{0\}$, we define the q,ω-interval by

$$I_{q,\omega}^{a,b} = [a,b]_{q,\omega} \;\; := \;\; \left\{q^ka+\omega[k]_q : k \in \mathbb{N}_0\right\} \cup \left\{q^kb+\omega[k]_q : k \in \mathbb{N}_0\right\} \cup \{\omega_0\}$$

$$
\begin{aligned}
&= [a,\omega_0]_{q,\omega} \cup [\omega_0, b]_{q,\omega} \\
&= (a,b)_{q,\omega} \cup \{a,b\} \;=\; [a,b)_{q,\omega} \cup \{b\} \;=\; (a,b]_{q,\omega} \cup \{a\}, \\
\text{and} \qquad I^{T}_{q,\omega} &:= I^{\omega_0,T}_{q,\omega} = [\omega_0, T]_{q,\omega}.
\end{aligned}
$$

Observe that for each $s \in [a,b]_{q,\omega}$, the sequence $\left\{\sigma^k_{q,\omega}(s)\right\}_{k=0}^{\infty} = \left\{q^k s + \omega[k]_q\right\}_{k=0}^{\infty}$ is uniformly convergent to ω_0.

We also define the forward jump operator as $\sigma^k_{q,\omega}(t) := q^k t + \omega[k]_q$ and the backward jump operator as $\rho^k_{q,\omega}(t) := \frac{t-\omega[k]_q}{q^k}$ for $k \in \mathbb{N}$.

Definition 2. *Let I be any closed interval of $\mathbb{R}$ which containing a, b and ω_0. Assumimg that $f : I \to \mathbb{R}$ is a given function, we define q,ω-integral of f from a to b by*

$$\int_a^b f(t) d_{q,\omega}t := \int_{\omega_0}^b f(t) d_{q,\omega}t - \int_{\omega_0}^a f(t) d_{q,\omega}t$$

where

$$\int_{\omega_0}^x f(t) d_{q,\omega}t := \big[x(1-q) - \omega\big] \sum_{k=0}^{\infty} q^k f\left(xq^k + \omega[k]_q\right), \quad x \in I.$$

Providing that the series converges at $x = a$ and $x = b$, we call f is q,ω-integrable on $[a,b]$ and the sum to the right hand side of above equation will be called the Jackson-Nörlund sum.

We note that the actual domain of the function f is defined on $[a,b]_{q,\omega} \subset I$.

We next introduce the fundamental theorem of Hahn calculus in the following lemma.

Lemma 1. *[17] Let $f : I \to \mathbb{R}$ be continuous at ω_0. Define*

$$F(x) := \int_{\omega_0}^x f(t) d_{q,\omega}t, \quad x \in I.$$

Then, F is continuous at ω_0. Furthermore, $D_{q,\omega}F(x)$ exists for every $x \in I$ and

$$D_{q,\omega}F(x) = f(x).$$

Conversely, we have

$$\int_a^b D_{q,\omega}F(t) d_{q,\omega}t = F(b) - F(a) \text{ for all } a, b \in I.$$

Lemma 2. *[26] Let $q \in (0,1)$, $\omega > 0$ and $f : I \to \mathbb{R}$ be continuous at ω_0. Then,*

$$\int_{\omega_0}^t \int_{\omega_0}^r f(s)\, d_{q,\omega}s\, d_{q,\omega}r \;=\; \int_{\omega_0}^t \int_{qs+\omega}^t f(s)\, d_{q,\omega}r\, d_{q,\omega}s.$$

Lemma 3. *[26] Let $q \in (0,1)$ and $\omega > 0$. Then,*

$$\int_{\omega_0}^{t} d_{q,\omega}s = t - \omega_0 \quad \textit{and} \quad \int_{\omega_0}^{t} [t - \sigma_{q,\omega}(s)]\, d_{q,\omega}s = \frac{(t-\omega_0)^2}{1+q}.$$

We next introduce fractioanal Hahn integral, fractional Hahn difference of Riemann-Liouville and Caputo types as follows.

Definition 3. *For $\alpha, \omega > 0,\ q \in (0,1)$ and f defined on $[\omega_0, T]_{q,\omega}$, the fractional Hahn integral is defined by*

$$\begin{aligned} \mathcal{I}^{\alpha}_{q,\omega} f(t) &:= \frac{1}{\Gamma_q(\alpha)} \int_{\omega_0}^{t} \left(t - \sigma_{q,\omega}(s)\right)_{q,\omega}^{\underline{\alpha-1}} f(s) d_{q,\omega}s, \\ &= \frac{[t(1-q)-\omega]}{\Gamma_q(\alpha)} \sum_{n=0}^{\infty} q^n \left(t - \sigma^{n+1}_{q,\omega}(t)\right)_{q,\omega}^{\underline{\alpha-1}} f\left(\sigma^n_{q,\omega}(t)\right), \end{aligned}$$

and $(\mathcal{I}^{0}_{q,\omega} f)(t) = f(t)$.

Definition 4. *For $\alpha, \omega > 0,\ q \in (0,1)$, $N-1 < \alpha < N, N \in \mathbb{N}$ and f defined on $[\omega_0, T]_{q,\omega}$, the fractional Hahn difference of the Riemann-Liouville type of order α is defined by*

$$\begin{aligned} D^{\alpha}_{q,\omega} f(t) &:= (D^{N}_{q,\omega} \mathcal{I}^{N-\alpha}_{q,\omega} f)(t), \\ &= \frac{1}{\Gamma_q(-\alpha)} \int_{\omega_0}^{t} \left(t - \sigma_{q,\omega}(s)\right)_{q,\omega}^{\underline{-\alpha-1}} f(s) d_{q,\omega}s. \end{aligned}$$

The fractional Hahn difference of the Caputo type of order α is defined by

$$\begin{aligned} {}^{C}D^{\alpha}_{q,\omega} f(t) &:= (\mathcal{I}^{N-\alpha}_{q,\omega} D^{N}_{q,\omega} f)(t), \\ &= \frac{1}{\Gamma_q(N-\alpha)} \int_{\omega_0}^{t} \left(t - \sigma_{q,\omega}(s)\right)_{q,\omega}^{\underline{N-\alpha-1}} D^{N}_{q,\omega} f(s) d_{q,\omega}s, \end{aligned}$$

and $D^{0}_{q,\omega} f(t) = {}^{C}D^{0}_{q,\omega} f(t) = f(t)$.

Lemma 4. *[32] Let $\alpha > 0, q \in (0,1), \omega > 0$ and $f : I^{T}_{q,\omega} \to \mathbb{R}$. Then,*

$$\mathcal{I}^{\alpha}_{q,\omega} D^{\alpha}_{q,\omega} f(t) = f(t) + C_1 (t-\omega_0)^{\alpha-1} + ... + C_N (t-\omega_0)^{\alpha-N},$$

for some $C_i \in \mathbb{R}, i = \mathbb{N}_{1,N}$ and $N-1 < \alpha \le N, N \in \mathbb{N}$.

Lemma 5. *[32] Let $\alpha > 0, q \in (0,1), \omega > 0$ and $f : I^{T}_{q,\omega} \to \mathbb{R}$. Then,*

$$\mathcal{I}^{\alpha}_{q,\omega} {}^{C}D^{\alpha}_{q,\omega} f(t) = f(t) + C_0 + C_1 (t-\omega_0) + ... + C_{N-1} (t-\omega_0)^{N-1},$$

for some $C_i \in \mathbb{R}, i = \mathbb{N}_{0,N-1}$ and $N-1 < \alpha \le N, N \in \mathbb{N}$.

Next, we give some auxiliary lemmas for simplifying our calculations.

Lemma 6. *[32] Let* $\alpha, \beta > 0$, $0 < q < 1$ *and* $\omega > 0$. *Then,*

$$\begin{aligned}\int_{\omega_0}^{t} \Big(t - \sigma_{q,\omega}(s)\Big)_{q,\omega}^{\underline{\alpha-1}} (s-\omega_0)_{q,\omega}^{\underline{\beta}}\, d_{q,\omega}s &= (t-\omega_0)^{\alpha+\beta} B_q(\beta+1,\alpha),\\ \int_{\omega_0}^{t}\int_{\omega_0}^{x} \Big(t - \sigma_{p,\omega}(x)\Big)_{p,\omega}^{\underline{\alpha-1}} \Big(x - \sigma_{q,\omega}(s)\Big)_{q,\omega}^{\underline{\beta-1}} d_{q,\omega}s\, d_{p,\omega}x &= \frac{(t-\omega_0)^{\alpha+\beta}}{[\beta]_q} B_p(\beta+1,\alpha).\end{aligned}$$

Lemma 7. *Let* $\alpha, \beta, \theta > 0$, $0 < q < 1$, $n \in \mathbb{Z}$. *Then,*

$$\begin{aligned}(a)\quad & \int_{\omega_0}^{t}\int_{\omega_0}^{y} \big(t-\sigma_{q,\omega}(y)\big)_{q,\omega}^{\underline{\theta-1}} \big(y-\sigma_{q,\omega}(x)\big)_{q,\omega}^{\underline{\beta-1}} (x-\omega_0)^{\alpha-n} d_{q,\omega}x d_{q,\omega}y\\ & = \frac{\Gamma_q(\alpha-n+1)\Gamma_q(\beta)\Gamma_q(\theta)}{\Gamma_q(\alpha-n+\beta+\theta+1)}(t-\omega_0)^{\alpha-n+\beta+\theta}.\\ (b)\quad & \int_{\omega_0}^{t}\int_{\omega_0}^{y}\int_{\omega_0}^{x} \big(t-\sigma_{q,\omega}(y)\big)_{q,\omega}^{\underline{\theta-1}} \big(y-\sigma_{q,\omega}(x)\big)_{q,\omega}^{\underline{\beta-1}} \big(x-\sigma_{q,\omega}(s)\big)_{q,\omega}^{\underline{\alpha-1}} d_{q,\omega}s d_{q,\omega}x d_{q,\omega}y\\ & = \frac{\Gamma_q(\alpha)\Gamma_q(\beta)\Gamma_q(\theta)}{\Gamma_q(\alpha+\beta+\theta+1)}(t-\omega_0)^{\alpha+\beta+\theta}.\end{aligned}$$

Proof.

$$\begin{aligned}(a)\quad & \int_{\omega_0}^{t}\int_{\omega_0}^{y} \big(t-\sigma_{q,\omega}(y)\big)_{q,\omega}^{\underline{\theta-1}} \big(y-\sigma_{q,\omega}(x)\big)_{q,\omega}^{\underline{\beta-1}} (x-\omega_0)^{\alpha-n} d_{q,\omega}x d_{q,\omega}y\\ & = \int_{\omega_0}^{t} \big(t-\sigma_{q,\omega}(y)\big)_{q,\omega}^{\underline{\theta-1}} \Big(\int_{\omega_0}^{y} \big(y-\sigma_{q,\omega}(x)\big)_{q,\omega}^{\underline{\beta-1}} (x-\omega_0)^{\alpha-n} d_{q,\omega}x\Big) d_{q,\omega}y. \qquad (1.1)\end{aligned}$$

By using lemma (6), we obtain

$$\int_{\omega_0}^{y} \big(y-\sigma_{q,\omega}(x)\big)_{q,\omega}^{\underline{\beta-1}} (x-\omega_0)^{\alpha-n} d_{q,\omega}x = (y-\omega_0)^{\beta+\alpha-n}\frac{\Gamma_q(\alpha-n+1)\Gamma_q(\beta)}{\Gamma_q(\alpha-n+1+\beta)}. \qquad (1.2)$$

Substituting (1.2) into (1.1), we obtain

$$\begin{aligned}& \int_{\omega_0}^{t}\int_{\omega_0}^{y} \big(t-\sigma_{q,\omega}(y)\big)_{q,\omega}^{\underline{\theta-1}} \big(y-\sigma_{q,\omega}(x)\big)_{q,\omega}^{\underline{\beta-1}} (x-\omega_0)^{\alpha-n} d_{q,\omega}x d_{q,\omega}y\\ & = \frac{\Gamma_q(\alpha-n+1)\Gamma_q(\beta)}{\Gamma_q(\alpha-n+1+\beta)} \int_{\omega_0}^{t} \big(t-\sigma_{q,\omega}(y)\big)_{q,\omega}^{\underline{\theta-1}} (y-\omega_0)^{\beta+\alpha-n} d_{q,\omega}y. \qquad (1.3)\end{aligned}$$

Since

$$\int_{\omega_0}^{t} \big(t-\sigma_{q,\omega}(y)\big)_{q,\omega}^{\underline{\theta-1}} (y-\omega_0)^{\alpha-n+\beta} d_{q,\omega}y = (t-\omega_0)^{\theta+\beta+\alpha-n}\frac{\Gamma_q(\beta+\alpha-n+1)\Gamma_q(\theta)}{\Gamma_q(\beta+\alpha-n+1+\theta)}, \qquad (1.4)$$

we get the following result after substituting (1.4) into (1.3) as

$$\int_{\omega_0}^{t}\int_{\omega_0}^{y}\left(t-\sigma_{q,\omega}(y)\right)_{q,\omega}^{\underline{\theta-1}}\left(y-\sigma_{q,\omega}(x)\right)_{q,\omega}^{\underline{\beta-1}}(x-\omega_0)^{\alpha-n}d_{q,\omega}x d_{q,\omega}y=\frac{\Gamma_q(\alpha-n+1)\Gamma_q(\beta)\Gamma_q(\theta)(t-\omega_0)^{\alpha-n+\beta+\theta}}{\Gamma_q(\alpha-n+\beta+\theta+1)}. \tag{1.5}$$

(b)
$$\begin{aligned}&\int_{\omega_0}^{t}\int_{\omega_0}^{y}\int_{\omega_0}^{x}\left(t-\sigma_{q,\omega}(y)\right)_{q,\omega}^{\underline{\theta-1}}\left(y-\sigma_{q,\omega}(x)\right)_{q,\omega}^{\underline{\beta-1}}\left(x-\sigma_{q,\omega}(s)\right)_{q,\omega}^{\underline{\alpha-1}}d_{q,\omega}s d_{q,\omega}x d_{q,\omega}y\\ =&\int_{\omega_0}^{t}\int_{\omega_0}^{y}\left(t-\sigma_{q,\omega}(y)\right)_{q,\omega}^{\underline{\theta-1}}\left(y-\sigma_{q,\omega}(x)\right)_{q,\omega}^{\underline{\beta-1}}\Big(\int_{\omega_0}^{x}\left(x-\sigma_{q,\omega}(s)\right)_{q,\omega}^{\underline{\alpha-1}}d_{q,\omega}s\Big)d_{q,\omega}x d_{q,\omega}y.\end{aligned} \tag{1.6}$$

Since $\int_{\omega_0}^{x}\left(x-\sigma_{q,\omega}(s)\right)_{q,\omega}^{\underline{\alpha-1}}d_{q,\omega}s=\frac{(x-\omega_0)^{\alpha}\Gamma_q(\alpha)}{\Gamma_q(\alpha+1)}$, we obtain

$$\begin{aligned}&\int_{\omega_0}^{t}\int_{\omega_0}^{y}\int_{\omega_0}^{x}\left(t-\sigma_{q,\omega}(y)\right)_{q,\omega}^{\underline{\theta-1}}\left(y-\sigma_{q,\omega}(x)\right)_{q,\omega}^{\underline{\beta-1}}\left(x-\sigma_{q,\omega}(s)\right)_{q,\omega}^{\underline{\alpha-1}}d_{q,\omega}s d_{q,\omega}x d_{q,\omega}y,\\ &=\int_{\omega_0}^{t}\int_{\omega_0}^{y}\left(t-\sigma_{q,\omega}(y)\right)_{q,\omega}^{\underline{\theta-1}}\left(y-\sigma_{q,\omega}(x)\right)_{q,\omega}^{\underline{\beta-1}}\Big(\frac{(x-\omega_0)^{\alpha}\Gamma_q(\alpha)}{\Gamma_q(\alpha+1)}\Big)d_{q,\omega}x d_{q,\omega}y,\\ &=\frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha+1)}\int_{\omega_0}^{t}\int_{\omega_0}^{y}\left(t-\sigma_{q,\omega}(y)\right)_{q,\omega}^{\underline{\theta-1}}\left(y-\sigma_{q,\omega}(x)\right)_{q,\omega}^{\underline{\beta-1}}(x-\omega_0)^{\alpha}d_{q,\omega}x d_{q,\omega}y\end{aligned} \tag{1.7}$$

By using lemma (6) (a), we have

$$\begin{aligned}&\int_{\omega_0}^{t}\int_{\omega_0}^{y}\int_{\omega_0}^{x}\left(t-\sigma_{q,\omega}(y)\right)_{q,\omega}^{\underline{\theta-1}}\left(y-\sigma_{q,\omega}(x)\right)_{q,\omega}^{\underline{\beta-1}}\left(x-\sigma_{q,\omega}(s)\right)_{q,\omega}^{\underline{\alpha-1}}d_{q,\omega}s d_{q,\omega}x d_{q,\omega}y\\ &=\frac{\Gamma_q(\alpha)\Gamma_q(\beta)\Gamma_q(\theta)}{\Gamma_q(\alpha+\beta+\theta+1)}(t-\omega_0)^{\alpha+\beta+\theta}.\end{aligned}$$

In the following, we present a lemma that deals with the linear variant of problem (0.1) and provides a representation of the solution.

Lemma 8. *Let* $\Omega\neq 0,\ \alpha,\beta,\theta\in(0,1],\ \omega>0,\ q\in(0,1),\ \omega_0=\frac{\omega}{1-q}$ *and* $h\in C\big([\omega_0,T]_{q,\omega},\mathbb{R}\big)$ *be given function. Then the problem*

$$\begin{aligned}&D_{q,\omega}^{\alpha}D_{q,\omega}^{\beta}u(t)=h(t),\\ &u(\omega_0)=u(T),\\ &\mathcal{I}_{q,\omega}^{\theta}g(\eta)u(\eta)=\varphi\big(u(\eta)\big),\end{aligned} \tag{1.8}$$

has the unique solution

$$\begin{aligned}u(t)\ =&\ \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)}\int_{\omega_0}^{t}\int_{\omega_0}^{x}\left(t-\sigma_{q,\omega}(x)\right)_{q,\omega}^{\underline{\beta-1}}\left(x-\sigma_{q,\omega}(s)\right)_{q,\omega}^{\underline{\alpha-1}}h(s)d_{q,\omega}s d_{q,\omega}x\\ &-\frac{(t-\omega_0)^{\beta-1}}{\Omega}\Big\{\mathcal{B}_{\eta}\mathbb{P}[h]+\mathcal{A}_T\big[\varphi\big(u(\eta)\big)-\mathbb{Q}[h]\big]\Big\}\end{aligned}$$

$$+(t-\omega_0)^{\beta+\alpha-1}\frac{\Gamma_q(\alpha)}{\Omega\Gamma_q(\alpha+\beta)}\Big\{\mathcal{A}_\eta\mathbb{P}[h]+(T-\omega_0)^{\beta-1}\big[\varphi\big(u(\eta)\big)-\mathbb{Q}[h]\big]\Big\}, \tag{1.9}$$

where the functionals and the constants are defined by

$$\mathcal{A}_T := (T-\omega_0)^{\beta+\alpha-1}\frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha+\beta)}, \tag{1.10}$$

$$\mathcal{A}_\eta := \frac{1}{\Gamma_q(\theta)}\int_{\omega_0}^{\eta} g(s)\big(\eta-\sigma_{q,\omega}(s)\big)_{q,\omega}^{\underline{\theta-1}}(s-\omega_0)^{\beta-1}d_{q,\omega}s, \tag{1.11}$$

$$\mathcal{B}_\eta := \frac{\Gamma_q(\alpha)}{\Gamma_q(\theta)\Gamma_q(\alpha+\beta)}\int_{\omega_0}^{\eta} g(s)\big(\eta-\sigma_{q,\omega}(s)\big)_{q,\omega}^{\underline{\theta-1}}(s-\omega_0)^{\beta+\alpha-1}d_{q,\omega}s, \tag{1.12}$$

$$\Omega := (T-\omega_0)^{\beta-1}\mathcal{B}_\eta-\mathcal{A}_T\mathcal{A}_\eta, \tag{1.13}$$

$$\mathbb{P}[h] := \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)}\int_{\omega_0}^{T}\int_{\omega_0}^{x}\big(T-\sigma_{q,\omega}(x)\big)_{q,\omega}^{\underline{\beta-1}}\big(x-\sigma_{q,\omega}(x)\big)_{q,\omega}^{\underline{\alpha-1}}h(s)d_{q,\omega}s\,d_{q,\omega}x, \tag{1.14}$$

$$\mathbb{Q}[h] := \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)\Gamma_q(\theta)}\int_{\omega_0}^{\eta}\int_{\omega_0}^{y}\int_{\omega_0}^{x} g(y)\big(\eta-\sigma_{q,\omega}(y)\big)_{q,\omega}^{\underline{\theta-1}}\big(y-\sigma_{q,\omega}(x)\big)_{q,\omega}^{\underline{\beta-1}}\times \big(x-\sigma_{q,\omega}(x)\big)_{q,\omega}^{\underline{\alpha-1}}h(s)d_{q,\omega}s\,d_{q,\omega}x\,d_{q,\omega}y. \tag{1.15}$$

Proof. Taking fractional Hahn integral of order α for the first equation of (1.8), we obtain

$$\begin{aligned} D_{q,\omega}^{\beta}u(t) &= \mathcal{I}_{q,\omega}^{\alpha}h(t)+C_1(t-\omega_0)^{\alpha-1} \\ &= \frac{1}{\Gamma_q(\alpha)}\int_{\omega_0}^{t}\big(t-\sigma_{q,\omega}(s)\big)_{q,\omega}^{\underline{\alpha-1}}h(s)\,d_{q,\omega}s+C_1(t-\omega_0)^{\alpha-1}. \end{aligned} \tag{1.16}$$

Taking fractional Hahn integral of order β for (1.16), we obtain

$$\begin{aligned} u(t) &= \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)}\int_{\omega_0}^{t}\int_{\omega_0}^{x}\big(t-\sigma_{q,\omega}(x)\big)_{q,\omega}^{\underline{\beta-1}}\big(x-\sigma_{q,\omega}(s)\big)_{q,\omega}^{\underline{\alpha-1}}h(s)\,d_{q,\omega}s\,d_{q,\omega}x \\ &\quad +C_1(t-\omega_0)^{\beta+\alpha-1}\frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha+\beta)}+C_0(t-\omega_0)^{\beta-1}. \end{aligned} \tag{1.17}$$

Substituting $t=\omega_0$ into (1.17), we have

$$u(\omega_0) = 0. \tag{1.18}$$

Substituting $t=T$ into (1.17), we have

$$\begin{aligned} u(T) &= \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)}\int_{\omega_0}^{T}\int_{\omega_0}^{x}\big(T-\sigma_{q,\omega}(x)\big)_{q,\omega}^{\underline{\beta-1}}\big(x-\sigma_{q,\omega}(s)\big)_{q,\omega}^{\underline{\alpha-1}}h(s)\,d_{q,\omega}s\,d_{q,\omega}x \\ &\quad +C_1(T-\omega_0)^{\beta+\alpha-1}\frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha+\beta)}+C_0(T-\omega_0)^{\beta-1}. \end{aligned} \tag{1.19}$$

Since $u(\omega_0) = u(T)$, we get

$$C_0(T-\omega_0)^{\beta-1} + C_1\mathcal{A}_T \;=\; -\mathbb{P}[h], \tag{1.20}$$

where $\mathcal{A}_T, \mathbb{P}[h]$ are defined as (1.10) and (1.14) respectively.
Multiplying (1.17) by $g(t)$ and taking fractional Hahn integral of order θ, we obtain

$$\begin{aligned}\mathcal{I}^{\theta}_{q,\omega}g(t)u(t) \;=\;& \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)\Gamma_q(\theta)}\int_{\omega_0}^{t}\int_{\omega_0}^{y}\int_{\omega_0}^{x} g(y)\big(t-\sigma_{q,\omega}(y)\big)_{q,\omega}^{\underline{\theta-1}}\big(y-\sigma_{q,\omega}(x)\big)_{q,\omega}^{\underline{\beta-1}} \times \\ & \big(x-\sigma_{q,\omega}(s)\big)_{q,\omega}^{\underline{\alpha-1}} h(s)d_{q,\omega}s\,d_{q,\omega}x\,d_{q,\omega}y \\ & + \frac{C_1\Gamma_q(\alpha)}{\Gamma_q(\theta)\Gamma_q(\alpha+\beta)}\int_{\omega_0}^{t} g(s)\big(t-\sigma_{q,\omega}(s)\big)_{q,\omega}^{\underline{\theta-1}}(s-\omega_0)^{\beta+\alpha-1}d_{q,\omega}s \\ & + \frac{C_0}{\Gamma_q(\theta)}\int_{\omega_0}^{t} g(s)\big(t-\sigma_{q,\omega}(s)\big)_{q,\omega}^{\underline{\theta-1}}(s-\omega_0)^{\beta-1}d_{q,\omega}s. \end{aligned} \tag{1.21}$$

Substituting $t=\eta$ into (1.21) and employing the condition $\mathcal{I}^{\theta}_{q,\omega}g(\eta)u(\eta) = \varphi\big(u(\eta)\big)$, we have

$$\begin{aligned}\varphi\big(u(\eta)\big) \;=\;& \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)\Gamma_q(\theta)}\int_{\omega_0}^{\eta}\int_{\omega_0}^{y}\int_{\omega_0}^{x} g(y)\big(\eta-\sigma_{q,\omega}(y)\big)_{q,\omega}^{\underline{\theta-1}}\big(y-\sigma_{q,\omega}(x)\big)_{q,\omega}^{\underline{\beta-1}} \times \\ & \big(x-\sigma_{q,\omega}(s)\big)_{q,\omega}^{\underline{\alpha-1}} h(s)d_{q,\omega}s\,d_{q,\omega}x\,d_{q,\omega}y \\ & + \frac{C_1\Gamma_q(\alpha)}{\Gamma_q(\theta)\Gamma_q(\alpha+\beta)}\int_{\omega_0}^{\eta} g(s)\big(\eta-\sigma_{q,\omega}(s)\big)_{q,\omega}^{\underline{\theta-1}}(s-\omega_0)^{\beta+\alpha-1}d_{q,\omega}s \\ & + \frac{C_0}{\Gamma_q(\theta)}\int_{\omega_0}^{\eta} g(s)\big(\eta-\sigma_{q,\omega}(s)\big)_{q,\omega}^{\underline{\theta-1}}(s-\omega_0)^{\beta-1}d_{q,\omega}s. \end{aligned}$$

Hence

$$C_0\mathcal{A}_\eta + C_1\mathcal{B}_\eta = \varphi\big(u(\eta)\big) - \mathbb{Q}[h], \tag{1.22}$$

where $\mathcal{A}_\eta, \mathcal{B}_\eta, \mathbb{Q}[h]$ are defined as (1.11), (1.12) and (1.15), respectively.
To find C_0 and C_1, we solve the system of eqautions (1.20) and (1.22). Then, we obtain

$$\begin{aligned} C_0 &= \frac{-\mathcal{B}_\eta\mathbb{P}[h] - \mathcal{A}_T\big[\varphi\big(u(\eta)\big) - \mathbb{Q}[h]\big]}{\Omega}, \\ C_1 &= \frac{(T-\omega_0)^{\beta-1}\big[\varphi\big(u(\eta)\big) - \mathbb{Q}[h]\big] + \mathcal{A}_\eta\mathbb{P}[h]}{\Omega}, \end{aligned}$$

where $\mathcal{A}_\eta, \mathcal{A}_T, \mathcal{B}_\eta, \Omega, \mathbb{P}[h], \mathbb{Q}[h]$ are defined as (1.10) – (1.15), respectively.

Substituting the constants C_0, C_1 into (1.17), we obtain the solution for (1.8), as shown in equation (1.9). □

We next introduce the Schauder's fixed point theorem used to prove the existence of a solution to (0.1) and (0.2).

Lemma 9. *[39] (Arzelá-Ascoli theorem) A set of function in $C[a,b]$ with the sup norm, is relatively compact if and only if is uniformly bounded and equicontinuous on $[a,b]$.*

Lemma 10. *[39] If a set is closed and relatively compact then it is compact.*

Lemma 11. *[40] (Schauder's fixed point theorem) Let (D,d) be a complete metric space, U be a closed convex subset of D, and $T: D \to D$ be the map such that the set $Tu : u \in U$ is relatively compact in D. Then the operator T has at least one fixed point $u^* \in U$: $Tu^* = u^*$.*

2. Existence and Uniqueness Results

In this section, we prove the existence results for problem (0.1). Let $\mathcal{C} = \mathcal{C}\left(I^T_{q,\omega}, \mathbb{R}\right)$ be a Banach space of all function u with the norm defined by

$$\|u\|_{\mathcal{C}} = \|u\| + \|\mathcal{D}^{\nu}_{q,\omega} u\|,$$

where $\|u\| = \max_{t \in I^T_{q,\omega}} \{|u(t)|\}$ and $\|\mathcal{D}^{\nu}_{q,\omega} u\| = \max_{t \in I^T_{q,\omega}} \{|\mathcal{D}^{\nu}_{q,\omega} u(t)|\}$.
By Lemma 8, replacing $h(t)$ by $F\big(t, u(t), \psi^{\gamma}_{q,\omega} u(t), D^{\nu}_{q,\omega} u(t)\big)$, we define an operator $\mathcal{A} : \mathcal{C} \to \mathcal{C}$ by

$$\begin{aligned}(\mathcal{A}u)(t) :=& \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)} \int_{\omega_0}^{t} \int_{\omega_0}^{x} \Big(t - \sigma_{q,\omega}(s)\Big)_{q,\omega}^{\underline{\beta-1}} \Big(x - \sigma_{q,\omega}(s)\Big)_{q,\omega}^{\underline{\alpha-1}} \times \\ & F\big(s, u(s), \psi^{\gamma}_{q,\omega} u(s), D^{\nu}_{q,\omega} u(s)\big) d_{q,\omega} s d_{q,\omega} x \\ & - \frac{(t-\omega_0)^{\beta-1}}{\Omega} \Big\{ \mathcal{B}_{\eta} \mathbb{P}^*[F_u] + \mathcal{A}_T \big[\varphi\big(u(\eta)\big) - \mathbb{Q}^*[F_u]\big] \Big\} \\ & + (t-\omega_0)^{\beta+\alpha-1} \frac{\Gamma_q(\alpha)}{\Omega \Gamma_q(\alpha+\beta)} \Big\{ \mathcal{A}_{\eta} \mathbb{P}^*[F_u] + (T-\omega_0)^{\beta-1} \big[\varphi\big(u(\eta)\big) - \mathbb{Q}^*[F_u]\big] \Big\},\end{aligned} \tag{2.1}$$

where the functionals $\mathbb{P}^*[F_u]$, $\mathbb{Q}^*[F_u]$ are defined by

$$\begin{aligned}\mathbb{P}^*[F_u] :=& \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)} \int_{\omega_0}^{T} \int_{\omega_0}^{x} \big(T - \sigma_{q,\omega}(x)\big)_{q,\omega}^{\underline{\beta-1}} \big(x - \sigma_{q,\omega}(s)\big)_{q,\omega}^{\underline{\alpha-1}} \times \\ & F\big(s, u(s), \psi^{\gamma}_{q,\omega} u(s), D^{\nu}_{q,\omega} u(s)\big) d_{q,\omega} s d_{q,\omega} x,\end{aligned} \tag{2.2}$$

$$\begin{aligned}\mathbb{Q}^*[F_u] :=& \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)\Gamma_q(\theta)} \int_{\omega_0}^{\eta} \int_{\omega_0}^{y} \int_{\omega_0}^{x} g(y) \big(\eta - \sigma_{q,\omega}(y)\big)_{q,\omega}^{\underline{\theta-1}} \big(y - \sigma_{q,\omega}(x)\big)_{q,\omega}^{\underline{\beta-1}} \times \\ & \big(x - \sigma_{q,\omega}(s)\big)_{q,\omega}^{\underline{\alpha-1}} F\big(s, u(s), \psi^{\gamma}_{q,\omega} u(s), D^{\nu}_{q,\omega} u(s)\big) d_{q,\omega} s d_{q,\omega} x d_{q,\omega} y,\end{aligned} \tag{2.3}$$

and the constants $\mathcal{A}_T, \mathcal{A}_\eta, \mathcal{B}_\eta$ and Ω are defined by (1.10)-(1.13), respectively.

We see that the problem (0.1) has solution if and only if the operator $\mathcal{A}$ has fixed point.

Theorem 1. *Assume that* $F : I^T_{q,\omega} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is continuous,* $\phi : I^T_{q,\omega} \times I^T_{q,\omega} \to [0, \infty)$ *is continuous with* $\phi_0 = \max\left\{\phi(t, s) : (t, s) \in I^T_{q,\omega} \times I^T_{q,\omega}\right\}$. *In addition, suppose that the following conditions hold:*

(H_1) *There exist constants* $L_i > 0$ *such that for each* $t \in I^T_{q,\omega}$ *and* $u_i, v_i \in \mathbb{R}$, $i = 1, 2, 3$,

$$\big|F[t, u_1, u_2, u_3] - F[t, v_1, v_2, v_3]\big| \le L_1\big|u_1 - v_1\big| + L_2\big|u_2 - v_2\big| + L_3\big|u_3 - v_3\big|.$$

(H_2) *There exist a positive constant* ω *such that for each* $u, v \in C$,

$$\big|\varphi(u) - \varphi(v)\big| \le \lambda\|u - v\|_C.$$

(H_3) *For each* $t \in I^T_{q,\omega}$, $0 < \hat{g} < g(t) < G$.

(H_4) $\mathcal{X} = \lambda\mathcal{O}^*_T + \big(\mathcal{L} + L_3\frac{(T-\omega_0)^{-\nu}}{\Gamma_q(1-\nu)}\big)\Theta^* < 1$,

where

$$\mathcal{L} := L_1 + L_2\frac{\phi_0(T-\omega_0)^\gamma}{\Gamma_q(\gamma+1)}, \tag{2.4}$$

$$\mathcal{O}_T := \frac{(T-\omega_0)^{\beta-1}}{\min|\Omega|}\Big\{A_T + \frac{\Gamma_q(\alpha)(T-\omega_0)^{\beta+\alpha-1}}{\Gamma_q(\alpha+\beta)}\Big\}, \tag{2.5}$$

$$\mathcal{O}_\eta := \frac{(T-\omega_0)^{\beta-1}}{\min|\Omega|}\Big\{\max\big|B_\eta\big| + \frac{\Gamma_q(\alpha)(T-\omega_0)^{\alpha}\max\big|A_\eta\big|}{\Gamma_q(\alpha+\beta)}\Big\}, \tag{2.6}$$

$$\Theta := \frac{\mathcal{O}_T G(\eta-\omega_0)^{\theta+\beta+\alpha}}{\Gamma_q(\beta+\alpha+\theta+1)} + \frac{(T-\omega_0)^{\alpha+\beta}}{\Gamma_q(\alpha+\beta+1)}\big(\mathcal{O}_\eta + 1\big), \tag{2.7}$$

$$\bar{\mathcal{O}_\eta} := (T-\omega_0)^{-\nu+\beta-1}\frac{1}{\min|\Omega|}\Big\{\max\big|A_\eta\big|\frac{\Gamma_q(\alpha)(T-\omega_0)^{\alpha}}{\Gamma_q(\beta+\alpha-\nu)} + \max\big|B_\eta\big|\frac{\Gamma_q(\beta)}{\Gamma_q(\beta-\nu)}\Big\}, \tag{2.8}$$

$$\bar{\mathcal{O}_T} := (T-\omega_0)^{-\nu+\beta-1}\frac{1}{\min|\Omega|}\Big\{\frac{\Gamma_q(\alpha)(T-\omega_0)^{\beta-1}(T-\omega_0)^{\alpha}}{\Gamma_q(\beta+\alpha-\nu)} + \max\big|A_T\big|\frac{\Gamma_q(\beta)}{\Gamma_q(\beta-\nu)}\Big\}, \tag{2.9}$$

$$\bar{\Theta} := \bar{\mathcal{O}_T}\frac{G(\eta-\omega_0)^{\theta+\beta+\alpha}}{\Gamma_q(\beta+\alpha+\theta+1)} + \bar{\mathcal{O}_\eta}\frac{(T-\omega_0)^{\beta+\alpha}}{\Gamma_q(\beta+\alpha+1)} + \frac{(T-\omega_0)^{\beta+\alpha-\nu}}{\Gamma_q(\beta+\alpha-\nu+1)}, \tag{2.10}$$

$$\mathcal{O}^*_T := \mathcal{O}_T + \bar{\mathcal{O}_T}, \tag{2.11}$$

$$\Theta^* := \Theta + \bar{\Theta}. \tag{2.12}$$

Then, problem (0.1) *has a unique solution.*

Proof. For each $t \in I^T_{q,\omega}$ and $u, v \in \mathcal{C}$, we find that

$$\Big|\big(\Psi^\gamma_{q,\omega}u\big)(t) - \big(\Psi^\gamma_{q,\omega}v\big)(t)\Big| \leq \frac{\phi_0}{\Gamma_q(\gamma)}\int_{\omega_0}^t \big(t-\sigma_{q,\omega}(s)\big)^{\underline{\gamma-1}}_{q,\omega}\big|u(s)-v(s)\big|d_{q,\omega}s, \\ = \frac{\phi_0(T-\omega_0)^\gamma}{\Gamma_q(\gamma+1)}\|u-v\|.$$

We set

$$\mathcal{F}|u-v|(t) := \Big|F[t,u(t),\psi^\gamma_{q,\omega}u(t),D^\nu_{q,\omega}u(t)] - F[t,v(t),\psi^\gamma_{q,\omega}v(t),D^\nu_{q,\omega}v(t)]\Big|.$$

Then, we obtain

$$\Big|\mathbb{P}^*[F_u]-\mathbb{P}^*[F_v]\Big| \leq \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)}\int_{\omega_0}^T\int_{\omega_0}^x \big(T-\sigma_{q,\omega}(x)\big)^{\underline{\beta-1}}_{q,\omega}\big(x-\sigma_{q,\omega}(s)\big)^{\underline{\alpha-1}}_{q,\omega}\mathcal{F}|u-v|(s)d_{q,\omega}sd_{q,\omega}x, \\ \leq \Big(L_1+L_2\frac{\phi_0(T-\omega_0)^\gamma}{\Gamma_q(\gamma+1)}+L_3\frac{(T-\omega_0)^{-\nu}}{\Gamma_q(1-\nu)}\Big)\|u-v\|_C\frac{(T-\omega_0)^{\alpha+\beta}}{\Gamma_q(\alpha+\beta+1)}, \\ \leq \frac{\big(\mathcal{L}+L_3\frac{(T-\omega_0)^{-\nu}}{\Gamma_q(1-\nu)}\big)}{\Gamma_q(\alpha+\beta+1)}(T-\omega_0)^{\alpha+\beta}\|u-v\|_{\mathcal{C}}.$$

Similarly, we obtain

$$\big|\mathbb{Q}^*[F_u]-\mathbb{Q}^*[F_v]\big| \leq \frac{G}{\Gamma_q(\alpha)\Gamma_q(\beta)\Gamma_q(\theta)}\int_{\omega_0}^\eta\int_{\omega_0}^y\int_{\omega_0}^x \big(\eta-\sigma_{q,\omega}(y)\big)^{\underline{\theta-1}}_{q,\omega}\big(y-\sigma_{q,\omega}(x)\big)^{\underline{\beta-1}}_{q,\omega}\times \\ \big(x-\sigma_{q,\omega}(s)\big)^{\underline{\alpha-1}}_{q,\omega}\mathcal{F}|u-v|(s)d_{q,\omega}sd_{q,\omega}xd_{q,\omega}y, \\ \leq \frac{G\Big[L_1+L_2\frac{\phi_0(T-\omega_0)^\gamma}{\Gamma_q(\gamma+1)}+L_3\Big]\|u-v\|_C(\eta-\omega_0)^{\theta+\beta+\alpha}}{\Gamma_q(\beta+\alpha+\beta+1)}, \\ \leq \frac{G(\mathcal{L}+L_3)(\eta-\omega_0)^{\alpha+\beta+\theta}}{\Gamma_q(\alpha+\beta+\theta+1)}\|u-v\|_{\mathcal{C}}.$$

Next, we find that

$$\big|(\mathcal{A}u)(t)-(\mathcal{A}v)(t)\big| \leq \frac{\big(\mathcal{L}+L_3\frac{(T-\omega_0)^{-\nu}}{\Gamma_q(1-\nu)}\big)}{\Gamma_q(\alpha+\beta+1)}(T-\omega_0)^{\alpha+\beta}\|u-v\|_{\mathcal{C}} \\ + \frac{\big(\mathcal{L}+L_3\frac{(T-\omega_0)^{-\nu}}{\Gamma_q(1-\nu)}\big)}{\min|\Omega|\Gamma_q(\alpha+\beta+1)}(T-\omega_0)^{\alpha+\beta}\|u-v\|_{\mathcal{C}}(T-\omega_0)^{\beta-1}\times \\ \Big\{\frac{(T-\omega_0)^\alpha\Gamma_q(\alpha)\max\big|\mathcal{A}_\eta\big|}{\Gamma_q(\alpha+\beta)}+\max\big|\mathcal{B}_\eta\big|\Big\} \\ + \frac{G\big(\mathcal{L}+L_3\frac{(T-\omega_0)^{-\nu}}{\Gamma_q(1-\nu)}\big)}{\min|\Omega|\Gamma_q(\alpha+\beta+\theta+1)}(\eta-\omega_0)^{\alpha+\beta+\theta}\|u-v\|_{\mathcal{C}}(T-\omega_0)^{\beta-1}\times$$

$$\Big\{\frac{(T-\omega_0)^{\alpha+\beta-1}\Gamma_q(\alpha)}{\Gamma_q(\alpha+\beta)}+\max\big|\mathcal{A}_T\big|\Big\}+\frac{\omega\|u-v\|_{\mathcal{C}}}{\min|\Omega|}(T-\omega_0)^{\beta-1}\times$$
$$\Big\{\frac{(T-\omega_0)^{\alpha+\beta-1}\Gamma_q(\alpha)}{\Gamma_q(\alpha+\beta)}+\max\big|\mathcal{A}_T\big|\Big\},$$
$$\leq\Big\{\omega\mathcal{O}_T+\big(\mathcal{L}+L_3\frac{(T-\omega_0)^{-\nu}}{\Gamma_q(1-\nu)}\big)\Theta\Big\}\|u-v\|_C. \tag{2.13}$$

Considering $\big(\mathcal{D}_q^{\nu}\mathcal{A}u\big)(t)$, we have

$$\begin{aligned}\big(\mathcal{D}_{q,\omega}^{\nu}\mathcal{A}u\big)(t) = {} & \frac{1}{\Gamma_q(-\nu)\Gamma_q(\alpha)\Gamma_q(\beta)}\int_{\omega_0}^{t}\int_{\omega_0}^{y}\int_{\omega_0}^{x}\big(t-\sigma_{q,\omega}(y)\big)_{q,\omega}^{-\nu-1}\big(y-\sigma_{q,\omega}(x)\big)_{q,\omega}^{\beta-1}\times\\
& \big(x-\sigma_{q,\omega}(s)\big)_{q,\omega}^{\alpha-1}F\big(s,u(s),\Psi_{q,\omega}^{\gamma}u(s),D_{q,\omega}^{\gamma}u(s)\big)d_{q,\omega}s d_{q,\omega}x d_{q,\omega}y\\
& +\frac{\mathbb{P}^*[Fu]}{\Omega}\Big\{A_\eta\frac{\Gamma_q(\alpha)}{\Gamma_q(\beta+\alpha-\nu)}(t-\omega_0)^{-\nu+\beta+\alpha-1}-B_\eta\frac{\Gamma_q(\beta)}{\Gamma_q(\beta-\nu)}\times\\
& (t-\omega_0)^{-\nu+\beta-1}\Big\}\\
& +\frac{\mathbb{Q}^*[Fu]}{\Omega}\Big\{A_T\frac{\Gamma_q(\beta)}{\Gamma_q(\beta-\nu)}(t-\omega_0)^{-\nu+\beta-1}-\frac{\Gamma_q(\alpha)(T-\omega_0)^{\beta-1}}{\Gamma_q(\beta+\alpha-\nu)}\times\\
& (t-\omega_0)^{-\nu+\beta+\alpha-1}\Big\}\\
& +\frac{\varphi\big(u(\eta)\big)}{\Omega}\Big\{\frac{(T-\omega_0)^{\beta-1}\Gamma_q(\alpha)}{\Gamma_q(\beta+\alpha-\nu)}(t-\omega_0)^{-\nu+\beta+\alpha-1}-A_T\frac{\Gamma_q(\beta)}{\Gamma_q(\beta-\nu)}\times\\
& (t-\omega_0)^{-\nu+\beta-1}\Big\}.\end{aligned} \tag{2.14}$$

Hence,

$$\begin{aligned}\Big|\big(\mathcal{D}_{q,\omega}^{\nu}\mathcal{A}u\big)(t)-\big(\mathcal{D}_{q,\omega}^{\nu}\mathcal{A}v\big)(t)\Big| \leq {} & \Big\{\omega\bar{\mathcal{O}}_T+\big(\mathcal{L}+L_3\frac{(T-\omega_0)^{-\nu}}{\Gamma_q(1-\nu)}\big)\Big[\bar{\mathcal{O}}_T\frac{G(\eta-\omega_0)^{\theta+\beta+\alpha}}{\Gamma_q(\beta+\alpha+\theta+1)}\\
& +\bar{\mathcal{O}}_\eta\frac{(T-\omega_0)^{\beta+\alpha}}{\Gamma_q(\beta+\alpha+1)}+\frac{(T-\omega_0)^{\beta+\alpha-\nu}}{\Gamma_q(\beta+\alpha-\nu+1)}\Big]\Big\}\|u-v\|_C,\\
\leq {} & \Big\{\omega\bar{\mathcal{O}}_T+\big(\mathcal{L}+L_3\frac{(T-\omega_0)^{-\nu}}{\Gamma_q(1-\nu)}\big)\bar{\Theta}\Big\}\|u-v\|_C.\end{aligned} \tag{2.15}$$

From (2.13) and (2.15), we find that

$$\|\mathcal{A}u-\mathcal{A}v\|_{\mathcal{C}}\leq\ \mathcal{X}\|u-v\|_{\mathcal{C}}.$$

Thus, the operator $\mathcal{A}$ is a contraction. Then, by the Banach contraction mapping principle, $\mathcal{A}$ has a fixed point which is the unique solution for (0.1).

We next show the existence of a solution to (0.1) by the following Schauder's fixed point theorem.

Theorem 2. *Let us assume that* $F : I^T_{q,\omega} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is continuous functions and* $\varphi : C\big(I^T_{q,\omega}, \mathbb{R}\big) \to \mathbb{R}$ *is given functional. Let us suppose that the following conditions hold:*

(H_5) There exists a positive constants M such that for each $t \in I^T_{q,\omega}$ and $u_i \in \mathbb{R}$, $i = 1, 2, 3,$

$$\big|F\left(t, u_1, u_2, u_3\right)\big| \le M.$$

(H_6) There exists a positve constants N such that for each $u \in \mathcal{C}$,

$$\big|\varphi(u)\big| \le N.$$

Then, problem (0.1) has at least one solution on $I^T_{q,\omega}$.
Proof. We organize the proof into three steps.

(i) Verify $\mathcal{A}$ maps bounded sets into bounded sets in $B_R = \{u \in C : \|u\|_C \le R\}$. Let us prove that for any $R > 0$, there exists a positive constant L such that for each $x \in \mathcal{B}_R$, we have $\|\mathcal{A}u\|_C \le L$. For each $t \in I^T_{q,\omega}$ and $u \in B_R$, we have.

$$\begin{aligned}\big|\mathbb{P}^*[F_u]\big| &= \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)}\int_{\omega_0}^{T}\int_{\omega_0}^{x}\big(T-\sigma_{q,\omega}(x)\big)_{q,\omega}^{\underline{\beta-1}}\big(x-\sigma_{q,\omega}(s)\big)_{q,\omega}^{\underline{\alpha-1}}\times\\ &\qquad\Big|F\big(s,u(s),\Psi^{\gamma}_{q,\omega}u(s),D^{\nu}_{q,\omega}u(s)\big)\Big|d_{q,\omega}s d_{q,\omega}x,\\ &\le \frac{M(T-\omega_0)^{\alpha+\beta}}{\Gamma_q(\alpha+\beta+1)}. \end{aligned} \tag{2.16}$$

$$\begin{aligned}\big|\mathbb{Q}^*[F_u]\big| &= \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)\Gamma_q(\theta)}\int_{\omega_0}^{\eta}\int_{\omega_0}^{y}\int_{\omega_0}^{x} g(y)\big(\eta-\sigma_{q,\omega}(y)\big)_{q,\omega}^{\underline{\theta-1}}\big(y-\sigma_{q,\omega}(x)\big)_{q,\omega}^{\underline{\beta-1}}\times\\ &\quad\big(x-\sigma_{q,\omega}(s)\big)_{q,\omega}^{\underline{\alpha-1}}\Big|F\big(s,u(s),\Psi^{\gamma}_{q,\omega}u(s),D^{\nu}_{q,\omega}u(s)\big)\Big|d_{q,\omega}s d_{q,\omega}x d_{q,\omega}y,\\ &\quad\big(x-\sigma_{q,\omega}(s)\big)_{q,\omega}^{\underline{\alpha-1}}d_{q,\omega}s d_{q,\omega}x d_{q,\omega}y,\\ &\le \frac{GM(\eta-\omega_0)^{\alpha+\beta+\theta}}{\Gamma_q(\alpha+\beta+\theta+1)}. \end{aligned} \tag{2.17}$$

From (2.16) and (2.17), we have

$$\begin{aligned}\big|(\mathcal{A}u)(t)\big| &\le \frac{M(T-\omega_0)^{\alpha+\beta}}{\Gamma_q(\alpha+\beta+1)} + \frac{M(T-\omega_0)^{\alpha+\beta}}{\Gamma_q(\alpha+\beta+1)\min|\Omega|}(T-\omega_0)^{\beta-1}\times\\ &\quad\Big\{\frac{\max|\mathcal{A}_\eta|(T-\omega_0)^{\alpha}\Gamma_q(\alpha)}{\Gamma_q(\alpha+\beta)} + \max|\mathcal{B}_\eta|\Big\}\\ &\quad+\frac{GM(\eta-\omega_0)^{\alpha+\beta+\theta}}{\Gamma_q(\alpha+\beta+\theta+1)\min|\Omega|}(T-\omega_0)^{\beta-1}\Big\{\mathcal{A}_T + \frac{(T-\omega_0)^{\beta-1}\Gamma_q(\alpha)}{\Gamma_q(\alpha+\beta)}(T-\omega_0)^{\alpha}\Big\}\end{aligned}$$

$$
\begin{aligned}
&+\frac{N}{\min|\Omega|}(T-\omega_0)^{\beta-1}\Big\{\mathcal{A}_T+\frac{(T-\omega_0)^{\beta-1}\Gamma_q(\alpha)}{\Gamma_q(\alpha+\beta)}(T-\omega_0)^{\alpha}\Big\},\\
&\le N\mathcal{O}_T+M\Big[\frac{G(\eta-\omega_0)^{\alpha+\beta+\theta}\mathcal{O}_T}{\Gamma_q(\alpha+\beta+\theta+1)}+\frac{(T-\omega_0)^{\alpha+\beta}}{\Gamma_q(\alpha+\beta+1)}\big(\mathcal{O}_\eta+1\big)\Big],\\
&\le N\mathcal{O}_T+M\Theta. \qquad (2.18)
\end{aligned}
$$

We find that

$$
\begin{aligned}
\big|(D^{\nu}_{q,\omega}\mathcal{A}u)(t)\big| &\le N\bar{\mathcal{O}_T}+M\Big[\frac{G(\eta-\omega_0)^{\alpha+\beta+\theta}}{\Gamma_q(\alpha+\beta+\theta+1)}\bar{\mathcal{O}_T}+(T-\omega_0)^{\alpha+\beta}\Big(\frac{\bar{\mathcal{O}_\eta}}{\Gamma_q(\alpha+\beta+1)}\\
&\quad+\frac{(T-\omega_0)^{-\nu}}{\Gamma_q(\alpha+\beta-\nu+1)}\Big)\Big],\\
&\le N\bar{\mathcal{O}_T}+M\bar{\Theta}. \qquad (2.19)
\end{aligned}
$$

Let $L=N\mathcal{O}^*_T+M\Theta^*$.
From (2.18) and (2.19), we obtain $\big\|(\mathcal{A}u)\big\|_{\mathcal{C}}\le L<\infty$ which implies that $\mathcal{A}$ is uniformly bounded.

(ii) Since F is continuous, we can conclude that the operator $\mathcal{A}$ is continous on B_R.

(iii) For any $t_1,t_2\in I^T_{q,\omega}$ with $t_1<t_2$, we find that

$$
\begin{aligned}
\big|(\mathcal{A}u)(t_1)-(\mathcal{A}u)(t_2)\big| &\le \frac{M}{\Gamma_q(\alpha+\beta+1)}\big|(t_2-\omega_0)^{\alpha+\beta}-(t_1-\omega_0)^{\alpha+\beta}\big|\\
&\quad+\frac{\big|(t_2-\omega_0)^{\alpha+\beta-1}-(t_1-\omega_0)^{\alpha+\beta-1}\big|\Gamma_q(\alpha)}{\min|\Omega|\Gamma_q(\alpha+\beta)}\Big[\mathcal{A}_\eta\mathbb{P}^*[F_u]\\
&\quad+(T-\omega_0)^{\beta-1}\big(\mathbb{Q}^*[F_u]+N\big)\Big]\\
&\quad+\frac{\big|(t_2-\omega_0)^{\beta-1}-(t_1-\omega_0)^{\beta-1}\big|}{\min|\Omega|}\Big[\mathcal{B}_\eta\mathbb{P}^*[F_u]+\mathcal{A}_T\big(\mathbb{Q}^*[F_u]+N\big)\Big],
\end{aligned}
\qquad (2.20)
$$

and

$$
\begin{aligned}
\big|(D^{\nu}_{q,\omega}\mathcal{A}u)(t_2)-(D^{\nu}_{q,\omega}\mathcal{A}u)(t_1)\big| &\le \frac{M}{\Gamma_q(\alpha+\beta-\nu+1)}\big|(t_2-\omega_0)^{\alpha+\beta-\nu}-(t_1-\omega_0)^{\alpha+\beta-\nu}\big|\\
&\quad+\frac{\big|(t_2-\omega_0)^{-\nu+\alpha+\beta-1}-(t_1-\omega_0)^{-\nu+\alpha+\beta-1}\big|\Gamma_q(\alpha)}{|\Omega|\Gamma_q(\alpha+\beta-\nu)}\times\\
&\quad\Big[\mathcal{A}_\eta\mathbb{P}^*[F_u]+(T-\omega_0)^{\beta-1}\big(\mathbb{Q}^*[F_u]+N\big)\Big]\\
&\quad+\frac{\big|(t_2-\omega_0)^{-\nu+\beta-1}-(t_1-\omega_0)^{-\nu+\beta-1}\big|\Gamma_q(\beta)}{|\Omega|\Gamma_q(\beta-\nu)}\times\\
&\quad\Big[\mathcal{B}_\eta\mathbb{P}^*[F_u]+\mathcal{A}_T\big(\mathbb{Q}^*[F_u]+N\big)\Big]. \qquad (2.21)
\end{aligned}
$$

The right-hand side of (2.20) and (2.21) tends to be zero when $|t_2 - t_1| \to 0$. Thus, $\mathcal{A}$ is relatively compact on B_R. Thus $\mathcal{A}[B_R]$ is equicontinuous set. By Arzela-Ascoli theorem in Lemmas (9) and (10), we find the $\mathcal{A} : C \to C$ is completely continuous. Hence, from Schauder fixed point theorem in Lemma (11), problem (0.1) has at least one solutions.

3. Hyers-Ulam Stability Analysis Result

In this section, we study the Hyers-Ulam Stability of problem (0.1). Let $\varepsilon > 0$ and $\delta : I^T_{q,\omega} \to \mathbb{R}$ be a continuous function. Consider

$$\begin{aligned}
&\left| D^{\alpha}_{q,\omega} D^{\beta}_{q,\omega} u(t) - F\left[t, u(t), \Psi^{\gamma}_{q,\omega} u(t), D^{\nu}_{q,\omega} u(t)\right] \right| \le \varepsilon\delta(t), \quad t \in I^T_{q,\omega}, \\
&u(\omega_0) = u(T), \\
&\mathcal{I}^{\theta}_{q,\omega} g(\eta) u(\eta) = \varphi(u), \quad \eta \in I^T_{q,\omega} - \{\omega_0, T\}.
\end{aligned} \tag{3.1}$$

Now, we give out the definition of Hyers-Ulam stability of problem (0.1).

Definition 5. *problem (0.1) is Hyers-Ulam stable with respect to problem (3.1), if there exists $A_F > 0$ such that*

$$|\bar{u} - \tilde{u}| \le \varepsilon A_F,$$

for all $t \in I^T_{q,\omega}$, where $\bar{u}$ is the solution of (3.1) and $\tilde{u}$ is the solution for problem (0.1).

Theorem 3. *Assume that $(H_1) - (H_4)$ hold, and $\max_{t \in I^T_{q,\omega}} \delta(t) \le 1$. Then, the problem (0.1) is Hyers-Ulam stable with respect to problem (3.1).*

Proof. Let $D^{\alpha}_{q,\omega} D^{\beta}_{q,\omega} \bar{u}(t) = F\left[t, \bar{u}(t), \Psi^{\gamma}_{q,\omega} \bar{u}(t), D^{\nu}_{q,\omega} \bar{u}(t)\right] + k(t)$. Consider

$$\begin{aligned}
&D^{\alpha}_{q,\omega} D^{\beta}_{q,\omega} \bar{u}(t) = F\left[t, \bar{u}(t), \Psi^{\gamma}_{q,\omega} \bar{u}(t), D^{\nu}_{q,\omega} \bar{u}(t)\right] + k(t), \quad t \in I^T_{q,\omega}, \\
&\bar{u}(\omega_0) = \bar{u}(T), \\
&\mathcal{I}^{\theta}_{q,\omega} g(\eta) \bar{u}(\eta) = \varphi(\bar{u}), \quad \eta \in I^T_{q,\omega} - \{\omega_0, T\}.
\end{aligned} \tag{3.2}$$

Similary to the problem in Theorem 1, problem (3.2) is equivalent to the following equation in Lemma 8.

$$\begin{aligned}
\bar{u}(t) \;=\; & \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)} \int_{\omega_0}^{t} \int_{\omega_0}^{x} \Big(t - \sigma_{q,\omega}(s)\Big)^{\underline{\beta-1}}_{q,\omega} \Big(x - \sigma_{q,\omega}(s)\Big)^{\underline{\alpha-1}}_{q,\omega} \times \\
& \Big\{ F\left[t, \bar{u}(s), \Psi^{\gamma}_{q,\omega} \bar{u}(s), D^{\nu}_{q,\omega} \bar{u}(s)\right] + k(s) \Big\}\, d_{q,\omega} s\, d_{q,\omega} x \\
& - \frac{(t-\omega_0)^{\beta-1}}{\Omega} \Big\{ \mathcal{B}_{\eta} \bar{\mathbb{P}}^*[F_{\bar{u}} + k] + \mathcal{A}_T \big[\varphi\big(u(\eta)\big) - \bar{\mathbb{Q}}^*[F_{\bar{u}} + k]\big] \Big\} \\
& + (t-\omega_0)^{\beta+\alpha-1} \frac{\Gamma_q(\alpha)}{\Omega\Gamma_q(\alpha+\beta)} \Big\{ \mathcal{A}_{\eta} \bar{\mathbb{P}}^*[F_{\bar{u}} + k] + (T-\omega_0)^{\beta-1} \big[\varphi\big(u(\eta)\big) - \bar{\mathbb{Q}}^*[F_{\bar{u}} + k]\big] \Big\},
\end{aligned} \tag{3.3}$$

where $\mathcal{A}_T, \mathcal{A}_\eta, \mathcal{B}_\eta$ and Ω are defined in (1.10)-(1.13), respectively, and the functionals $\bar{\mathbb{P}}^*[F_{\bar{u}}+k]$, $\bar{\mathbb{Q}}^*[F_{\bar{u}}+k]$ are defined by

$$\begin{aligned} \bar{\mathbb{P}}^*[F_{\bar{u}}+k] &:= \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)} \int_{\omega_0}^{T} \int_{\omega_0}^{x} \big(T-\sigma_{q,\omega}(x)\big)_{q,\omega}^{\underline{\beta-1}} \big(x-\sigma_{q,\omega}(s)\big)_{q,\omega}^{\underline{\alpha-1}} \times \\ & \quad \left\{ F\left[t, \bar{u}(s), \Psi_{q,\omega}^{\gamma}\bar{u}(s), D_{q,\omega}^{\nu}\bar{u}(s)\right] + k(s) \right\} d_{q,\omega}s d_{q,\omega}x, \qquad (3.4) \\ \bar{\mathbb{Q}}^*[F_{\bar{u}}+k] &:= \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)\Gamma_q(\theta)} \int_{\omega_0}^{\eta} \int_{\omega_0}^{y} \int_{\omega_0}^{x} g(y)\big(\eta-\sigma_{q,\omega}(y)\big)_{q,\omega}^{\underline{\theta-1}} \big(y-\sigma_{q,\omega}(x)\big)_{q,\omega}^{\underline{\beta-1}} \times \\ & \quad \big(x-\sigma_{q,\omega}(s)\big)_{q,\omega}^{\underline{\alpha-1}} \left\{ F\left[t, \bar{u}(s), \Psi_{q,\omega}^{\gamma}\bar{u}(s), D_{q,\omega}^{\nu}\bar{u}(s)\right] + k(s) \right\} d_{q,\omega}s d_{q,\omega}x d_{q,\omega}y. \qquad (3.5) \end{aligned}$$

Now, we define the operator as

$$(\tilde{\mathcal{A}}u)(t) = (\mathcal{A}u)(t) + \mathcal{K}(t), \qquad (3.6)$$

where

$$\begin{aligned} \mathcal{K}(t) &= \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)} \int_{\omega_0}^{t} \int_{\omega_0}^{x} \big(t-\sigma_{q,\omega}(x)\big)_{q,\omega}^{\underline{\beta-1}} \big(x-\sigma_{q,\omega}(s)\big)_{q,\omega}^{\underline{\alpha-1}} k(s) d_{q,\omega}s d_{q,\omega}x \\ & -\frac{(t-\omega_0)^{\beta-1}}{\Omega} \Big\{ \mathcal{B}_\eta \mathbb{P}[k] + \mathcal{A}_T\big[k(\eta) - \mathbb{Q}[k]\big] \Big\} \\ & +(t-\omega_0)^{\beta+\alpha-1} \frac{\Gamma_q(\alpha)}{\Omega\Gamma_q(\alpha+\beta)} \Big\{ \mathcal{A}_\eta \mathbb{P}[k] + (T-\omega_0)^{\beta-1}\big[k(\eta) - \mathbb{Q}[k]\big] \Big\}, \qquad (3.7) \end{aligned}$$

and the functionals $\mathbb{P}[k]$ and $\mathbb{Q}[k]$ are defined by

$$\begin{aligned} \mathbb{P}[k] &:= \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)} \int_{\omega_0}^{T} \int_{\omega_0}^{x} \big(T-\sigma_{q,\omega}(x)\big)_{q,\omega}^{\underline{\beta-1}} \big(x-\sigma_{q,\omega}(x)\big)_{q,\omega}^{\underline{\alpha-1}} k(s) d_{q,\omega}s d_{q,\omega}x, \qquad (3.8) \\ \mathbb{Q}[k] &:= \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)\Gamma_q(\theta)} \int_{\omega_0}^{\eta} \int_{\omega_0}^{y} \int_{\omega_0}^{x} g(y)\big(\eta-\sigma_{q,\omega}(y)\big)_{q,\omega}^{\underline{\theta-1}} \big(y-\sigma_{q,\omega}(x)\big)_{q,\omega}^{\underline{\beta-1}} \times \\ & \quad \big(x-\sigma_{q,\omega}(x)\big)_{q,\omega}^{\underline{\alpha-1}} k(s) d_{q,\omega}s d_{q,\omega}x d_{q,\omega}y. \qquad (3.9) \end{aligned}$$

Note that

$$\|\tilde{\mathcal{A}}u - \tilde{\mathcal{A}}v\| = \|\mathcal{A}u - \mathcal{A}v\|. \qquad (3.10)$$

Then the existence of a solution of (0.1) implies the existence of a solution to (3.2). It follows from Theorem 1 that $\tilde{\mathcal{A}}$ is a contraction. Thus there is a unique fixed point $\bar{u}$ of $\tilde{\mathcal{A}}$, and $\tilde{u}$ of $\mathcal{A}$.
Since $t \in I_{q,\omega}^{T}$ and $\max_{t\in I_{q,\omega}^{T}} \delta(t) \le 1$, we obtain

$$\|\mathcal{K}\| = \max_{t\in I_{q,\omega}^{T}} |\mathcal{K}(t)| \le \varepsilon\hat{\chi}, \qquad (3.11)$$

where

$$\hat{\chi} = \mathcal{O}_T^* + \left(\frac{\phi_0(T-\omega_0)^{\gamma}}{\Gamma_q(\gamma+1)} + \frac{(T-\omega_0)^{-\nu}}{\Gamma_q(1-\nu)} \right) \Theta^*, \tag{3.12}$$

$\mathcal{O}_T^*$ and Θ^* are defined by (2.11)-(2.12), respectively.

Hence, we get

$$\|\bar{u}-\tilde{u}\| = \|\tilde{\mathcal{A}}\bar{u}-\mathcal{A}\tilde{u}\| = \|\mathcal{A}\bar{u}-\mathcal{A}\tilde{u}+\mathcal{K}(t)\| = \|\mathcal{A}\bar{u}-\mathcal{A}\tilde{u}\| + \|\mathcal{K}(t)\| = \chi\|\bar{u}-\tilde{u}\| + \varepsilon\hat{\chi}. \tag{3.13}$$

By condition (H_4), we obtain

$$\|\bar{u}-\tilde{u}\| \leq \frac{\varepsilon\hat{\chi}}{1-\chi}. \tag{3.14}$$

Let $A_F = \frac{\hat{\chi}}{1-\chi}$, then

$$\|\bar{u}-\tilde{u}\| \leq \varepsilon A_F. \tag{3.15}$$

This completes the proof. □

4. An Illustrative Example

In this section, we consider the following fractional Hahn BVP as

$$\begin{aligned} D_{\frac{3}{4},\frac{2}{3}}^{\frac{1}{2}} D_{\frac{3}{4},\frac{2}{3}}^{\frac{1}{4}} u(t) \quad &= \quad \frac{1}{(10e^3+t^2)(1+|u(t)|)} \left[e^{-2t}\left(u^3+2|u|\right) + e^{-(3\pi+\cos^2\pi t)} \left| \Psi_{\frac{3}{4},\frac{2}{3}}^{\frac{2}{3}} u(t) \right| \right. \\ &\left. + e^{-(2\pi+\sin^2\pi t)} \left| D_{\frac{3}{4},\frac{2}{3}}^{\frac{3}{4}} u(t) \right| \right], \ \eta \in [\frac{8}{3}, 15]_{\frac{3}{4},\frac{2}{3}} - \{\frac{8}{3}, 15\}, \end{aligned}$$

with periodic fractional Hahn-integral boundary condition

$$\begin{aligned} u(\frac{8}{3}) \quad &= \quad u(15), \\ \mathcal{I}_{\frac{3}{4},\frac{2}{3}}^{\frac{1}{3}} \left(10e + \cos\left(\frac{17183}{3072}\right)\right)^3 u\left(\frac{17183}{3072}\right) \quad &= \quad \sum_{i=0}^{\infty} \frac{C_i|u(t_i)|}{1+|u(t_i)|}, \quad t_i = \sigma_{\frac{3}{4},\frac{2}{3}}^{i}(15), \end{aligned}$$

where C_i is given constants with $\frac{1}{1000} \leq \sum_{i=0}^{\infty} C_i \leq \frac{\pi}{1000}$ and $\phi(t,s) = \frac{e^{-|t-s|}}{(t+e)^3}$.

Letting $\alpha = \frac{2}{3}$, $\beta = \frac{1}{4}$, $\gamma = \frac{1}{2}$, $\nu = \frac{2}{5}$, $\theta = \frac{1}{3}$, $q = \frac{3}{4}$, $\omega = \frac{2}{3}$, $T = 15$, $\eta = \sigma_{\frac{3}{4},\frac{2}{3}}^{5}(15) = \frac{17183}{3072}$, $g(t) = (10e+\cos t)^3$ and

$F\left[t,u(t),\Psi^{\gamma}_{q,\omega}u(t),D^{\nu}_{q\omega}u(t)\right] = \frac{1}{(10e^3+t^2)(1+|u(t)|)}\Bigg[e^{-2t}\left(u^3+2|u|\right)+e^{-(3\pi+\cos^2\pi t)}\left|\Psi^{\frac{2}{3}}_{\frac{3}{4},\frac{2}{3}}u(t)\right|+$
$e^{-(2\pi+\sin^2\pi t)}\left|D^{\frac{3}{4}}_{\frac{3}{4},\frac{2}{3}}u(t)\right|\Bigg].$

Using above values, we find that

$$\phi_0 = 0.006404,\quad |\mathbf{A}_T| = 1.015231,\quad |\mathbf{A}_\eta| \leq 32225.953611,\quad |\mathbf{B}_\eta| \leq 38302.628676 \quad \text{and} \quad |\Omega| \geq 26896.840698.$$

For all $t\in I^5_{\frac{3}{4},\frac{2}{3}}$ and $u,v\in\mathbb{R}$, we find that

$$\begin{aligned}&\left|F\left[t,u,\Psi^{\gamma}_{q,\omega}u,D^{\nu}_{q,\omega}u\right]-F\left[t,v,\Psi^{\gamma}_{q,\omega}v,D^{\nu}_{q,\omega}v\right]\right|\\ &\leq \frac{1}{10e^3}|u-v|+\frac{1}{100e^{3+3\pi}}\left|\Psi^{\gamma}_{q,\omega}u-\Psi^{\gamma}_{q,\omega}v\right|+\frac{1}{10e^{3+2\pi}}\left|D^{\nu}_{p,q}u-D^{\nu}_{p,q}v\right|.\end{aligned}$$

Thus, (H_1) holds with $L_1 = 0.004979$, $L_2 = 4.01779\times10^{-8}$ and $L_3 = 9.2974\times10^{-6}$. So $\mathcal{L} = 0.0049787$.

For all $u,v\in\mathcal{C}$,

$$|\varphi(u)-\varphi(v)| \leq \frac{\pi}{1000}\|u-v\|_{\mathcal{C}}.$$

Thus, (H_2) holds with $\lambda = 0.003142$.
In addition, (H_3) holds with $g = 22303.7079$, $G = 22361.6208$.
Since

$$\mathcal{O}^*_T = 1.24855\times10^{-5} \quad \text{and} \quad \Theta^* = 34.336692,$$

therefore, (H_4) holds with

$$\mathcal{X} = 0.1710574 < 1.$$

Hence, by Theorem 1 this problem has a unique solution.

In view of Theorem 3, we have $\hat{\chi} = 12.1764085$ and

$$A_F \approx 14.689085.$$

Therefore, the BVP is Hyers-Ulam stable.

To obtain the numerical results, we set $F\left[t,u(t),\Psi^{\gamma}_{q,\omega}u(t),D^{\nu}_{q\omega}u(t)\right] = 1$, $g = 22350$, $\varphi = -1.5\pi$ and $T = 50$. Using these values while varying θ, we present the graph of the solution to the problem in Figure 1. The results show that the values of $u(t)$ increase as t approaches ω_0. However, at $t=\omega_0$, we have $u(\omega_0)=0$ due to the boundary condition.

This numerical illustration explicitly validates the theoretical existence and uniqueness criteria, demonstrating the framework's capability to handle complex non-linearities on the Hahn time scale. Furthermore, the confirmation of Hyers-Ulam stability suggests that this model is robust against perturbations, making it potentially applicable to real-world discrete systems found in control theory and quantum physics. □

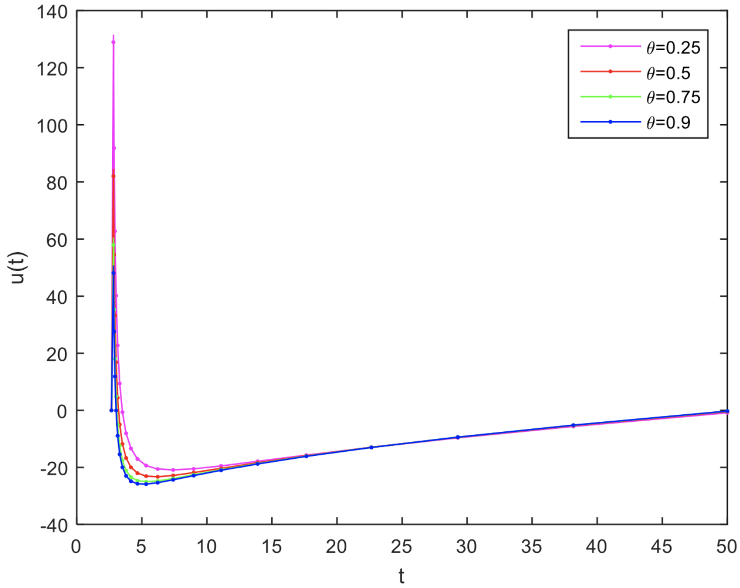

Figure 1: The graph of the solution, where the order of the fractional integral (θ) is varied.

5. Conclusion

In this work, we have successfully demonstrated the uniqueness and existence of solutions for a nonlocal sequential fractional Hahn integro-difference boundary value problem via the Banach and Schauder fixed point theorems, respectively. Furthermore, we have established the Hyers-Ulam stability of the proposed problem.

The distinct novelty of this research lies in the structural complexity of the governing equation, which features a sequential combination of three fractional Hahn difference operators and two fractional Hahn integrals. While recent literature on Hahn calculus has largely restricted its focus to lower-order problems or single-term fractional difference equations, our work addresses a higher-order, composite operator structure. By integrating multiple difference operators with integro-difference terms, we provide a nontrivial generalization of existing BVP frameworks. This approach not only extends the scope of fractional Hahn calculus but also demonstrates the robustness of fixed-point techniques in handling multi-term, nonlocal sequential problems that were previously unexplored.

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