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Strong Fuzzy Planar Graphs: Theoretical Foundations and Applications in Traffic Network Planning Under Uncertainty

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Abstract. This paper introduces strong fuzzy planar graphs (SFPLGs), extending fuzzy graph theory with a quantitative planarity measure $\vartheta(\Omega) = \frac{1}{1+\sum_{i=1}^{n} \Lambda(\theta_i)}$ that classifies networks as strong or weak based on controlled edge crossings. Formal definitions establish fuzzy strong-weak arcs, face memberships, dual graph constructions, and key theorems, including the 0.67 threshold that prohibits strong-strong intersections and maintains planarity values through isomorphism. Theoretical results reconcile classical Kuratowski's graphs with fuzzy gradations. The framework proves effective in the planning of the traffic network, modelling a 10 urban core intersections with vertex memberships of $0.70 - 0.90$ and edge strengths revealing connectivity bottlenecks $CONN$ limited by weak segments $5-10, 9-10$ at 0.70. Strong edges form reliable backbones, while weak links identify upgrade priorities, balancing costs with necessary intersections in environments with uncertain capacities. SFPLGs provide transportation engineers with interpretable tools for durable infrastructure design, with zero-crossing embeddings verifying planarity and edge analysis guiding investments. Future work will investigate dynamic traffic data, multi-layer networks, and intuitionistic variants.

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Key Words and Phrases: Fuzzy set, fuzzy graph, fuzzy planar graph, strong fuzzy planar graph, fuzzy dual graph, traffic network, uncertainty modelling

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1. Introduction

Fuzzy planar graphs extend the traditional idea of planar graphs by integrating fuzzy set theory principles. In a fuzzy planar graph, each vertex and edge has a membership value between 0 and 1 , indicating the degree of belonging or connection [1]. Fuzzy sets were originally introduced by Zadeh [2] as a way to manage uncertainty in sets and

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variables. Classical planar graphs, which can be drawn on a plane without crossing edges, offer a binary classification: a graph is either planar or non-planar. However, real-world networks often display partial planarity with controlled intersection scenarios where strict planarity may not be necessary or achievable, and some overlaps are acceptable or even required [3].

Fuzzy planar graphs incorporate membership values to vertices and edges, extending classical planarity to handle partial overlaps in real-world networks. Unlike binary planar/nonplanar classification, SFPLGs use a computed planarity value $\vartheta(\Omega) = \frac{1}{1+\sum_{i=1}^{n} \Lambda(\theta_i)} > 0.5.$ to quantify structural robustness under uncertainty. This work establishes theoretical foundations, including fuzzy dual graphs and 0.67 -thresholds that prevent strong-strong intersections, then applies SFPLGs to traffic planning where road importance varies, and minor crossings are often necessary.

Recent developments in fuzzy and intuitionistic fuzzy mathematics have opened new avenues for handling uncertainty in structured systems [4]. Multi-criteria evaluation frameworks using intuitionistic fuzzy sets [5] enable complex decision-making in environments with both membership and non-membership uncertainties. Simultaneously, algebraic generalisations such as fuzzy Γ-semimodules [6] and anti-fuzzy algebraic structures [7] demonstrate how algebraic operations can support uncertainty modelling across diverse mathematical contexts. The present work contributes to this landscape by extending planarity concepts from classical graph theory into the fuzzy domain with quantitative rigour, thereby bridging graph-theoretic and fuzzy-algebraic perspectives.

Pal et al. [3] noted that planar graph properties vary in degree based on the membership values of edges within and between themselves. Kuratowski [8] established that a planar graph must not contain subdivisions of K_5 and $K_{3,3}$. Harary [9] explored interesting features of planar graphs, non-planar graphs, and dual graphs. The concept of a fuzzy graph was first introduced by Rosenfeld [10] in 1975 and subsequently modified by Subramani et al. [11].

Samanta et al. [12] introduced a new class of fuzzy planar graphs with a quantified planarity value, extending the fuzzy graph theory to capture graded planarity. Since then, fuzzy graphs have been applied to various domains and extended in multiple directions. Picture fuzzy planar graphs, complex Pythagorean fuzzy planar graphs, m-polar fuzzy planar graphs, and bipolar fuzzy planar graphs have all been developed to handle increasingly complex forms of uncertainty [13–19]. Vague graphs and their applications to facility location and network monitoring have also been explored [20, 21]. Recent work on intuitionistic fuzzy trees and picture fuzzy graphs demonstrates the continued expansion of uncertainty modelling frameworks [22, 23].

Furthermore, the contemporary significance of fuzzy logic in modelling complex systems characterized by uncertainty is underscored by the insights of Zadeh, notably in the context of the Internet of Things (IoT). The inherent vagueness and imprecision managed by fuzzy set theory motivate the extension of classical graph models to fuzzy planar graphs, enabling robust modelling of partially certain real-world networks found in communication, transportation, and biomedical domains [24].

The paper is organised as follows. Section 2 introduces the preliminary concepts, including essential definitions and notations for fuzzy graphs and fuzzy planar graphs. In Section 3, we develop key ideas for fuzzy planar graphs, such as the classification into fuzzy strong and weak fuzzy arcs, the definition of the fuzzy planarity value, fuzzy faces and strong fuzzy planar graphs. Section 4 presents advanced theoretical results, including fuzzy dual graph theory, isomorphism properties, and characterisations of strong fuzzy planar graphs, with practical implications and modern applications across various fields. Section 5 examines classical Kuratowski's graphs and their role within the fuzzy framework, providing mathematical justification for their fuzzy planarity values and reconciling classical nonplanarity with fuzzy classification. In Section 6, we demonstrate the application of the strong fuzzy planar graph framework to planning of traffic networks under uncertainty. Finally, Section 7 summarises the key theoretical and practical contributions, discusses future research directions involving intuitionistic fuzzy extensions and their applications in dynamic network settings, and emphasises the importance of strong fuzzy planar graphs as a versatile tool in both theoretical and applied fuzzy graph research.

The primary contributions of this work include: (1) introducing a rigorous quantitative planarity threshold that classifies fuzzy planar graphs into strong or weak categories based on a calculated fuzzy planarity value; (2) establishing theoretical foundations through formal definitions, theorems, and illustrative examples, including fuzzy faces and fuzzy dual graphs; (3) extending classical graph theory concepts into the fuzzy domain with demonstrated mathematical rigour and practical applicability; and (4) applying the proposed framework to traffic network planning under uncertainty. This comprehensive approach bridges mathematical theory and real-world applications, making significant advancements in fuzzy graph theory and its interdisciplinary applications.

2. Preliminaries

We recall a few essential definitions that are foundational for developing the concept of fuzzy planar graphs and their associated theorems.

In establishing the mathematical foundations for fuzzy planar graphs, the formal framework of fuzzy graphs and fuzzy hypergraphs, as detailed by Mordeson and Mathew is meticulously followed. Their comprehensive treatment provides the necessary theoretical foundation for defining fuzzy vertices, edges, membership functions, and fuzzy planarity measures that underpin this work [25, 26].

Definition 1. *[11] A graph is a triple (σ, μ, I), where σ is a non-empty set of vertices, μ is a set of edges, and $I : \mu \to \sigma \times \sigma$ maps each edge to a pair of vertices.*

Example 1. *Let $\sigma = \{C_1, C_2, C_3, C_4, C_5\}$ be a vertex set, $\mu = \{d_1, d_2, d_3, d_4, d_5, d_6\}$ be an edge set with $I : \mu \to \sigma \times \sigma$ be as $\mu(d_1) = (C_1, C_2), \mu(d_2) = (C_2, C_3), \mu(d_3) = (C_3, C_3), \mu(d_4) = (C_3, C_4), \mu(d_5) = (C_4, C_5)$ and $\mu(d_6) = (C_1, C_5)$.*

This defines a graph $G = (\sigma, \mu, I)$

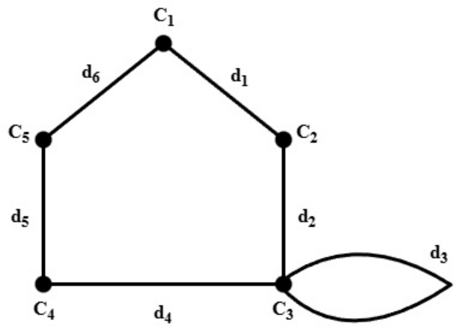


Figure 1: Illustration of a Simple Graph Structure.

Definition 2. *[11] Let S be any non-empty set and L be any set with a function $I : \mathrm{L} \rightarrow \mathrm{S} \times \mathrm{S}$. Let σ be a Fuzzy subset of S, and let μ be a Fuzzy subset of L. For every element $a \in L$, if $\mathrm{I(a)} = \mathrm{(x, y)}$, then the condition*

$$\mu(a) \leq \min\{\sigma(x), \sigma(y)\}$$

must be satisfied. Then the ordered triple $\Omega = (\sigma, \mu, I)$. It is called a fuzzy graph. Here, the elements of σ are called the fuzzy vertices and the elements of μ are called the fuzzy edges.

If $I(a) = (c_1, d_1)$, then $(c_1, \sigma(c_1))$ and $(d_1, \sigma(d_1))$ are said to be adjacent $_{\mathrm{F}}\mathrm{Vs}$, and the fuzzy edge $(a, \mu(\mathrm{a}))$. It is an incident with both. Two distinct fuzzy edges $(a_1, \mu(a_1))$ and $(a_2, \mu(a_2))$, are said to be adjacent fuzzy edges if they are incident on a common fuzzy vertex. Now define the subsets $\sigma^* = \{x \in S \mid \sigma(x) > 0\}$ and $\mu^* = \{a \epsilon L \mid \mu(a) > 0\}$. Then, the ordered triple $\Omega^* = (\sigma^*, \mu^*, I)$ is called the underlying crisp graph of the fuzzy graph Ω.

Definition 3. *[11] Let $\Omega = (\sigma, \mu, I)$ be a Fuzzy graph. A fuzzy edge ($a, \mu(\mathrm{a})$) is called a fuzzy loop if its end vertices are the same; that is, if $I(a) = (x, x)$ for some $x \in S$.*

In other words, a fuzzy edge is a fuzzy loop when it joins a fuzzy vertex to itself.

Definition 4. *[11] Let $\Omega = (\sigma, \mu, I)$ a Fuzzy graph. Then, Ω is called a Fuzzy multigraph if there exist two or more Fuzzy edges that are incident on the same pair of fuzzy vertices.*

In other words, a fuzzy graph is a fuzzy multigraph if multiple fuzzy edges connect the same pair of fuzzy vertices.

Definition 5. *[11] Let $\Omega = (\sigma, \mu, I)$ be a Fuzzy graph. Then, Ω is called a fuzzy simple graph if it contains neither fuzzy loops nor fuzzy multiple edges. A fuzzy simple graph is one in which no fuzzy edge is a loop, and no two fuzzy vertices are connected by more than one edge.*

Example 2. *Consider the Fuzzy graph $\Omega = (\sigma, \mu, I)$ where $S = \{a_1, b_1, c_1, d_1, e_1\}$ is a set of vertices and $L = \{x, y, z, u, v, w, p\}$ is a set of edges. The mapping $\mathrm{I} : \mathrm{L} \rightarrow \mathrm{S} \times \mathrm{S}$ is defined as $\mathrm{I}(x) = (a_1, b_1), I(y) = (b_1, b_1), I(z) = (b_1, c_1), I(u) = (c_1, d_1), I(v) = (c_1, d_1), I(w) = (d_1, e_1)$ and $\mathrm{I}(p) = (a_1, e_1)$.*

The fuzzy subset of vertices is given by $\sigma = \{(a_1, 0.5), (b_1, 0.4), (c_1, 0.6), (d_1, 0.7), (e_1, 0.3)\}$. *A fuzzy relation* R *on* S *with respect to* σ *as follows:*

$$R = \left\{ \begin{array}{c} ((a_1,a_1),0.5),((a_1,b_1),0.4),((a_1,c_1),0.5),((a_1,d_1),0.5),((a_1,e_1),0.3), \\ ((b_1,a_1),0.4),((b_1,b_1),0.4),((b_1,c_1),0.4),((b_1,d_1),0.4),((b_1,e_1),0.3), \\ ((c_1,a_1),0.5),((c_1,b_1),0.4),((c_1,c_1),0.6),((c_1,d_1),0.6),((c_1,e_1),0.3), \\ ((d_1,a_1),0.5),((d_1,b_1),0.4),((d_1,c_1),0.5),((d_1,d_1),0.7),((d_1,e_1),0.3), \\ ((e_1,a_1),0.3),((e_1,b_1),0.3),((e_1,c_1),0.3),((e_1,d_1),0.3),((e_1,e_1),0.3) \end{array} \right\}$$

The fuzzy subset μ *on* L *is given by assigning membership values to the edges as follows:*

$$\mu = \{(x, 0.3), (y, 0.2), (z, 0.1), (u, 0.4), (v, 0.5), (w, 0.15), (p, 0.25)\}$$

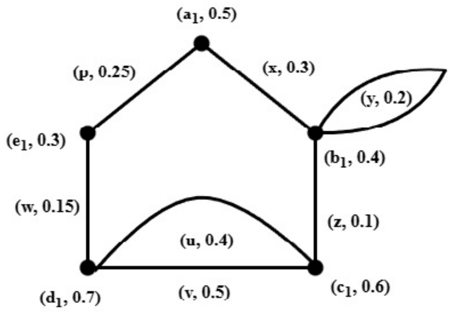


Figure 2: Graphical illustration of a fuzzy planar graph.

Based on the fuzzy graph $\Omega = (\sigma, \mu, I)$ *and its representation in Figure 2, we observe the following structural properties:*
(i) $(a_1, 0.5), (b_1, 0.4), (c_1, 0.6), (d_1, 0.7)$ *and* $(e_1, 0.3)$ *are Fuzzy vertices.*
(ii) $(x, 0.3), (y, 0.2), (z, 0.1), (u, 0.4), (v, 0.5), (w, 0.15)$ *and* $(p, 0.25)$ *are Fuzzy edges.*
(iii) $(a_1, 0.5)(b_1, 0.4)$, $(b_1, 0.4)(c_1, 0.6)$, $(c_1, 0.6)(d_1, 0.7)$, $(d_1, 0.7)(e_1, 0.3)$ *and* $(e_1, 0.3)(a_1, 0.5)$ *are adjacent Fuzzy vertices.*
(iv) Each fuzzy edge connects two fuzzy vertices and is incident on both. For instance, edge $(x, 0.3)$ *is an incident with* $(a_1, 0.5)$ *and* $(b_1, 0.4)$.
(v) Two edges such as $(x, 0.3)$ *and* $(p, 0.25)$ *are adjacent fuzzy edges as they share a common vertex.*
(vi) The edge $(y, 0.2)$ *forms a fuzzy loop, since it connects the vertex* $(b_1, 0.4)$ *to itself.*
(vii) The edges$(u, 0.4)$ *and* $(v, 0.5)$ *are multiple fuzzy edges connecting the same pair of vertices,* $(c_1, 0.6)$ *and* $(d_1, 0.7)$.
(viii) Therefore, the given fuzzy graph is not a fuzzy simple graph, due to the presence of a loop and multiple edges. However, it satisfies the conditions of a fuzzy planar graph, as it can be drawn in the plane without any edge intersections.

Definition 6. *[11] Let* $\Omega = (\sigma, \mu, I)$ *be a Fuzzy graph. The degree of a fuzzy vertex* $\beta \epsilon S$, *denoted by* $d(\beta)$, *is defined as*

$$d(\beta) = \sum_{e \in \Im^{-1}(\alpha,\beta)} \mu(\mathrm{a}) + 2 \sum_{e \in \Im^{-1}(\beta,\beta)} \mu(\mathrm{a}).$$

That is, the degree of a fuzzy vertex β is the sum of the membership values of all fuzzy edges incident with β, where each fuzzy loop contributes twice its membership value.

Definition 7. *[13] Let $\Omega = (\sigma, \mu, I)$ be a Fuzzy graph. Then Ω is called a fuzzy regular graph if the degree of every fuzzy vertex $\beta \in S$ is the same; that is, $d(\beta) = r$, for all $\beta \epsilon S$, where r is constant.*

In this case, Ω is also referred to as a fuzzy r-regular graph.

Definition 8. *[13] Let $\Omega = (\sigma, \mu, I)$ be a Fuzzy graph. Then Ω is called a fuzzy complete graph ($\mathrm{FC_{om}G}$) if every pair of distinct fuzzy vertices is adjacent, and for each edge $a \in L$ with $I(a) = (\alpha, \beta)$, the membership value of the fuzzy edge satisfies:*

$$\mu(a) = R(\alpha, \beta),$$

where R is a fuzzy relation in S defined with respect to the fuzzy vertex set σ.

These definitions and examples provide a sufficient theoretical basis for understanding the construction and behaviour of fuzzy planar graphs explored in the subsequent sections.

3. Fundamental concepts of Fuzzy Planar Graph

3.1. Fuzzy Planar Graph

A fuzzy planar graph extends fuzzy graph theory by incorporating a quantified measure of planarity. This is accomplished by classifying each fuzzy arc (edge) as either strong or weak based on its relationship to its end vertices.

Definition 9. *[12] Let $\Omega = (\sigma, \mu, I)$ be a fuzzy graph, suppose $(\mathrm{a}, \mu(\mathrm{a}))$ is an arc such that $I(\mathrm{a}) = (\xi, \varsigma)$. Then, $(\mathrm{a}, \mu(\mathrm{a}))$ is defined as a fuzzy strong* arc $(_{\mathrm{F}}\mathrm{S_t A})$ *if its membership value is at least half of the minimum membership value of its end vertices:*

$$\mu(\mathrm{a}) \geq \frac{1}{2}\{\min\{\sigma(\xi), \sigma(\varsigma)\}\}$$

If this inequality does not hold, the arc is called a fuzzy weak arc $(\mathrm{FW_e A})$.

Example 3. *Consider the fuzzy graph shown in Figure 3, the arcs $(z_1, 0.6)$,$(z_4, 0.4)$, $(z_5, 0.42)$, and $(z_7, 0.52)$ are classified as fuzzy strong arcs because each arc's membership value satisfies the required strength condition relative to its end vertices. In contrast, the arcs $(z_2, 0.2), (z_3, 0.25)$ and $(z_6, 0.2)$ are identified as fuzzy weak arcs because their membership values do not meet the threshold.*

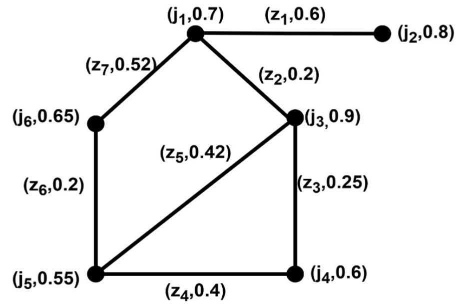


Figure 3: Illustration of a fuzzy graph with both fuzzy strong arcs and fuzzy weak arcs. The arcs $(z_1, 0.6)$,$(z_4, 0.4)$,$(z_5, 0.42)$, and $(z_7, 0.52)$ satisfy the strength condition, while $(z_2, 0.2)$,$(z_3, 0.25)$ and $(z_6, 0.2)$ do not.

Definition 10. *[12] Let $\Omega = (\sigma, \mu, I)$ be a fuzzy graph, and let (a, $\mu(\mathrm{a})$) be a fuzzy arc such that $I(\mathrm{a}) = (\xi, \varsigma)$, where ξ and ς are its end vertices. The strength of the fuzzy* $\mathrm{arc}(\mathrm{a}, \mu(\mathrm{a}))$ *is defined by:*

$$S(a) = \frac{\mu(a)}{\min\{\sigma(\xi), \sigma(\varsigma)\}}.$$

Definition 11. *[12] Let $\Omega = (\sigma, \mu, \mathrm{I})$ be a fuzzy graph, and let $(\mathrm{a}, \mu(\mathrm{a}))$ and $(\mathrm{b}, \mu(\mathrm{b}))$ be two fuzzy arcs such that $I(a) = (\xi, \varsigma)$ and $I(b) = (\varpi, \omega)$. Suppose these arcs intersect at a point θ. Then, the value assigned to the intersection point θ is defined as:*

$$\Lambda(\theta) = \frac{S(a) + S(b)}{2}.$$

Example 4. *Consider the fuzzy graph shown in Figure 4, where the vertices j_1, j_2, j_3, j_4, j_5 and j_6 are connected by fuzzy arcs z_1 through z_8. The membership values of these arcs are: $(z_1, 0.4), (z_2, 0.4), (z_3, 0.33), (z_4, 0.3), (z_5, 0.11), (z_6, 0.22), (z_7, 0.2)$ and $(z_8, 0.1)$.*

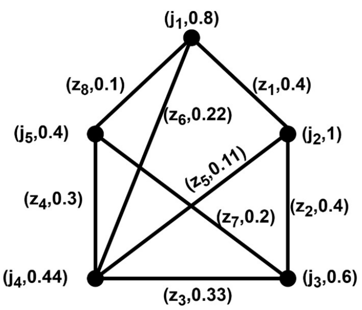


Figure 4: A fuzzy graph showing the calculation of arc strengths and the determination of intersection point values. Strengths are computed using the ratio of arc membership to minimum vertex membership, while intersection values are derived by averaging the strengths of intersecting arcs.

The strengths of these arcs are computed using the definition: $S(z_1) = 0.50, S(z_2) = 0.67, S(z_3) = 0.75, S(z_4) = 0.75, S(z_5) = 0.25, S(z_6) = 0.50, S(z_7) = 0.50, S(z_8) = 0.25$.

For intersecting arcs, the value of an intersection point is given by the mean of their strengths. For instance:

$$\Lambda(\theta_1) = 0.375, \Lambda(\theta_2) = 0.50.$$

This demonstrates how intersecting fuzzy arcs contribute to determining the planarity value.

Definition 12. *Let Ω be a fuzzy planar graph, and let $\theta_1, \theta_2, \ldots, \theta_n$ be its intersection points with corresponding values $\Lambda(\theta_1), \Lambda(\theta_2), \ldots, \Lambda(\theta_p)$. The fuzzy planarity value of Ω is defined by*

$$\vartheta(\Omega) = \frac{1}{1 + \sum_{i=1}^{n} \Lambda(\theta_i)}$$

If the fuzzy planarity value $\vartheta(\Omega)$ exceeds a specified threshold (e.g., 0.5), then Ω is classified as a strong fuzzy planar graph; otherwise, it remains a fuzzy planar graph.

Example 5. *Consider the fuzzy planar graph Ω shown in Figure 5.*

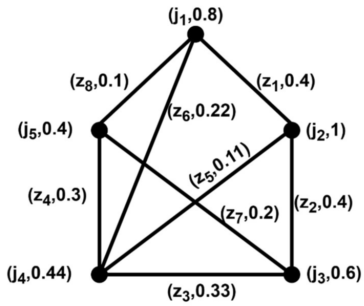


Figure 5: Illustration of a fuzzy planar graph with multiple intersection points. This figure demonstrates how the fuzzy planarity value $\vartheta(\Omega)$ is calculated based on the combined values of intersecting points.

Suppose the values of its two intersection points are $\Lambda(\theta_1) = 0.375$ and $\Lambda(\theta_2) = 0.5$. Applying the definition of fuzzy planarity, the fuzzy planarity value of Ω is calculated as:

$$\vartheta(\Omega) = \frac{1}{1 + 0.375 + 0.50} = \frac{1}{1.875} = 0.533.$$

Since $\vartheta(\Omega) > 0.5$, the graph Ω is classified as a strong fuzzy planar graph ($SFPLG$) according to the threshold condition.

Remark 1. *Embedding Dependence:*

The fuzzy planarity value $\vartheta(\Omega)$ depends on the specific planar embedding (drawing) of the graph. The same graph yields different $\vartheta(\Omega)$ values under different layouts, as demonstrated using Example 5 and Figure 5.

Table 1: Different embeddings of Example 3.3 graph yield different $\vartheta(\Omega)$ values.

Drawing Type	**Crossings**	**Intersection Values**	**Fuzzy planarity value** $\vartheta(\Omega)$	**Classification**
Crossing (Fig 3.3)	2	$\Lambda(\theta_1) = 0.375$ and $\Lambda(\theta_2) = 0.5$	$\vartheta(\Omega) = 1/(1 + 0.875)$ $= 0.533$	SFPLG ($\vartheta(\Omega) > 0.5$)
Improved Layout	1	$\Lambda(\theta_1) = 0.375$	$\vartheta(\Omega) = 1/(1 + 0.375)$ $= 0.727$	SFPLG ($\vartheta(\Omega) > 0.5$)
Optimal Layout	0	None	$\vartheta(\Omega) = 1/(1 + 0)$ $= 1.000$	Classical Planar

In traffic applications, physical GPS straight-line coordinates naturally yield $\vartheta(\Omega) = 1$ when roads admit planar layouts. The Fuzzy Planarity Value $\vartheta(\Omega)$ depends on the specific embedding (drawing) of the graph. For applications, use embeddings with minimum-intersections via force-directed algorithms (e.g., NetworkX spring layout [27] or MATLAB layout G, force), yielding $\vartheta = 1$ for planar road networks.

Remark 2. *The fuzzy planarity value $\vartheta(\Omega)$ provides a numerical measure of how planar a fuzzy graph remains in the presence of intersecting arcs. If $\vartheta(\Omega) > 0.5$, the graph retains a high degree of planarity and is classified as a strong fuzzy planar graph (SFPLG). If the value is less than or equal to 0.5, the graph is still a fuzzy planar graph (FPLG) but does not meet the criterion for strong planarity.*

Remark 3. *The fuzzy planarity value $\vartheta(\Omega)$ always lies in the open interval (0,1]. As the number or strength of intersections increases, the sum $\sum_{i=1}^{n} \Lambda(\theta_i)$ increases, which decreases $\vartheta(\Omega)$. Therefore, more intersecting arcs reduce the degree of planarity of the fuzzy planar graph.*

Definition 13. *A fuzzy planar graph $\Omega = (\sigma, \mu, I)$ is called a ε-fuzzy planar graph if its fuzzy planarity satisfies $\vartheta(\Omega) > \varepsilon$, where $\varepsilon \in (0, 1)$.*

Remark 4. *The definition of SFPLG uses the strict inequality $\vartheta(\Omega) > 0.5$. This choice is motivated by the following:*

1. In fuzzy systems, the threshold 0.5 represents the neutrality point, where membership equals non-membership.

2. The inequality $\vartheta(\Omega) > 0.5$ ensures the planarity measure strictly dominates the non-planarity measure, providing a rigorous quantitative distinction between strong and weak fuzzy planar graphs.

3. While this threshold may appear restrictive, it creates a natural partition: graphs with borderline planarity ($\vartheta(\Omega) \approx 0.5$) are classified as weak FPLGs, allowing practitioners to apply appropriate analysis techniques. Examples 3.3 and 3.4 illustrate this distinction with $\vartheta(\Omega) \approx 0.533$ $(SFPLG)$ and $\vartheta(\Omega) = 0.5$ $(weak - FPLG)$.

4. Alternative frameworks that use fuzzy or soft threshold (e.g, $\vartheta(\Omega) \geq 0.5 - \varepsilon$) could be investigated in future work for applications that require smoother transitions.

3.2. Strong Fuzzy Planar Graphs

Strong fuzzy planar graphs represent a fundamental advancement in graph theory, extending classical planar graph concepts to handle uncertainty and partial memberships. Unlike traditional planar graphs that strictly prohibit edge crossings, strong fuzzy planar graphs allow controlled intersections based on membership values and edge strengths.

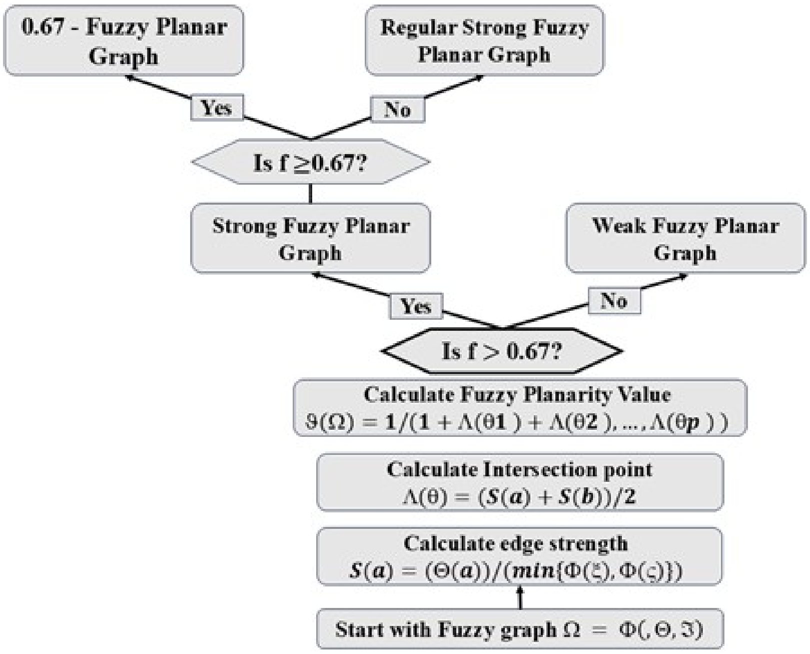


Figure 6: Summarises the workflow for classifying strong fuzzy planar graphs.

Definition 14 (Fuzzy Planarity Value)**.** *Let* Ω *be a fuzzy planar graph with intersection points* $\theta_1, \theta_2, \ldots, \theta_n$*. For a fuzzy arc* uv *with endpoints* u *and* v*, define its strength by*

$$\Lambda(uv) = \frac{\mu(uv)}{\min\{\sigma(u), \sigma(v)\}}.$$

If two fuzzy arcs u_1v_1 *and* u_2v_2 *intersect at* θ_i*, the intersection value is*

$$\Lambda(\theta_i) = \frac{\Lambda(u_1v_1) + \Lambda(u_2v_2)}{2}.$$

Then the fuzzy planarity value of Ω *is defined as*

$$\vartheta(\Omega) = \frac{1}{1 + \sum_{i=1}^{n} \Lambda(\theta_i)}, \quad 0 < \vartheta(\Omega) \leq 1.$$

Definition 15. *A fuzzy planar graph* $\Omega = (\sigma, \mu, I)$ *is called a strong fuzzy planar graph if and only if* $\vartheta(\Omega) > 0.5$*.*

Example 6. *Consider the fuzzy planar graph shown in Figure 3.3. The arc strengths are* $\Lambda(z_5) = 0.44 < 0.5$, $\Lambda(z_6) = 0.44 < 0.5$, $\Lambda(z_7) = 0.4 < 0.5$. *The intersection values are* $\Lambda(\theta_1) = 0.5$, $\Lambda(\theta_2) = 0.5$. *Thus, Therefore, the fuzzy planarity value of* Ω *is:*

$$\vartheta(\Omega) = \frac{1}{1 + 0.5 + 0.5} = \frac{1}{2} = 0.5.$$

Since $\vartheta(\Omega) = 0.5$, *the graph* Ω *is classified as a fuzzy planar graph* $(FPLG)$ *but does not satisfy the condition for a strong fuzzy planar graph* $(SFPLG)$.

Remark 5. *The strict inequality* $\vartheta(\Omega) > 0.5$ *distinguishes strong planarity from neutral cases, ensuring the planarity measure dominates non-planarity in fuzzy systems.*

Definition 16. *Let* $\Omega = (\sigma, \mu, I)$ *be a fuzzy planar graph (FPLG). A fuzzy face (FFa) of a fuzzy planar graph* Ω*is a region bounded by a set of fuzzy arcs in a planar embedding of* Ω. *The membership value of a fuzzy face is defined by:*

$$\mu(F) = \min\left\{\frac{\mu(a)}{\min\{\sigma(\xi), \sigma(\zeta)\}}, a \in \mathcal{Z}, I(a)\right\} = (\xi, \zeta), equivalently, \mu(F) = \min\{S(a), a \in \mathcal{Z}\}.$$

In other words, it is the minimum strength of the fuzzy arcs that bound the face.

A fuzzy face is called a strong fuzzy face if its membership value is greater than 0.5, and a weak fuzzy face otherwise. Every FPLG has an unbounded exterior region, called the outer fuzzy face. All other bounded regions are called inner fuzzy faces.

Definition 17. *Let* $\Omega = (\sigma, \mu, I)$ *be a fuzzy planar graph. A fuzzy face* F *of* Ω *is called an* ε*-fuzzy face if its membership value satisfies* $\mu(F) > \varepsilon$, *where* $\varepsilon \in (0, 1)$.

Example 7. *Consider the fuzzy planar graph shown in Figure 3.1. The strengths of its fuzzy arcs are calculated as follows:*

$S(z_1) = 0.6/0.7 = 0.857$, $S(z_2) = 0.2/0.7 = 0.2857$, $S(z_3) = 0.25/0.6 = 0.4167$, $S(z_4) = 0.4/0.55 = 0.7636$,

$S(z_5) = 0.42/0.55 = 0.7636$, $S(z_6) = 0.2/0.55 = 0.3636$, $S(z_7) = 0.52/0.65 = 0.8$.

The membership values of the fuzzy faces are then determined as the minimum strength of the bounding arcs:

$\mu(F_1) = \min\{S(z_2), S(z_5), S(z_6), S(z_7)\} = \min\{0.2857, 0.7636, 0.3636, 0.8\} = 0.2857$,

$\mu(F_2) = \min\{S(z_3), S(z_4), S(z_5)\} = \min\{0.4167, 0.7636, 0.7636\} = 0.4167$,

$\mu(F_3) = \min\{S(z_1), S(z_2), S(z_3), S(z_4), S(z_5)\} = \min\{0.857, 0.2857, 0.4167, 0.7636, 0.7636\} = 0.2857$.

Since all these membership values are less than 0.5, the fuzzy faces $F_1, F_2,$ *and* F_3 *are classified as weak fuzzy faces. Furthermore, they are also 0.2857-fuzzy faces, since the lowest membership value does not exceed this threshold.*

Definition 18. *Let* $\Omega = (\sigma, \mu, I)$*be an* ε*-fuzzy planar graph with* $\varepsilon = 0.55$, *and let* $F_1, F_2, \ldots, F_p$ *be the strong fuzzy faces of* Ω. *Let* $F = \{F_1, F_2, \ldots, F_p\}$ *be this set of faces. The fuzzy dual graph of* Ω *is a fuzzy planar graph* $\Omega_1 = (\sigma_1, \mu_1, I_1)$, *where the vertex* *membership function is defined by* $\sigma_1 : F \longrightarrow [0,1]$, $\mu_1(F_i) = \max\{\mu(a) : a \in \{$*boundary arcs* $F_i\}\}$.

If $(a, \mu(a))$ *is a fuzzy arc common to faces* F_i *and* F_j*, then its membership value in the dual graph is* $\mu_1(a) = \mu(a)$*. The arc* $(a, \mu_1(a))$ *connects the vertices* F_i *and* F_j *in* Ω_1*. If* $(a, \mu(a))$ *is a pendant fuzzy arc of a face* F_i*, then* $\mu_1(a) = \mu(a)$ *and* $(a, \mu_1(a))$ *forms a self-loop on the vertex* F_i *in* Ω_1*.*

Theorem 1. *If* Ω *is a strong fuzzy planar graph, the number of intersection points between strong arcs is at most one.*

Proof. Let Ω be strong, and suppose two distinct intersection points $\theta 1, \theta 2$ occur between strong arcs. For a strong arc, $S(.) \geq 0.5$, hence $\Lambda(\theta_k) \geq 0.5$ for each k. Therefore,

$$\vartheta(\Omega) \leq \frac{1}{1 + 0.5 + 0.5} = 0.5,$$

Contradicting $\vartheta(\Omega) > 0.5$. Thus, at most one such intersection can occur.

Remark 6. *The classification hierarchy is:*

$$0.67 - FPLG \subset SFPLG \subset FPLG.$$

Example 8. *Let* $\sigma(a) = 0.7$*,* $\sigma(b) = 0.9$*,* $\sigma(c) = 0.75$*,* $\sigma(d) = 0.8$ *and arcs* (a, b) *with* $\mu = 0.4$*,* (c, d) *with*$\mu = 0.45$*, intersecting once. The arc strengths as follows:*

$$S(a, b) = 0.4/0.7 \approx 0.571, S(c, d) = 0.45/0.75 = 0.6.$$

Intersection value:

$$\Lambda = \frac{0.571 + 0.6}{2} \approx 0.585.$$

Planarity value:

$$\vartheta(\Omega) = \frac{1}{1 + 0.585} \approx 0.632 > 0.5.$$

Thus Ω *is SFPLG.*

Definition 19. *Let* Ω *is a 0.67–fuzzy planar graph if:*

$$\vartheta(\Omega) \geq 0.67.$$

From Theorem 2 below, this means no strong–strong arc intersections occur.

Theorem 2. *If* $\vartheta(\Omega) \geq 0.67$*, then* Ω *contains no intersection points between two strong arcs.*

Proof. If a strong–strong intersection exists, $\Lambda \geq 0.5$, giving:

$$\vartheta(\Omega) \leq \frac{1}{1+0.5} \approx 0.666\cdots < 0.67,$$

a contradiction.

Remark 7. *a. 0.67-FPLG ⊂ SFPLG (0.67-graphs are special strong graphs with no strong-strong intersections).*

b. $\vartheta(\Omega) = 1$ *corresponds to classical planarity (no intersections).*

c. The 0.67 threshold represents the boundary above which strong-strong arc intersections become impossible, creating a bridge between fuzzy and classical graph theory.

Example 9. *If all intersections involve at least one weak arc* ($S < 0.5$)*,* $\vartheta(\Omega)$ *can be* ≥ 0.67*. When all arcs are weak,* $\mathcal{P}(\Omega)$ *often exceeds 0.7, ensuring no strong crossings.*

4. Advanced Theoretical Results

4.1. Dual Graph Theory

We now introduce the dual of ϵ -fuzzy planar graph. In a fuzzy dual graph, vertices correspond to the strong fuzzy faces of the ϵ - fuzzy planar graph, and each fuzzy edge between two vertices corresponds to each edge in the boundary between two faces of ϵ -fuzzy planar graph. The formal definition is given below.

Definition 20. [12] *Let* $\Omega = (\sigma, \mu, I)$ *be an* ϵ*-fuzzy planar graph, where the set of strong fuzzy faces is* $\{F_1, F_2, ..., F_p\}$*. The fuzzy dual graph* $\Omega_1 = (\sigma_1, \mu_1, I_1)$ *is defined as follows:*
i). Vertices: Each strong fuzzy face F_i *of* Ω *is represented as a vertex in* Ω_1*.*
ii). Vertex Membership: The membership value of the vertex F_i *in the dual is*
$\sigma_1(F_i) = \max\{\sigma(a) : a \text{ is a boundary arc of } F_i\}$.
iii). Edges: For every fuzzy arc a common to faces F_i *and* F_j *of* Ω*, there is an edge between the vertices* F_i *and* F_j *in* Ω_1*. The membership value of this dual edge is the same as the original:* $\mu_1(a) = \mu(a)$*.*
iv). Self-loops: If a fuzzy arc is a pendant (only part of one face F_i*), a self-loop is created at* F_i *with membership* $\mu_1(a) = \mu(a)$*.*

Example 10. *Consider Figure 7 and 8,*

For this graph, let $\mu(j_1) = 0.7$, $\mu(j_2) = 0.9$, $\mu(j_3) = 0.6$, $\mu(j_4) = 0.55$, $\mu(j_5) = 0.65$ and $I = \{((j_1, j_2), 0.63), ((j_2, j_3), 0.5), ((j_3, j_4), 0.4), ((j_4, j_5), 0.3), ((j_5, j_1), 0.52), ((j_2, j_4), 0.42)\}$.

Thus, the 0.67-fuzzy planar graph has the following fuzzy faces F_1 (bounded by $((j_1, j_2), 0.63)$, $((j_2, j_4), 0.42)$, $((j_4, j_5), 0.3)$) F_2 (bounded by $((j_2, j_3), 0.5)$, $((j_3, j_4), 0.4)$, $((j_2, j_4), 0.42)$) and outer fuzzy face F_3 (surrounded by $((j_1, j_2), 0.63)$, $((j_2, j_3), 0.5)$, $((j_3, j_4), 0.4)$, $((j_4, j_5), 0.3)$, $((j_5, j_1), 0.52)$, $((j_2, j_4), 0.42)$).

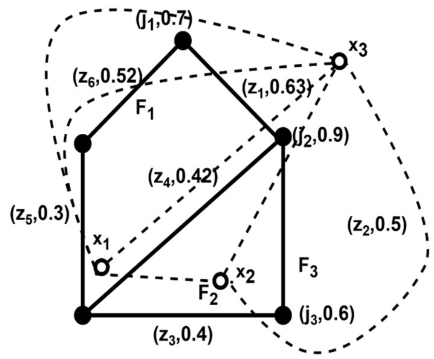


Figure 7: Example of fuzzy dual graph.

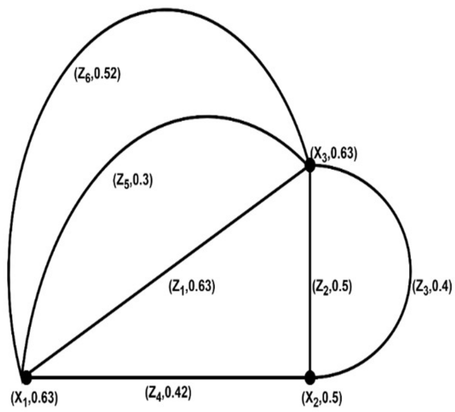


Figure 8: Example of fuzzy dual graph.

The fuzzy dual graph is constructed as follows. Here, all the fuzzy faces are strong fuzzy faces.

For each strong fuzzy face, we consider a vertex for the fuzzy dual graph. Thus, the vertex set $\sigma' = \{x_1, x_2, x_3\}$ where the vertex x_i is taken corresponding to the strong fuzzy face F_i, $i = 1, 2, 3$. So $\sigma'(x_1) = \max\{0.63, 0.42, 0.3\} = 0.63$, $\sigma'(x_2) = \max\{0.5, 0.4, 0.42\} = 0.5$, $\sigma'(x_3) = \max\{0.63, 0.5, 0.4, 0.3, 0.52, 0.42\} = 0.63$. The membership values of other edges of the fuzzy dual graph are calculated as follows:

$$(x_1, x_2)_v = (j_2, j_4)_{\mu^1} = 0.42,$$

$$(x_1, x_3)_v = \{\min(j_1, j_2)_{\mu^1}, (j_4, j_5)_{\mu^1}, (j_5, j_1)_{\mu^1}\} = 0.3,$$

$$(x_2, x_3)_v = \min\{(j_2, j_3)_{\mu^1}, (j_3, j_4)_{\mu^1}\} = 0.4.$$

Thus, the edge set of the fuzzy dual graph is $I' = \{((x_1, x_2), 0.42), ((x_1, x_3), 0.3), ((x_2, x_3), 0.4)\}$. In Figure 7, the fuzzy dual graph $\Omega' = (\sigma', \mu', I')$ of Ω is drawn by a dotted line.

Theorem 3. *Let Ω be a 0.67-fuzzy planar graph without weak edges. The number of vertices, the number of fuzzy edges and the number of strong faces of Ω are denoted by n, p, m respectively. Also let Ω' be the fuzzy dual graph of Ω.*

Then,
(i) the number of vertices of Ω' is equal to m,
(ii) the number of edges of Ω' is equal to p,
(iii) the number of fuzzy faces of Ω' is equal to n.

Proof. The results follow directly from the definition of the fuzzy dual graph.

Theorem 4. *Let Ω' be a fuzzy dual graph of a 0.67-fuzzy planar graph Ω. The number of strong fuzzy faces in Ω' is less than or equal to the number of vertices of Ω.*

Proof. Here, Ω' is a fuzzy dual graph of a 0.67-fuzzy planar graph Ω. Let Ω have n vertices and Ω' has m strong fuzzy faces. Now, Ω may have weak edges and strong edges. To construct a fuzzy dual graph, weak edges are to be eliminated. Thus, it Ω has some weak edges, some vertices may have all their adjacent edges as weak edges. Let the number of such vertices be t. These vertices do not bound any strong fuzzy faces. If we remove these vertices and adjacent edges, then the number of vertices is $n - t$. Again, from Theorem 3, $m = n - t$. Hence, in general $m \leq n$. This concludes that the number of strong fuzzy faces in Ω' is less than or equal to the number of vertices of Ω.

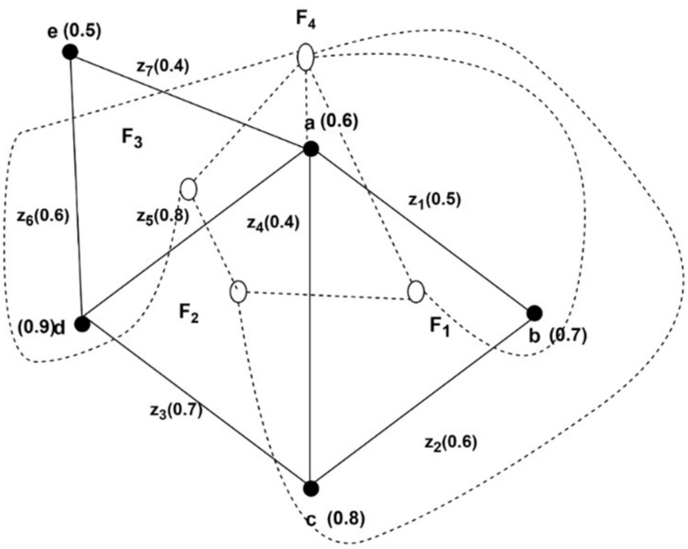


Figure 9: Example of a fuzzy dual graph with a strong face.

In Figure 9, a 0.67 - fuzzy planar graph $\Omega = (\sigma, \mu, I)$ where $\sigma = \{\mu, b, c, d\}$ is above.

For this graph, let $\mu(a) = 0.6$, $\mu(b) = 0.7$, $\mu(c) = 0.8$, $\mu(d) = 0.9$, $\mu(e) = 0.5$ and $I = \{((a, b), 0.5), ((a, c), 0.4), ((a, d), 0.8), ((b, c), 0.6), ((c, d), 0.7), ((d, e), 0.6), ((a, e), 0.4)\}$.

We consider a vertex for the fuzzy dual graph. Thus, the vertex set

$\sigma' = \{x_1, x_2, x_3, x_4\}$ where the vertex x_i is taken corresponding to the strong fuzzy face F_i, $i = 1, 2, 3, 4$.

The corresponding fuzzy dual graph is $\Omega' = (\sigma', \mu', I')$ where $\sigma' = \{x_1, x_2, x_3, x_4\}$, $\mu'(x_1) = 0.7$, $\mu'(x_2) = 0.7$, $\mu'(x_3) = 0.8$, $\mu'(x_4) = 0.7$ and $I' = \{((x_1, x_2), 0.7)$, $((x_1, x_4), 0.8)$, $((x_1, x_4), 0.9)$, $((x_1, x_4), 0.9), ((x_2, x_3), 1)$, $((x_3, x_4), 0.8)$, $((3, x_4), 1)\}$. Here number of strong fuzzy faces is four (see Figure 9).

Theorem 5. *Let $\Omega = (\sigma, \mu, I)$ be a 0.67 -fuzzy planar graph without weak edges, and the fuzzy dual graph of Ω be $\Omega' = (V', \sigma', e')$. The membership values of the fuzzy edges of Ω' are equal to the membership values of the fuzzy edges of Ω.*

Proof. Let $\Omega = (\sigma, \mu, I)$ be a 0.67 -fuzzy planar graph without weak edges. The fuzzy dual graph of Ω is $\Omega' = (\sigma', \mu', I')$ which is a 0.67 - fuzzy planar graph, as there is no point of intersection between any edges. Let $\{F_1, F_2, \ldots, F_k\}$ be the set of strong fuzzy faces of Ω.

From the definition of the fuzzy dual graph, we know that $(x_i, x_j)\nu^l = (u, v)\mu_l^j$ where $(u, v)_l$ is an edge in the boundary between two strong fuzzy faces F_i and F_j and $l = 1, 2, \ldots, s$, where s is the number of common edges in the boundary between F_i and F_j.

The number of fuzzy edges of two fuzzy graphs Ω and Ω' are the same as Ω has no weak edges.

For each fuzzy edge of ψ there is a fuzzy edge in Ω' with the same membership value.

4.2. Isomorphism of Fuzzy Planar Graphs

Isomorphism between fuzzy graphs is an equivalence relation. Importantly, if there is an isomorphism between two fuzzy graphs and one is planar, then the other is also planar.

Theorem 6. *Let Ω be a fuzzy planar graph. If there exists an isomorphism $h : \Omega \to \xi$ where ξ is a fuzzy graph, ξ can be drawn as a fuzzy planar graph with the same planarity value of Ω.*

Proof. Let ψ be a fuzzy planar graph, and there exists an isomorphism $h : \Omega \to \xi$ where ξ is a fuzzy graph. Now, an isomorphism preserves edge and vertex weights. Also, the order and size of fuzzy graphs are preserved in isomorphic fuzzy graphs [13]. So, the order and size of ξ will be equal to Ω. Then, ξ can be drawn similarly to Ω. Hence, the number of intersections between edges and the fuzzy planarity value of ξ will be the same as Ω. This concludes that ξ can be drawn as a fuzzy planar graph with the same fuzzy planarity value.

In crisp graph theory, the dual graph of a planar graph is the planar graph itself. In the fuzzy graph concept, the fuzzy dual graph of a fuzzy dual graph is not isomorphic to a fuzzy planar graph. The membership values of the vertices of the fuzzy dual graph are the maximum membership values of its bounding edges of the corresponding fuzzy faces in the fuzzy planar graph. Thus, vertex weight is not preserved in the fuzzy dual graph. But edge weight is preserved in the fuzzy dual graph. This result is established in the following theorem.

Theorem 7. *Let Ω_2 be the fuzzy dual graph of fuzzy dual graph of a 0.67-fuzzy planar graph Ω without weak edges. Then there exists a co-weak isomorphism between Ω and $\Omega 2$.*

Proof. Let $\Omega = (\sigma, \mu, I)$ be a 0.67 -fuzzy planar graph without weak edges. Also let, $\psi 1$ be the fuzzy dual graph of Ω and Ω_2 be the fuzzy dual graph of $\Omega 1$. Now we have to establish a co-weak isomorphism between Ω_2 and Ω. As the number of vertices of Ω_2 is equal to that of strong fuzzy faces of Ω_1. Again, the number of strong fuzzy faces is equal to the number of vertices of Ω. Thus, the number of vertices of Ω_2 and Ω are same. Also, the number of edges of a fuzzy planar graph and its dual graph is the same. By the definition of fuzzy dual graph, the edge membership value of an edge in a fuzzy dual graph is equal to the edge membership value of an edge in a fuzzy planar graph. Thus, we can construct a co-weak isomorphism from Ω_2 to Ω.

Hence, the result is true.

Theorem 8. *Let ξ_1 and ξ_2 be two isomorphic fuzzy graphs with fuzzy planarity values f_1 and f_2, respectively. Then $f_1 = f_2$.*

The proof of the theorem is an immediate consequence of Theorem. 4.5.

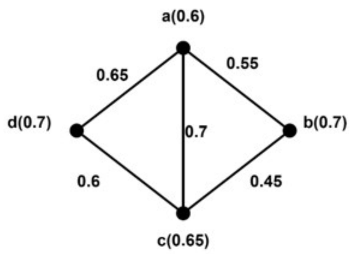
Theorem 9. *Let ξ_1 and ξ_2 be two weakly isomorphic fuzzy graphs with fuzzy planarity values f_1 and f_2, respectively. $f_1 = f_2$ if the edge membership values of corresponding intersecting edges are the same.*

Proof. Here $\xi_1 = (V, \sigma_1, \mu_1)$ and $\xi_2 = (V, \sigma_2, \mu_2)$ are two weak isomorphic fuzzy graphs with fuzzy planarity values f_1 and f_2 respectively. As two fuzzy graphs are weak isomorphic, $\sigma_1(x) = \sigma_2(y)$ for some x in ξ_1 and y in ξ_2. Let the graphs have one point of intersection. Let two intersecting edges be (a_1, b_1) and (c_1, d_1) in ξ_1. Also, two corresponding edges in ξ_2 be (a_2, b_2) and (c_2, d_2). Then, intersecting value of the point is $\frac{\mu(a_1,b_1)}{\sigma(a_1 \wedge \sigma(b_1)} + \frac{\mu(c_1,d_1)}{\sigma(c_1 \wedge \sigma(d_1)}$ given by $\frac{\frac{\sigma(a_1)\wedge\sigma(b_1)}{2} + \frac{\sigma(c_1)\wedge\sigma(d_1)}{2}}{2}$. The intersecting value of the corresponding point in ξ_2 is given as $\frac{\frac{\mu(a_2,b_2)}{\sigma(a_2)\wedge\sigma(b_2)} + \frac{\mu(c_2,d_2)}{\sigma(c_2)\wedge\sigma(d_2)}}{2}$.

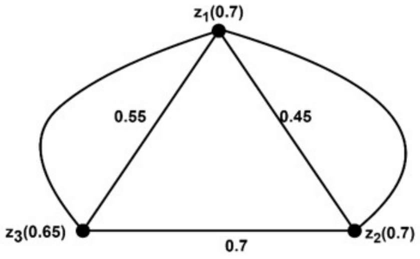
Now, $f_1 = f_2$, if $\mu(a_1, b_1) = \mu(a_2, b_2)$. The number of intersection points may increase. But, if the sum of the intersecting value of ξ_1 is equal to that of ξ_2, fuzzy planarity values of the graphs must be equal. Thus, for equality of f_1 and f_2, the edge membership values of intersecting edges of ξ are equal to the edge membership values of the corresponding edges in ξ_2.

Theorem 10. *Let ξ_1 and ξ_2 be two co-weak isomorphic fuzzy graphs with fuzzy planarity values f_1 and f_2, respectively. $f_1 = f_2$ if the minimum of membership values of the end vertices of corresponding intersecting edges is the same.*

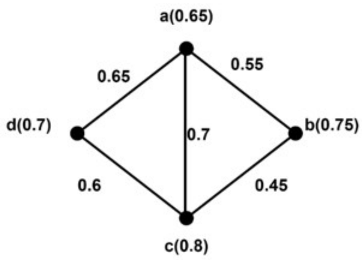
Proof. Here $\xi_1 = (V, \sigma 1, \mu_1)$ and $\xi_2 = (V, \sigma_2, \mu_2)$ are two co-weak isomorphic fuzzy graphs with fuzzy planarity values f_1 and f_2 respectively. As two fuzzy graphs are co-weak isomorphic, $\mu_1(x, y) = \mu_2(z, t)$ for some edge (x, y) in ξ_1 and (z, t) in ξ_2. Let the graphs have one point of intersection. Let two intersecting edges be $(a_1, b_1\ )$ and (c_1, d_1) in ξ_1. Also, two corresponding edges in $\xi 2$ be (a_2, b_2) and (c_2, d_2). Then, intersecting value of the point is given by $\frac{\frac{\mu(a_1,b_1)}{\sigma(a_1)\wedge\sigma(b_1)}+\frac{\mu(c_1,d_1)}{\sigma(c_1)\wedge\sigma(d_1)}}{2}$.



(i) A 0.67 Fuzzy Planar Graph Ω



(ii) Fuzzy Planar Graph Ω_1 of Ω



(iii) Fuzzy Planar Graph Ω_2 of Ω_1

Figure 10: Dual of dual is co-weak isomorphic to a planar graph in fuzzy graph theory.

The intersecting value of the corresponding point in ξ_2 is given as $\frac{\mu(a_2,b_2)}{\sigma(a_2)\wedge\sigma(b_2)}+\frac{\mu(c_2,d_2)}{\sigma(c_2)\wedge\sigma(d_2)}$. Now, the fuzzy planarity values $f_1 = f_2$, if $\sigma_1(a_1) \wedge \sigma(b_1) = \sigma_2(a_2) \wedge \sigma_2(b_2)$. The number of points of intersection may increase. But if the sum of the intersecting value of ξ_1 is equal to that of ξ_2, fuzzy planarity values of the graphs must be equal. Thus, for equality of f_1 and f_2, the minimum membership value of the end vertices of an edge in ξ_1 is equal to that of a corresponding edge in ξ_2.

4.3. Real-World Applications of Strong Fuzzy Planar Graphs

Strong fuzzy planar graphs find extensive applications across multiple domains:

Transportation Networks: In subway system design, tunnels represent edges with varying signal strengths between nodes. Strong fuzzy planar graphs can model these networks where connection quality varies, and some signal interference (crossings) is acceptable for less critical connections [28].

Circuit Design: Electronic circuit layouts benefit from fuzzy planarity concepts when designing printed circuit boards. Wire crossings increase manufacturing complexity and cost, but weak connections can be crossed without significant penalties [29, 30].

Image Processing: Fuzzy planar graphs excel in image segmentation applications. The fuzzy soft planar graph approach has been shown to outperform traditional fuzzy planar graph models in image segmentation tasks, particularly for complex images requiring gradual transitions between regions [31].

Communication Networks: Satellite communication systems utilize varying signal strengths between nodes. Strong fuzzy planar graphs can model these networks where connection quality varies, and some signal interference (crossings) is acceptable for less critical connections [28].

4.4. Theoretical Significance and Extensions

The theory of strong fuzzy planar graphs represents a significant advancement in handling uncertainty in network topology. Unlike classical planar graphs that provide binary classification (planar or non-planar), fuzzy planar graphs offer a continuous spectrum of planarity values, enabling more nuanced analysis of real-world networks [18].

4.5. Duality Relationships

The fuzzy dual graph construction preserves essential structural properties while adapting classical duality concepts to the fuzzy domain. For 0.67-fuzzy planar graphs, the dual relationship maintains edge membership values, establishing a robust theoretical foundation for graph transformations [18].

4.6. Comparative Analysis with Classical Theory

Strong fuzzy planar graphs extend classical planar graph theory in several crucial ways:

1. Flexibility: Allow controlled edge crossings based on membership values
2. Gradual Transitions: Provide continuous planarity measures rather than binary classification
3. Real-world Applicability: Better model networks with varying connection strengths
4. Uncertainty Handling: Incorporate imprecision inherent in practical applications

The 0.67 threshold is particularly significant, marking the boundary above which no strong edge intersections occur and effectively bridging fuzzy and classical planar graph concepts [18].

5. Kuratowski's Graphs and Fuzzy Planar Graphs

In classical graph theory, Kuratowski's theorem characterizes planar graphs as those which do not contain subdivisions of $K5$ (the complete graph on five vertices) or $K_{3,3}$ (the complete bipartite graph on two sets of three vertices) as subgraphs. Both $K5$ and $K_{3,3}$ are notoriously non-planar, meaning they cannot be embedded on a plane without edge crossings [32].

Note: In the fuzzy graph framework, the complete graphs $\tilde{K}_5$ and $\tilde{K}_{3,3}$, formed by fuzzy membership functions assigned to the vertices and edges of these classical graphs, exhibit a fuzzy planarity value $\vartheta = 0.5$. This statement is justified below. According to the foundational planarity formula, a single intersection point $N_p = 1$ leads to:

$$f = \frac{1}{1 + N_p} = \frac{1}{2} = 0.5$$

Extending this, the general fuzzy planarity formula is:

$$\vartheta(\Omega) = \frac{1}{1 + \sum_{i=1}^{n} \Lambda(\theta_i)}$$

For Kuratowski's graphs, this reduces to the base case where the sum of intersection values for the single intersection is 1. This yields the neutral fuzzy planarity measure of 0.5, classifying $\tilde{K}_5$ and $\tilde{K}_{3,3}$ as fuzzy planar graphs (rather than strong fuzzy planar graphs).

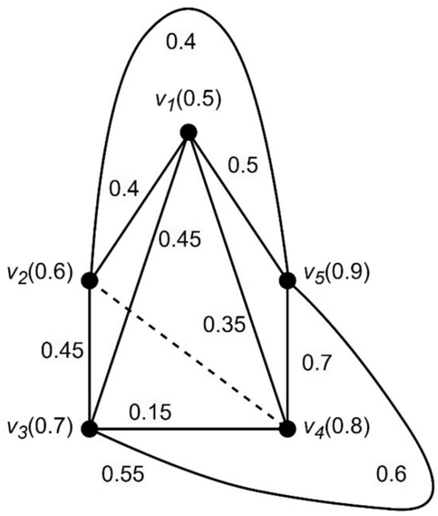


Figure 11: Example of a fuzzy planar graph with $f > 0.5$.

This highlights an important conceptual difference between classical and fuzzy graph theory: classically non-planar structures can still exhibit measurable fuzzy planarity. Our framework, therefore, generalizes classical planarity by quantifying the degree of planarity, rather than enforcing an absolute binary classification.

6. Applications of Strong Fuzzy Planar Graphs in Traffic Planning

This section demonstrates applications of strong fuzzy planar graphs to traffic planning under uncertainty. This section illustrates how strong fuzzy planar graphs support traffic planning under uncertainty. Let $G = (V, E)$ represent a road network, where V denotes intersections and E denotes road segments. A fuzzy graph on G is $\psi = (V, \sigma, E, \mu)$, where $\sigma : V \to [0, 1]$ measures intersection importance and $\mu : E \to [0, 1]$ satisfies $\mu(uv) \leq \min\{\sigma(u), \sigma(v)\}$ for all $uv \in E$.

The fuzzy planarity value is

$$f(\psi) = \frac{1}{1 + N_p},$$

where N_p is the number of edge intersection points in a plane drawing of G. The fuzzy graph ψ is called a strong fuzzy planar if $f(\psi) > 0.5$. Strong edges satisfy $\mu(uv) \geq \min\{\sigma(u), \sigma(v)\}$ and form the robust backbone $\psi_s = (V, \sigma, E_s, \mu|_{E_s})$.

6.1. High-intersection network example

Consider a dense urban core with ten major intersections $V = \{v_1, v_2, ..., v_{10}\}$. The crisp graph $G = (V, E)$ forms a planar two-row structure with $E = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_1v_6, v_2v_7, v_3v_8, v_4v_9, v$

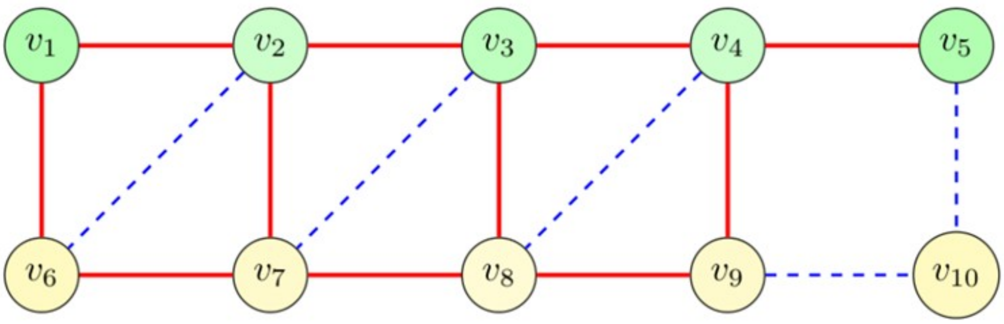


Figure 12: Strong fuzzy planar road network..

Figure 12, depicts the corresponding strong fuzzy planar road network on $V = \{v_1, ..., v_{10}\}$. Green nodes indicate high-importance intersections ($\sigma \geq 0.8$), while yellow nodes represent medium-importance ones. Red thick edges denote strong edges ($\mu(uv) \geq \min\{\sigma(u), \sigma(v)\}$), and blue dashed edges denote weak edges. Since G admits a crossing-free embedding, $N_p = 0$ and hence $f(\psi) = 1 > 0.5$, so ψ is a strong fuzzy planar.

Table 6.1 lists the vertex and edge membership values derived from traffic data. For the pair $(x, y) = (v1, v10)$, representative candidate paths and their minimum edge memberships are as follows:

$P1 : v_1 - v_2 - v_3 - v_4 - v_5 - v_{10}$, $\min\mu = 0.70$;
$P2 : v_1 - v_6 - v_7 - v_8 - v_9 - v_{10}$, $\min\mu = 0.71$;
$P3 : v_1 - v_2 - v_7 - v_8 - v_9 - v_{10}$, $\min\mu = 0.71$.

Table 2: Vertex and edge membership values for the high-intersection network.

Vertex	$\sigma(v_i)$	**Edges**	$\mu(uv)$
v_1	0.90	v_1v_2	0.82
v_2	0.80	v_2v_3	0.78
v_3	0.85	v_3v_4	0.80
v_4	0.80	v_4v_5	0.83
v_5	0.90	v_1v_6	0.75
v_6	0.70	v_2v_7	0.76
v_7	0.75	v_3v_8	0.79
v_8	0.80	v_4v_9	0.77
v_9	0.78	v_5v_{10}	0.70
v_{10}	0.72	v_6v_7	0.72
		v_7v_8	0.74
		v_8v_9	0.76
		v_9v_{10}	0.71
		v_2v_6	0.70
		v_3v_7	0.73
		v_4v_8	0.75
		v_5v_9	0.74
		v_6v_{10}	0.70

6.2. Fuzzy connectivity and route analysis

The fuzzy connectivity index between vertices x and y is defined by

$$\mathrm{CONN}_\psi(x,y) = \max_{P\in\mathcal{P}(x,y)} \min_{uv\in P} \mu(uv),$$

where $\sigma(v)$ denotes vertex membership and $\mu(uv)$ denotes edge membership, with $\mu(uv) \leq \min\{\sigma(u),\sigma(v)\}$ for all edges uv. For $x = v_1$ and $y = v_{10}$, the paths above give $\mathrm{CONN}_\psi(v_1, v_{10}) = 0.71$, with the value limited by the weak edges v_5v_{10}, v_9v_{10}, and v_2v_6.

Define the edge cost by $c(uv) = 1 - \mu(uv)$. For any path P, the total cost is

$$C(P) = \sum_{uv\in P} c(uv).$$

Let P^* be a minimum-cost path between x and y, that is, a path minimizing $C(P)$. Then

$$\mathrm{CONN}_\psi(x,y) \leq \min_{uv\in P^*} \mu(uv),$$

with equality if there exists a maximum-connectivity path that also has minimum cost. Sketch.

For any path P,

$$\min_{uv \in P} \mu(uv) \leq \text{CONN}_\psi(x, y)$$

by definition of CONN_ψ as the maximum of such minima. Equality holds when a minimum-cost path P^* also attains the global maximum of $\min_{uv \in P} \mu(uv)$, i.e., when cost-optimal routing aligns with maximum fuzzy connectivity. In traffic terms, if $\min_{uv \in P^*} \mu(uv)$ is significantly smaller than $\text{CONN}_\psi(x, y)$, then P^* uses unnecessarily weak segments; upgrading such edges (for example, $v5v10$) can simultaneously improve connectivity and reduce effective cost.

The strong edges (red in Figure 1) form ψ_s, the reliable backbone of the network. Weak edges (blue, dashed) are natural candidates for capacity upgrades or alternative routing strategies. Since the embedding is planar ($N_p = 0$), the visualization is clear and interpretable for planners. This framework highlights critical junctions (such as $v5$ and $v10$ as cut vertices in ψ_s), prioritizes weak links for bypass or reinforcement, and supports designs that balance connectivity with congestion-related costs in urban traffic planning.

7. Conclusion

This research advances fuzzy graph theory through strong fuzzy planar graphs (SFPLGs), establishing a rigorous 0.5 planarity threshold that classifies fuzzy networks as strong or weak based on controlled edge intersections. Key theoretical contributions include formal constructions of fuzzy faces and dual graphs, isomorphism properties that preserve planarity values, and the critical 0.67 threshold that prohibits strong-strong arc crossings, bridging classical Kuratowski's non-planarity with fuzzy gradations. The framework demonstrates practical efficacy in traffic planning, where urban road graphs with varying intersection importance $(0.70 - 0.90)$ and edge capacities reveal connectivity bottlenecks, such as weak segments $(5-10, 9-10at0.70)$, and prioritize infrastructure upgrades while accommodating necessary intersections. SFPLGs provide transportation engineers with interpretable tools that balance fuzzy connectivity ($CONN = 0.71$ for key OD pairs) against crossing costs, enabling robust network design under capacity uncertainty. The prioritisezero-crossing urban core example ($\gamma = 1 > 0.5$) confirms strong planarity for planar embeddings while identifying upgrade targets through edge strength analysis. Future directions include developing dynamic SFPLG extensions for real-time traffic data, multilayer road network models, intuitionistic fuzzy variants for hesitancy, and soft threshold adaptations $(e.g., 0.5\epsilon)$ for adaptive urban planning systems. Integration with GIS platforms could operationalise these models for city planners, while machine learning hybrids might predict evolving membership values from traffic patterns. The methodology's adaptability indicates wider uses in power grid design, telecommunication routing, and supply chain networks, particularly where partial planarity under uncertainty is beneficial. Strong fuzzy planar graphs thus become a key tool connecting precise mathematical theory with real-world decision-making in complex systems.

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References

[1] S. Samanta and M. Pal. Fuzzy planar graphs. *IEEE Transactions on Fuzzy Systems*, 23(6):1936–1942, 2015.

[2] L. A. Zadeh. Fuzzy sets. *Information and Control*, 8(3):338–353, 1965.

[3] M. Pal, S. Samanta, and G. Ghorai. Fuzzy planar graphs. In *Modern Trends in Fuzzy Graph Theory*, pages 99–114. Springer Singapore, Singapore, 2020.

[4] K. T. Atanassov. *On Intuitionistic Fuzzy Sets Theory*. Springer-Verlag, 2012.

[5] N. Palaniappan. Multi-criteria comparison in intuitionistic fuzzy systems. *Applied Soft Computing*, 118:108542, 2022.

[6] R. Srivastava and R. Yager. Fuzzy γ-semimodules and their applications. *Journal of Intelligent & Fuzzy Systems*, 35(2):1451–1465, 2018.

[7] A. Z. Khameneh and A. Kilicman. Anti-fuzzy subgroups and strong anti-fuzzy subgroups. *Soft Computing*, 22(17):5687–5695, 2018.

[8] M. Berthold. Constructing fuzzy graphs from examples. *Intelligent Data Analysis*, 3(1):37–53, 1999.

[9] F. Harary. Graph theoretic methods in the management sciences. *Management Science*, 5(4):387–403, 1959.

[10] A. Rosenfeld. Fuzzy graphs. *Fuzzy Sets and their Applications to Cognitive and Decision Processes*, pages 77–95, 1975.

[11] C. Subramani. Notes on i-fuzzy graph. *Journal of Computer and Mathematical Sciences*, 9(11):1601–1608, 2018.

[12] S. Samanta, A. Pal, and M. Pal. New concepts of fuzzy planar graphs. *International Journal of Advanced Research in Artificial Intelligence*, 3(1), 2014.

[13] B. A. M. Nagoor Gani. Order and size in fuzzy graphs. *Bulletin of Pure and Applied Sciences*, 22E(1):145–148, 2003.

[14] B. Bera, S. Amanathulla, and S. K. Mahato. A comprehensive study of picture fuzzy planar graphs with real-world applications. *Journal of Uncertain Systems*, 16(4), 2023.

[15] M. Akram, A. Bashir, and S. Samanta. Complex pythagorean fuzzy planar graphs. *International Journal of Applied and Computational Mathematics*, 6(3):58, 2020.

[16] G. Ghorai and M. Pal. A study on m-polar fuzzy planar graphs. *International Journal of Computing Science and Mathematics*, 7(3):283, 2016.

[17] M. Akram, M. Sarwar, and W. A. Dudek. Bipolar fuzzy planar graphs. In *Bipolar Fuzzy Sets*, pages 223–251. 2021.

[18] T. Mahapatra, S. Sahoo, G. Ghorai, and M. Pal. Interval valued m-polar fuzzy planar graph and its application. *Artificial Intelligence Review*, 54(3):1649–1675, 2021.

[19] D. Jaisankar, S. Ramalingam, N. Deivanayagampillai, and T. Walelign. Fuzzy outerplanar graphs and its applications. *International Journal of Computational Intelligence Systems*, 17(1):231, 2024.

[20] H. Rashmanlou, S. Samanta, M. Pal, and R. A. Borzooei. A study on vague graphs. *SpringerPlus*, 5(1):1234, 2016.

[21] Y. Talebi and H. Rashmanlou. New concepts of domination sets in vague graphs with applications. *International Journal of Computing Science and Mathematics*, 10(4):375, 2019.

[22] R. Yongsheng, Saeed Kosari, A. A. Talebi, A. Mahdavi, and H. Rashmanlou. New concepts of intuitionistic fuzzy trees with applications. *International Journal of Computational Intelligence Systems*, 14:175, 2021.

[23] Xiaolong Shi, Saeed Kosari, Ali Asghar Talebi, Seyed Hossein Sadati, and Hossein Rashmanlou. Investigation of the main energies of picture fuzzy graph and its applications. *International Journal of Computational Intelligence Systems*, 15:31, 2022.

[24] Obbu Ramesh, Nainaru Tarakaramu, G. Y. Lakshmi, S. S. Basha, S. Iskandarov, A. Toyirov, J. Yuldoshev, S. Sabirov, and K. K. Prashanth. Planarity of intuitionistic pythagorean fuzzy graphs and its application in decision making. *European Journal of Pure and Applied Mathematics*, 18(3):6372, 2025.

[25] J. N. Mordeson and S. Mathew. *Fuzzy Graphs and Fuzzy Hypergraphs*. Springer International Publishing, 2 edition, 2018.

[26] L. A. Zadeh. Fuzzy logic and the internet of things. *IEEE Computational Intelligence Magazine*, 10(4):14–17, 2015.

[27] A. A. Hagberg, D. A. Schult, and P. J. Swart. Exploring network structure, dynamics, and function using networkx. In *Proceedings of the SciPy Conference*, pages 11–15, 2008.

[28] J. Z. Wu and G. Beliakov. Some ideas about graphic representations of discrete fuzzy measures. arXiv preprint, 2023.

[29] Angel Benjamin and Angel Dharmakkan. Fuzzy graph-based machine learning optimization for permeable pavement systems in smart cities of thoothukudi. *Scientific Reports*, 15:35123, 2025.

[30] Shriram Kalathian, B. S. Ramalingam, and N. Deivanayagampillai. A fuzzy planar subgraph formation model for partitioning very large-scale integration networks. *Decision Analytics Journal*, 9:100339, 2023.

[31] W. A. Khan, A. E. M. Niaz, and T. T. Nguyen. A fuzzy soft planar graph with application in image segmentation. *Scientific Reports*, 15:21809, 2025.

[32] V. K. Balakrishnan. *Graph Theory*. McGraw-Hill, 1997.