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Existence Results for Neutral and Second-Order Functional Differential Equations with Causal Operators in L^p_{loc} Spaces

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Abstract. This paper studies the particular class of second order functional differential equations involving casual operators on a function space $L^p_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^n)$. Previous studies [1, 2] discussed this equations in the following different function spaces $C(\mathbb{R}_+, \mathbb{R}^n)$ and $L^2_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^n)$. We establish the existence and uniqueness of solutions for both linear and nonlinear cases. Our worked based on the resolvent kernel method, Hölder's inequality, and successive approximation techniques. Finally, we provide examples to illustrate our results.

2020 Mathematics Subject Classifications: 34K40, 34K05, 45D05

Key Words and Phrases: Neutral functional equations, causal operators, global existence, Volterra integral equations.

1. Introduction

Neutral functional differential equations with causal operators play an important role in various area in mathematics and engineering, such that the analysis systems with memory or recurrence, fluid dynamics, heat exchange, and phenomena described by the Volterra integral equations. The research of these equations has a long history, rooted in the early studies on integral equations and causal operators [3–8]. The study of neutral functional equations with causal operators on the semi-axis was given in [1], and the existence and uniqueness solution were obtained in the space $C(\mathbb{R}_+, \mathbb{R}^n)$. Later, the approaches was extended to the Hilbert space $L^2_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^n)$ [2], where they applied the Cauchy–Schwarz type estimates to the inner product structure.

The present paper extends the same class of equations to the general Banach space $L^p_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^n)$, for $1 < p < \infty$. Our approach lies in replacing Hilbert-space methods with estimates based on Hölder's inequality. This technique allows the existence and uniqueness theory to be extended to a wider class of function spaces. Both linear and nonlinear cases

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are handled where the nonlinear analysis depending on successive approximations under a variable Lipschitz condition.

the study of neutral functional equations and related operator methods has been growing in recent years. For example, neutral equations with sequential fractional operators were investigated in [9]. In addition, [10] addressed neutral equations with infinite delay and applications to compartmental systems. Additional research focus on applying causal operator methods to the study and development of the equations in Banach spaces [11], and nonlinear functional integral equations considered by approximation techniques in $C[0,1]$ [12]. These works demonstrate the growing of the area, but they did not discuss the studies or the general results in the $L^p_{\rm loc}$ framework.

In this paper, We study the existence and uniqueness of solutions for neutral functional differential equations with causal operators in $L^p_{\rm loc}$, $1 < p < \infty$. In the linear case, we develop a resolvent kernel approach based on Hölder estimates, to ensure that the Volterra integral equation is well-posed. In the nonlinear case, we use successive approximations and a Grönwall-type inequality to prove existence and uniqueness under a Lipschitz condition with variable coefficient in $L^1_{\rm loc}$.

The paper is organized as follows. Section 2 recalls basic definitions and notations. Section 3 formulates the problem in equivalent integral form. Section 4 discusses the linear case, while Section 5 studies the nonlinear case. Section 7 provides examples that explain the application of the main results.

2. Preliminaries

Let $L^p_{\rm loc}(\mathbb{R}_+, \mathbb{R}^n)$ denote the space of measurable functions $x : \mathbb{R}_+ \to \mathbb{R}^n$ such that $\|x\|_{L^p([0,T])} < \infty$ for every $T > 0$. A causal operator T is an operator acting on $L^p_{\rm loc}$ such that $(Tx)(t)$ depends only on $\{x(s) : 0 \le s \le t\}$.

Consider the following problem

$$\frac{d}{dt}\left[x'(t) - (Lx)(t)\right] = (Vx)(t), \quad t \in \mathbb{R}_+, \tag{1}$$

with initial conditions

$$x(0) = x_0 \in \mathbb{R}^n, \qquad x'(0) = v_0 \in \mathbb{R}^n. \tag{2}$$

Here L and V are causal operators on $L^p_{\rm loc}$. With suitable assumptions, equation (1) can be written as an integral equation. For example, if V is linear,

$$\int_0^t (Vx)(s)\,ds = \int_0^t k(t,s)x(s)\,ds, \quad t \in \mathbb{R}_+, \tag{3}$$

with a measurable kernel $k(t,s)$.

The associated Cauchy operator $X(t,s)$ admits the following representation

$$X(t,s) = I + \int_s^t \tilde{k}(t,u)\,du, \tag{4}$$

where $\tilde{k}$ is the resolvent kernel corresponding to $k_0(t,s)$ which is the kernel of L. Integrating (1) from 0 to t yields

$$x'(t) - (Lx)(t) = v_0 + \int_0^t (Vx)(s)\,ds. \tag{5}$$

Substituting (3) leads to

$$x(t) = f(t) + \int_0^t k_1(t,s)x(s)\,ds, \tag{6}$$

with

$$f(t) = X(t,0)x_0 + \int_0^t X(t,s)v_0\,ds, \tag{7}$$

$$k_1(t,s) = \int_s^t X(t,u)k(u,s)\,du. \tag{8}$$

Thus the Cauchy problem (1)–(2) is equivalent to a Volterra-type integral equation with kernel k_1.

3. Linear Case

We now focus on the situation where both L and V are linear, continuous, and causal operators acting on $L^p_{\text{loc}}(\mathbb{R}_+,\mathbb{R}^n)$, with $1 < p < \infty$. In this case, the Cauchy problem (1)–(2) can be reduced to a Volterra-type integral equation.

As shown in Section 2, the problem (1)–(2) is equivalent to

$$x(t) = f(t) + \int_0^t k_1(t,s)x(s)\,ds, \qquad t \in \mathbb{R}_+, \tag{9}$$

with

$$f(t) = X(t,0)x_0 + \int_0^t X(t,s)v_0\,ds, \tag{10}$$

$$k_1(t,s) = \int_s^t X(t,u)k_0(u,s)\,du, \qquad (t,s) \in \Delta, \tag{11}$$

where $\Delta = \{(t,s) : 0 \le s \le t < \infty\}$, $X(t,s)$ is the Cauchy operator associated with L, and k_0 is the kernel corresponding to the representation of L.
From the continuity of L we know that

$$\int_0^t (Lx)(s)\,ds = \int_0^t k_0(t,s)x(s)\,ds, \qquad t \in \mathbb{R}_+, \tag{12}$$

with $k_0 \in L^p_{\text{loc}}(\Delta; \mathcal{L}(\mathbb{R}^n,\mathbb{R}^n))$.
The Cauchy operator admits the representation

$$X(t,s) = I + \int_s^t \tilde{k}(t,v)\,dv,$$

where $\tilde{k}$ is the resolvent kernel of k_0. Substituting this into (11) gives

$$k_1(t,s) = \int_s^t \Big(I + \int_s^t \tilde{k}(t,v)\,dv \Big) k_0(u,s)\,du. \tag{13}$$

To prove that $k_1 \in L^p_{\text{loc}}(\Delta)$, fix $T > 0$ and consider

$$\int_0^T \int_0^T |k_1(t,s)|^p \,dt\,ds. \tag{14}$$

From (13) we estimate

$$|k_1(t,s)| \le \int_s^t \left| \int_s^u \tilde{k}(t,v)\,dv \right| |k_0(u,s)|\,du + \int_s^t |k_0(u,s)|\,du.$$

Now apply Hölder's inequality with conjugate exponents $p, q > 1$ $(1/p + 1/q = 1)$ to obtain

$$\left| \int_s^t \int_s^u \tilde{k}(t,v) k_0(u,s)\,dv\,du \right| \le \left(\int_s^t \int_s^u |\tilde{k}(t,v)|^p \,dv\,du \right)^{1/p} \left(\int_s^t \int_s^u |k_0(u,s)|^q \,dv\,du \right)^{1/q}. \tag{15}$$

Since $\tilde{k}, k_0 \in L^p_{\text{loc}}(\Delta)$, the integrals on the right-hand side of (15) are finite for $t, s \le T$. Therefore,

$$\int_0^T \int_0^T |k_1(t,s)|^p \,dt\,ds < \infty. \tag{16}$$

Thus we conclude that

$$k_1 \in L^p_{\text{loc}}(\Delta; \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)).$$

By standard results for Volterra integral equations, the kernel k_1 admits a resolvent $\tilde{k}_1 \in L^p_{\text{loc}}(\Delta)$, and the unique solution of (9) is given by

$$x(t) = f(t) + \int_0^t \tilde{k}_1(t,s) f(s)\,ds, \qquad t \in \mathbb{R}_+. \tag{17}$$

Since f defined in (10) belongs to $L^p_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^n)$, it follows that $x \in L^p_{\text{loc}}$. Moreover, x is locally absolutely continuous on $\mathbb{R}_+$ and $x'(t) - (Lx)(t)$ exists almost everywhere.

Theorem 1. *Let L and V be linear, continuous, and causal operators on $L^p_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^n)$ with $1 < p < \infty$. Suppose that their kernels k_0 and k belong to $L^p_{\text{loc}}(\Delta)$. Then the Cauchy problem* (1)–(2) *admits a unique solution $x \in L^p_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^n)$, which is locally absolutely continuous and satisfies the representation* (17).

4. Nonlinear Case

We now turn to the situation where L remains linear, continuous, and causal, but the operator V is nonlinear. The analysis is based on the approach of successive approximations in L^p_{loc}.

In this case, the Cauchy problem (1)–(2) is equivalent to

$$x(t) = f(t) + \int_0^t X(t,s) \int_0^s (Vx)(u)\,du\,ds, \qquad t \in \mathbb{R}_+, \tag{18}$$

where $f(t)$ is given by (10).

We assume that V satisfies a Lipschitz-type condition of the form

$$\|(Vx)(t) - (Vy)(t)\| \le \lambda(t)\|x(t) - y(t)\|, \qquad t \in \mathbb{R}_+, \tag{19}$$

for all $x, y \in L^p_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n)$, where $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing function with $\lambda \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$.

Define the sequence $\{x_k\}_{k\ge 0}$ by

$$x_0(t) = f(t), \tag{20}$$

$$x_{k+1}(t) = f(t) + \int_0^t X(t,s) \int_0^s (Vx_k)(u)\,du\,ds, \qquad k \ge 0. \tag{21}$$

Subtracting the relations for x_{k+1} and x_k gives

$$x_{k+1}(t) - x_k(t) = \int_0^t X(t,s) \int_0^s \big[(Vx_k)(u) - (Vx_{k-1})(u)\big]\,du\,ds. \tag{22}$$

By (19), we obtain

$$\|(Vx_k)(u) - (Vx_{k-1})(u)\| \le \lambda(u)\|x_k(u) - x_{k-1}(u)\|.$$

Hence, using (22),

$$\|x_{k+1}(t) - x_k(t)\| \le \int_0^t \|X(t,s)\| \int_0^s \lambda(u)\|x_k(u) - x_{k-1}(u)\|\,du\,ds. \tag{23}$$

Define

$$y_k(t) = \sup_{0\le s\le t} \|x_k(s) - x_{k-1}(s)\|.$$

Then (23) yields

$$y_{k+1}(t) \le M \int_0^t \lambda(s) y_k(s)\,ds, \tag{24}$$

where $M = \sup_{0\le s\le t} \|X(t,s)\|$ is finite on bounded intervals due to the properties of the Cauchy operator.
Iterating (24) and applying Grönwall's inequality, we obtain

$$y_{k+1}(t) \le \frac{A(M\Lambda(t))^k}{k!}, \qquad \Lambda(t) = \int_0^t \lambda(s)\,ds,$$

for some constant $A > 0$ depending on the initial difference. This shows that $\{x_k\}$ is a Cauchy sequence in $L^p([0,T])$ for any $T > 0$, hence it converges to a function x in L^p_{loc}.

Theorem 2. *Let L be linear, continuous, and causal on $L^p_{\text{loc}}(\mathbb{R}_+,\mathbb{R}^n)$, with $1 < p < \infty$. Let V be a causal operator satisfying the Lipschitz condition* (19) *with $\lambda \in L^1_{\text{loc}}(\mathbb{R}_+)$. Then the problem* (1)–(2) *has a unique solution $x \in L^p_{\text{loc}}(\mathbb{R}_+,\mathbb{R}^n)$. Moreover, x is locally absolutely continuous and satisfies* (18).

5. Examples

In this section, we provide examples with explicit kernels to illustrate our results.

5.1. Example: Convolution-Type Kernel

Consider the operator V defined by

$$(Vx)(t) = \int_0^t e^{-(t-s)}x(s)\,ds, \qquad t \ge 0. \tag{25}$$

The associated kernel is

$$k(t,s) = e^{-(t-s)}I_n, \qquad (t,s) \in \Delta,$$

where I_n is the $n \times n$ identity matrix.

For any finite $T > 0$, we compute

$$\int_0^T\int_0^t |k(t,s)|^p\,ds\,dt = \int_0^T\int_0^t e^{-p(t-s)}\,ds\,dt = \int_0^T \frac{1-e^{-pt}}{p}\,dt < \infty.$$

Thus $k \in L^p_{\text{loc}}(\Delta)$ for all $1 < p < \infty$. Since the assumptions of Theorem 1 are satisfied, the linear problem with kernel (25) admits a unique solution $x \in L^p_{\text{loc}}(\mathbb{R}_+,\mathbb{R}^n)$.

5.2. Example: Nonlinear Perturbation

Let

$$(Vx)(t) = \phi\left(\int_0^t e^{-(t-s)}x(s)\,ds\right),$$

where $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz continuous with constant $L > 0$. Then

$$\|(Vx)(t) - (Vy)(t)\| \le L\int_0^t e^{-(t-s)}\|x(s)-y(s)\|\,ds.$$

This estimate shows that condition (19) holds with

$$\lambda(t) = L \int_0^t e^{-(t-s)}\, ds = L(1 - e^{-t}),$$

which belongs to $L^1_{\text{loc}}(\mathbb{R}_+)$. Hence Theorem 2 applies, and the problem has a unique solution in L^p_{loc}.

5.3. Non-Example: Nonintegrable Kernel

Consider instead

$$k(t,s) = \frac{1}{t-s}, \qquad 0 < s < t.$$

For this kernel,

$$\int_0^T \int_0^t |k(t,s)|^p\, ds\, dt = \int_0^T \int_0^t \frac{1}{|t-s|^p}\, ds\, dt = \infty, \qquad \text{for } p \geq 1.$$

Thus $k \notin L^p_{\text{loc}}(\Delta)$ for any $p \geq 1$. In this case, the assumptions of Theorem 1 fail, and the existence of a resolvent kernel cannot be guaranteed. This example shows the sharpness of the integrability requirement on the kernel.

6. Conclusion

In this paper, we investigated a class of neutral functional differential equations with causal operators acting on the Banach space $L^p_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^n)$, $1 < p < \infty$. We established the results for both the linear and nonlinear cases. For the linear case, we proved that under suitable assumptions on the kernels associated with the operators L and V, the Cauchy problem admits a unique solution in L^p_{loc}. To prove the result, we reformulated the problem as a Volterra integral equation and showed that the generated kernel k_1 belongs to L^p_{loc} and applied the resolvent kernel method. However, for the nonlinear case, we introduced a Lipschitz-type condition with coefficient $\lambda(t) \in L^1_{\text{loc}}$ and established existence and uniqueness of solutions by the method of successive approximations. In addition, we used integral inequalities and a Grönwall inequality to show convergence. Finally, we illustrated both cases with explicit examples and a non-example. Future work may address asymptotics in $BC(\mathbb{R}_+, \mathbb{R}^n)$, extensions to Orlicz spaces, or delay equations that are special cases of equations involving causal operators or abstract Volterra operators.

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