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Exploring the Applications of Grill Rough Topological Structures Generated by Different Minimal Neighborhood Types

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Abstract. A variety of grill-based topologies are developed and contrasted with earlier topologies. The results demonstrate that the present ones exceed their predecessors. This study distinguishes itself by highlighting the advantages of certain topologies and identifying both the minimum and maximum values. These structural topologies are later utilized to conduct a more thorough investigation of extended rough sets. Compared to earlier models, the suggested approximate models reduce vagueness and uncertainty, which makes them especially important when applied to rough sets ($rhss$). Furthermore, the suggested models differ from their predecessors in that they exhibit all of Pawlak's properties, including the capability of contrasting various approximations ($Aprs$), and have the quality of monotonicity across all relations. In addition, the importance of new discoveries was highlighted by demonstrating their use for human health. Besides examining its limitations, the benefits of the chosen technique were assessed. The paper ends with a summary of the main ideas of the proposed methodology and recommendations for future research paths.

2020 Mathematics Subject Classifications: 54A05, 54C55, 54D80

Key Words and Phrases: Upper-lower Apr, grill, minimal neighborhood, ($rhss$)

1. Introduction

In the 1980s, Pawlak [1, 2] created rough set theory ($rhst$) to deal with uncertainty in medical and technical data processing, as well as other domains. It is very beneficial for assessing inadequate or confusing information systems, as well as categorizing data. According to the facts at hand, each class comprises a collection of items that are comparable to one another. To deal with uncertainty and confusion, the theory makes use of two main ($Aprs$). In situations where exact limits are unclear, this distinction aids in data management and analysis. Through ($Aprs$) of groups that cannot be properly specified, it also offers a formal method of handling ambiguous or incomplete information. $rhst$

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has been expanded in a number of ways to handle a greater variety of uncertainty and complexity in data processing [3-6]. Although the equivalence relation (eqr) is the main emphasis of classic $rhst$, its generalizations seek to improve and broaden this methodology to handle more complicated kinds of data and do away with the equivalency requirement. $rhst$ can now handle a wider variety of complicated and varied datasets thanks to these generalizations. This also allows it to be used for a broader range of issues in domains such as data analysis, machine learning, decision support systems, smart cities, and medicine, among others. Practitioners as well as researchers can obtain more precise and significant insights from data with different levels of uncertainty by broadening the use of $rhst$, which also makes decision-making easier. Topology was used to achieve one of these generalizations. The connection between rough sets ($rhss$) and topology was initially identified by [7, 8]. Here, the closure is linked to the upper (up) approximation (Apr), whereas the topological notion of the interior is linked to the lower (lw) Apr [9–14]. A grill is a nonempty collection of closed sets with hereditary property and limited additivity [15]. This idea was first put forth by Choquet [15]. The grill and ideas, nets, and filters have a number of similarities. Many theories and facets have been covered by [16, 17] and [18]. It facilitates the enhancement of the topological framework, which substitutes numerical values for evaluating attributes such as affection, intellect, aesthetic appeal, and academic achievement. Furthermore, by employing the notion of grill modifications in the border region (Br), up, and lw-$Aprs$, it expands the topological structure and creates novel limits in nano topological spaces [17]. When it comes to removing ambiguity from raw sets, grills are proven to be helpful [19, 20]. Therefore, the introduction of novel grill-based rhs approaches is one of the main reasons for this effort. In other words, when the grill is the universal set, it generates a unique scenario for its generic rhs model counterparts. As a result, some interesting research looked into the $rhst$ that grills describe [21]. Neighborhoods (nbs) are a fundamental topological notion to understand and evaluate set $Aprs$. nbs and any relation was used to generate Apr spaces in [22, 23].

Neighborhood (nb) systems are used to generalize $rhst$ by describing $Aprs$ through a nb rather than an equivalency class. Some of the nb types used to define the lw and up-Apr were union, intersection [24, 25], equal nbs [26, 27], minimal and maximal nbs [28, 29], cardinality nbs [30, 31], right and left nbs [22, 23], rough nb ideal [32], and nbs with minimal left(MNL) and minimal right(MNR) [33, 34]. In the interim, Abo-Tabl [35] created the $apps$ using MNR nbs, which are created using reflexive relations (rfr) that form the basis of topological space. Three more Apr categories were created more recently by Dai et al. [36] via maximal right nbs determined by similarity relations (sir). Stated differently, they offer a broad framework that is unrestricted in terms of the types of binary relations (br) that can be established. Interestingly, it $rhst$ has proven to be a valuable tool for describing information content in a wide range of frameworks and applications in a wide range of domains [37-41]. The topological properties $rhss$ were studied in [42]. This led to the combination of topological and rhs theories, which became the focus of a number of researchs [43-47]. Furthermore, topological generalizations such as minimal structures [48], supra topology [49], infra topology [50], nano-topology [17], and bitopology [51] were engaged in this relationship, and rh Apr spaces using grills and maximal *rh-nbs* [18]. This study investigates the purpose behind the specialized expansion utilizing grills by distinct forms of minimal *nbs* to reduce the boundary regions. This work focuses on building different topologies using grills and highlights the linkages between these topologies and *rhss*, acknowledging the crucial role grills play in influencing topological *rhs* difficulties. The six sections that make up this article are presented as follows: A list of important definitions is provided in Section 2. The many topologies that grills can create are examined in Section 3. Unlike previous methods [36, 52], it presents comparisons among diverse topologies and determines the smallest and largest, whereas other methods just compare topologies within distinct sets. Conditions for determining equivalencies between these topologies are established in the section's conclusion. Section 4 describes the attributes of the new *apps*, which are examined using the recommended topologies. In contrast to the previous ones, they have the property of monotonicity and satisfy all of Pawlak's properties without limitations [53, 54]. In Section 5, a medical use is also suggested. Consequently, these techniques make it simple and very accurate for medical professionals to diagnose heart failure. The usefulness and effectiveness of the suggested models are demonstrated, highlighting the crucial part grills play in decision-making. Consequently, these techniques enable physicians to make a quick and accurate diagnosis of heart failure. The discussion is in Section 6, and the conclusion section marks the end of this study.

2. Preliminaries

Definition 1. [25, 37, 38, 39] *Assume that* $\pi_1 \in \Pi$ *and let* $\times$ *be any (bir) on a finite set* $\Pi \neq \phi$*. The subsequent terms are elucidated:*

(1) $\mathcal{N}_r(\pi_1) = \{\pi_2 \in \Pi : \pi_1 \times \pi_2\}$.

(2) $\mathcal{N}_l(\pi_1) = \{\pi_2 \in \Pi : \pi_2 \times \pi_1\}$.

(3)

$$\mathcal{N}_{\langle r \rangle}(\pi_1) = \begin{cases} \sqcap_{\pi_1 \in \mathcal{N}_r(\pi_2)} \mathcal{N}_r(\pi_2) & : \ \exists\, \mathcal{N}_r(\pi_2) \textit{ containing } \pi_1, \\ \phi & : \ \textit{otherwise}. \end{cases}$$

(4)

$$\mathcal{N}_{\langle l \rangle}(\pi_1) = \begin{cases} \sqcap_{\pi_1 \in \mathcal{N}_l(\pi_2)} \mathcal{N}_l(\pi_2) & : \ \exists\, \mathcal{N}_l(\pi_2) \textit{ containing } \pi_1, \\ \phi & : \ \textit{otherwise}. \end{cases}$$

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(5) $\mathcal{N}_i(\pi_1) = \mathcal{N}_r(\pi_1) \sqcap \mathcal{N}_l(\pi_1)$.

(6) $\mathcal{N}_u(\pi_1) = \mathcal{N}_r(\pi_1) \sqcup \mathcal{N}_l(\pi_1)$.

(7) $\mathcal{N}_{\langle i\rangle}(\pi_1) = \mathcal{N}_{\langle r\rangle}(\pi_1) \sqcap \mathcal{N}_{\langle l\rangle}(\pi_1)$.

(8) $\mathcal{N}_{\langle u\rangle}(\pi_1) = \mathcal{N}_{\langle r\rangle}(\pi_1) \sqcup \mathcal{N}_{\langle l\rangle}(\pi_1)$.

Definition 2. [25] *The triple* $(\Pi, \times, \zeta_\iota)$ *is referred to as a* ι*-nb space (shortened to* ι*-NS).* $\iota \in \{r, \langle r\rangle, l, \langle l\rangle, i, \langle i\rangle, u, \langle u\rangle\}$*, and* ζ_ι *is a map from* Π *to* 2^Π *which links each member of* Π *to its* ι*-NS.*

Theorem 1. [16] *For a* ι*-NS* $(\Pi, \times, \zeta_\iota)$*, and* $\mathcal{L}$ *be a grill. The family*

$\tau^{\mathcal{L}}_{\mathcal{N}_\iota} = \{C \sqsubseteq \Pi : \forall \pi_1 \in C, \mathcal{N}_\iota(\pi_1) \sqcap C^c \notin \mathcal{L}\}$ *a* $\mathcal{N}^{\mathcal{L}}_\iota$ *topology on* Π *where* C^c *represents* C*'s complementary set.*

Definition 3. [55] *Consider* $(\Pi, \times, \zeta_\iota)$ *is a* ι*-NS, and* $\mathcal{L}$ *be a grill on* Π*. The* $\mathcal{L}$*-*$\mathcal{N}_\iota$*-up app* $\overline{\times}^{\mathcal{L}}_{\mathcal{N}_\iota}$*, and* $\mathcal{L}$*-*$\mathcal{N}_\iota$*-lo app* $\underline{\times}^{\mathcal{L}}_{\mathcal{N}_\iota}$ *of* $C \sqsubseteq \Pi$ *are*

$$\overline{\times}^{\mathcal{L}}_{\mathcal{N}_\iota}(C) = \sqcap\{Y : Y^c \in \tau^{\mathcal{L}}_{\mathcal{N}_\iota} : C \sqsubseteq Y\} = CL^{\mathcal{L}}_{\mathcal{N}_\iota}(C).$$

$$\underline{\times}^{\mathcal{L}}_{\mathcal{N}_\iota}(C) = \sqcup\{G \in \tau^{\mathcal{L}}_{\mathcal{N}_\iota} : G \sqsubseteq C\} = Int^{\mathcal{L}}_{\mathcal{N}_\iota}(C).$$

Definition 4. [56] *Let the relation* $\times$ *be arbitrary (arr) on a universe* Π*. The maximal right nb of an* $\pi_1 \in \Pi$ *is delineated as such.*

$$\mathcal{M}_r(\pi_1) = \sqcup_{\pi_1 \in \mathcal{N}_r(\pi_2)} \mathcal{N}_r(\pi_2)$$

Definition 5. [55] *Let* $(\Pi, \times, \zeta_\iota)$ *be a* ι*-NS and* $\mathcal{L}$ *be a grill on* Π*. The* $\mathcal{N}^{\mathcal{L}^*}_\iota$ *lower and* $\times^{\mathcal{L}^*}_\iota$*-up apps of the set* C *delineated as such..*

$$\underline{\times}^{\mathcal{L}^*}_{\mathcal{M}_\iota}(C) = \{\pi_1 \in \Pi : \mathcal{M}_\iota(\pi_1) \sqcap C^c \notin \mathcal{L}\}.$$

$$\overline{\times}^{\mathcal{L}^*}_{\mathcal{M}_\iota}(C) = \{\pi_1 \in \Pi : \mathcal{M}_\iota(\pi_1) \sqcap C \in \mathcal{L}\}.$$

Definition 6. [55] *Let* $(\Pi, \times, \zeta_\iota)$ *be a* ι*-NS and* $\mathcal{L}$ *be a grill on* Π*. The* $\mathcal{N}^{\mathcal{L}^{**}}_\iota$ *lo and* $\times^{\mathcal{L}^{**}}_\iota$*-up apps of the set* C *is delineated as such.*

$$\underline{\times}^{\mathcal{L}^{**}}_{\mathcal{M}_\iota}(C) = \{\pi_1 \in \Pi : \mathcal{M}_\iota(\pi_1) \sqcap C^c \notin \mathcal{L}\}.$$

$$\overline{\times}^{\mathcal{L}^{**}}_{\mathcal{M}_\iota}(C) = C \sqcup \overline{\times}^{\mathcal{L}^*}_{\mathcal{M}_\iota}(C).$$

Definition 7. [55] *Let* $(\Pi, \times, \zeta_\iota)$ *be a* ι*-NS and* $\mathcal{L}$ *be a grill on* Π*. The* $\mathcal{N}^{\mathcal{L}^{***}}_\iota$ *lo and* $\times^{\mathcal{L}^{***}}_\iota$*-up apps of the set* C *articulated as follows.*

$$\underline{\times}^{\mathcal{L}^{***}}_{\mathcal{M}_\iota}(C) = \sqcup\{\mathcal{M}_\iota(\pi_1) : \mathcal{M}_\iota(\pi_1) \sqcap C^c \in \mathcal{L}\}.$$

$$\overline{\times}^{\mathcal{L}^{***}}_{\mathcal{M}_\iota}(C) = [\underline{\times}^{\mathcal{L}^{***}}_{\mathcal{M}_\iota}(C^c)]^c.$$

Definition 8. [52]

Assume that $\pi_1 \in \Pi$ and let $\times$ be a br on a finite set $\Pi \neq \phi$. The minimal nb($\mathcal{MN}$) of $\pi_1 \in \Pi$ are as follows:

(i) $\mathcal{MN}_r(\pi_1) = \sqcap_{\pi_2}\{\mathcal{N}_r(\pi_2) : \pi_2 \times \pi_1\}$.

(2) $\mathcal{MN}_l(\pi_1) = \sqcap_{\pi_2}\{\mathcal{N}_l(\pi_2) : \pi_1 \times \pi_2\}$.

(3) $\mathcal{MN}_u(\pi_1) = \mathcal{MN}_r(\pi_1) \sqcup \mathcal{MN}_l(\pi_1)$.

(4) $\mathcal{MN}_i(\pi_1) = \mathcal{MN}_r(\pi_1) \sqcap \mathcal{MN}_l(\pi_1)$.

(5) $\mathcal{MN}_{\langle r\rangle}(\pi_1) = \sqcap_{\pi_1 \in \mathcal{MN}_r(\pi_2)} \mathcal{MN}_r(\pi_2)$.

(6) $\mathcal{MN}_{\langle l\rangle}(\pi_1) = \sqcap_{\pi_1 \in \mathcal{MN}_l(\pi_2)} \mathcal{MN}_l(\pi_2)$.

(7) $\mathcal{MN}_{\langle i\rangle}(\pi_1) = \mathcal{MN}_{\langle r\rangle}(\pi_1) \sqcap \mathcal{MN}_{\langle l\rangle}(\pi_1)$.

(8) $\mathcal{MN}_{\langle u\rangle}(\pi_1) = \mathcal{MN}_{\langle r\rangle}(\pi_1) \sqcup \mathcal{MN}_{\langle l\rangle}(\pi_1)$.

Definition 9. [52]

Let $\mathcal{MN}_\iota(\pi_1)$ be a $\mathcal{MN}$ system with br $\times$, where $\pi_1 \in \Pi$, and $\iota \in \{r, l, u, i\}$. Then, $(\Pi, \times, \mathcal{MN}_\iota)$ is an app space (shortly, $\mathcal{MN}_\iota$-app space).

Lemma 1. [52] *If $\pi_2 \in \mathcal{MN}_\iota(\pi_1)$, then $\mathcal{MN}_\iota(\pi_2) \sqsubseteq \mathcal{MN}_\iota(\pi_1)$, where $\pi_1, \pi_2 \in \Pi$, and $\iota \in \{r, l, i\}$.*

Lemma 2. [52] *Let $\times$ be a symmetric relation (sr) and $\pi_2 \in \mathcal{MN}_u(\pi_1)$. Then, $\mathcal{MN}_u(\pi_2) \sqsubseteq \mathcal{MN}_u(\pi_1)$, for all $\pi_1, \pi_2 \in \Pi$.*

Proposition 1. [52]

If $\times$ is reflexive relation(rr) and $\pi_2 \in \Pi$, then

(1) $\mathcal{MN}_r(\pi_2) \sqsubseteq \mathcal{N}_r(\pi_2)$

(2) $\mathcal{MN}_l(\pi_2) \sqsubseteq \mathcal{N}_l(\pi_2)$

(3) $\mathcal{MN}_i(\pi_2) \sqsubseteq \mathcal{N}_r(\pi_2)$

(4) $\mathcal{MN}_i(\pi_2) \sqsubseteq \mathcal{N}_l(\pi_2)$

(5) $\mathcal{MN}_u(\pi_2) \sqsubseteq \mathcal{N}_r(\pi_2) \sqcup \mathcal{N}_l(\pi_2)$

Definition 10. [52]

In $(\Pi, \times, \mathcal{MN}_\iota)$ with $C \sqsubseteq \Pi$, then $\mathcal{MN}_\iota$-lower and $\mathcal{MN}_\iota$-up apps of C are defined by

$$\underline{\times}_{\mathcal{MN}_\iota}(C) = \{\pi_1 \in \Pi : \mathcal{MN}_\iota(\pi_1) \sqsubseteq C\}.$$

$$\overline{\times}_{\mathcal{MN}_\iota}(C) = \{\pi_1 \in \Pi : \mathcal{MN}_\iota(\pi_1) \sqcap C \neq \phi\}.$$

Theorem 2. [52]

If $(\Pi, \times, \mathcal{MN}_\iota)$ *is a* $\mathcal{MN}_\iota$*-app space and* $\times$ *is a br, the families*

$$\tau_{\mathcal{MN}_\iota} = \{C \sqsubseteq \Pi : \mathcal{MN}_\iota(\pi_1) \sqsubseteq C, \pi_1 \in C\} \text{ are topology on } \Pi,$$
$$\forall \iota \in \{r, l, u, i, \langle r\rangle, \langle l\rangle, \langle u\rangle, \langle i\rangle\}.$$

Definition 11. [52]

Let $(\Pi, \times, \mathcal{MN}_\iota)$ *be a* $\mathcal{MN}_\iota$*-app space, the* $\mathcal{MN}_\iota$*-lo app,* $\mathcal{MN}_\iota$*-up app,* $\mathcal{MN}_\iota$*-accuracy, and* $\mathcal{MN}_\iota$*-boundary of* C *are as follows:*

$$\underline{\times}_{\mathcal{MN}_\iota}(C) = \sqcup\{G \in \tau_{\mathcal{MN}_\iota} : G \sqsubseteq C\} = Int_{\mathcal{MN}_\iota}(C),$$

$$\overline{\times}_{\mathcal{MN}_\iota}(C) = \sqcap\{Y : Y^c \in \tau_{\mathcal{MN}_\iota} : C \sqsubseteq Y\} = Cl_{\mathcal{MN}_\iota}(C),$$

$$\mathcal{A}_{\mathcal{MN}_\iota}(C) = \frac{|\underline{\times}_{\mathcal{MN}_\iota}(C)|}{|\overline{\times}_{\mathcal{MN}_\iota}(C)|}, \text{ where } C \neq \phi,$$

$$\mathcal{B}_{\mathcal{MN}_\iota}(C) = \overline{\times}_{\mathcal{MN}_\iota}(C) \backslash \underline{\times}_{\mathcal{MN}_\iota}(C).$$

3. Grill topology induced by various types of minimum neighborhoods

[52] proposed topologies based on right minimal *nbds*, and they also presented three other topologies based on distinct minimal *nbds*. This part looks at the links between various topologies and generalizes them using grills. As a conceptual extension of the topologies in [52], Theorem 3.5 creates the topologies by combining the minimal *nbds* and grills. Also, this section's objective is to provide three types of *rh app* extensions.

Definition 12. *Let* $(\Pi, \times, \mathcal{MN}_\iota)$ *be a* $\mathcal{MN}_\iota$*-app space and* $\mathcal{L}$ *be a grill on* Π*. The* $\mathcal{MN}_\iota^{\mathcal{L}^*}$*-lo and* $\mathcal{MN}_\iota^{\mathcal{L}^*}$*-up app of* C *are defined by*

$$\underline{\times}^{\mathcal{L}^*}_{\mathcal{MN}_\iota}(C) = \{\pi_1 \in \Pi : \mathcal{MN}_\iota(\pi_1) \sqcap C^c \notin \mathcal{L}\}.$$

$$\overline{\times}^{\mathcal{L}^*}_{\mathcal{MN}_\iota}(C) = \{\pi_1 \in \Pi : \mathcal{MN}_\iota(\pi_1) \sqcap C \in \mathcal{L}\}.$$

$$\mathcal{A}^{\mathcal{L}^*}_{\mathcal{MN}_\iota}(C) = \frac{|\underline{\times}^{\mathcal{L}^*}_{\mathcal{MN}_\iota}(C)|}{|\overline{\times}^{\mathcal{L}^*}_{\mathcal{MN}_\iota}(C)|}, \text{ where } \overline{\times}^{\mathcal{L}^*}_{\mathcal{MN}_\iota}(C) \neq \phi,.$$

$$\mathcal{B}^{\mathcal{L}^*}_{\mathcal{MN}_\iota}(C) = \overline{\times}^{\mathcal{L}^*}_{\mathcal{MN}_\iota}(C) \backslash \underline{\times}^{\mathcal{L}^*}_{\mathcal{MN}_\iota}(C).$$

Definition 13. *Let* $(\Pi, \times, \mathcal{MN}_\iota)$ *be a* $\mathcal{MN}_\iota$*-app space and* $\mathcal{L}$ *be a grill on* Π*. The* $\mathcal{MN}_\iota^{\mathcal{L}^{**}}$*-lw and* $\mathcal{MN}_\iota^{\mathcal{L}^{**}}$*-up-apps of* C *are defined by*

$$\underline{\times}^{\mathcal{L}^{**}}_{\mathcal{MN}_\iota}(C) = \{\pi_1 \in C : \mathcal{MN}_\iota(\pi_1) \sqcap C^c \notin \mathcal{L}\}.$$

$$\overline{\times}^{\mathcal{L}^{**}}_{\mathcal{MN}_\iota}(C) = C \sqcup \overline{\times}^{\mathcal{L}^{*}}_{\mathcal{MN}_\iota}(C).$$

$$\mathcal{A}^{\mathcal{L}^{**}}_{\mathcal{MN}_\iota}(C) = \frac{|\underline{\times}^{\mathcal{L}^{**}}_{\mathcal{MN}_\iota}(C)|}{|\overline{\times}^{\mathcal{L}^{**}}_{\mathcal{MN}_\iota}(C)|}, \text{ where } \overline{\times}^{\mathcal{L}^{**}}_{\mathcal{MN}_\iota}(C) \neq \phi,.$$

$$\mathcal{B}^{\mathcal{L}^{**}}_{\mathcal{MN}_\iota}(C) = \overline{\times}^{\mathcal{L}^{**}}_{\mathcal{MN}_\iota}(C) \backslash \underline{\times}^{\mathcal{L}^{**}}_{\mathcal{MN}_\iota}(C).$$

Definition 14. *Let* $(\Pi, \times, \mathcal{MN}_\iota)$ *be a* $\mathcal{MN}_\iota$*-app space and* $\mathcal{L}$ *be a grill on* Π. *The* $\mathcal{MN}^{\mathcal{L}^{***}}_\iota$*-up and* $\mathcal{MN}^{\mathcal{L}^{***}}_\iota$*-lw-apps of* C *are defined by*

$$\underline{\times}^{\mathcal{L}^{***}}_{\mathcal{MN}_\iota}(C) = \sqcup\{\mathcal{MN}_\iota(\pi_1) : \mathcal{MN}_\iota(\pi_1) \sqcap C^c \in \mathcal{L}\}.$$

$$\overline{\times}^{\mathcal{L}^{***}}_{\mathcal{MN}_\iota}(C) = [\underline{\times}^{\mathcal{L}^{***}}_{\mathcal{MN}_\iota}(C^c)]^c.$$

$$\mathcal{A}^{\mathcal{L}^{***}}_{\mathcal{MN}_\iota}(C) = \frac{|\underline{\times}^{\mathcal{L}^{***}}_{\mathcal{MN}_\iota}(C)|}{|\overline{\times}^{\mathcal{L}^{***}}_{\mathcal{MN}_\iota}(C)|}, \textit{ where } \overline{\times}^{\mathcal{L}^{***}}_{\mathcal{MN}_\iota}(C) \neq \phi,.$$

$$\mathcal{B}^{\mathcal{L}^{***}}_{\mathcal{MN}_\iota}(C) = \overline{\times}^{\mathcal{L}^{***}}_{\mathcal{MN}_\iota}(C) \backslash \underline{\times}^{\mathcal{L}^{***}}_{\mathcal{MN}_\iota}(C).$$

Theorem 3. [52] *Let* $(\Pi, \times, \zeta_\iota)$ *be a* ι*-NS and* $\pi_1 \in \Pi$. *Then the following statement are true:*
(1) $\mathcal{MN}_{\langle\iota\rangle}(\pi_1) \sqsubseteq \mathcal{MN}_\iota(\pi_1)$, $\iota \in \{r, l, i, u\}$*;*

(2) $\mathcal{MN}_r(\pi_1) = \mathcal{MN}_l(\pi_1) = \mathcal{MN}_i(\pi_1) = \mathcal{MN}_u(\pi_1)$*, and*

$\mathcal{MN}_{\langle r\rangle}(\pi_1), \mathcal{MN}_{\langle l\rangle}(\pi_1) = \mathcal{MN}_{\langle i\rangle}(\pi_1) = \mathcal{MN}_{\langle u\rangle}(\pi_1)$ *when* $\times$ *is symmetric;*

(3) $\mathcal{MN}_{\langle\iota\rangle}(\pi_1) = \mathcal{MN}_\iota(\pi_1)$ $\forall\iota \in \{r, l, i, u\}$ *at what time* $\times$ *exhibits symmetry and transitivity.;*

(4) Al types of $\mathcal{MN}_{\langle\iota\rangle}(\pi_1)$ *are equal when* $\times$ *is equivalence.*

Theorem 4. *If* $(\Pi, \times, \mathcal{MN}_\iota)$ *is a* $\mathcal{MN}_\iota$*-NS,* $\mathcal{L}$ *is a grill on* Π*, and* $\times$ *is a br, the families*

$\tau^{\mathcal{L}}_{\mathcal{MN}_\iota} = \{C \sqsubseteq \Pi : \mathcal{MN}_\iota(\pi_1) \sqcap C^c \notin \mathcal{L}, \forall \pi_1 \in C\}$ *represents as* $\mathcal{MN}^{\mathcal{L}}_\iota$*-topology on* Π *about the grill* $\mathcal{L}$*,* $\forall\iota \in \{r, l, u, i\}$.

Proof.
(1) Π, ϕ clearly belongs to $\tau^{\mathcal{L}}_{\mathcal{MN}_\iota}$.

(2) let $C_1, C_2 \in \tau^{\mathcal{L}}_{\mathcal{MN}_\iota}$, $\pi_1 \in C_1 \sqcap C_2$. Then

$\mathcal{MN}_\iota(\pi_1) \sqcap C_1^c \notin \mathcal{L}$, and $\mathcal{MN}_\iota(\pi_1) \sqcap C_2^c \notin \mathcal{L}$

$\Rightarrow (\mathcal{MN}_\iota(\pi_1) \sqcap C_1^c) \sqcup (\mathcal{MN}_\iota(\pi_1) \sqcap C_2^c) \notin \mathcal{L}$

$\Rightarrow \mathcal{MN}_\iota(\pi_1) \sqcap (C_1^c \sqcup C_2^c) \notin \mathcal{L}$

$\Rightarrow \mathcal{MN}_\iota(\pi_1) \sqcap (C_1 \sqcap C_2)^c \notin \mathcal{L}$

$\Rightarrow C_1 \sqcap C_2 \in \mathcal{L}$.

(3) Let $C_i \in \tau^{\mathcal{L}}_{\mathcal{MN}_\iota}$ $\forall i \in I$, and $\pi_1 \in \sqcup_{i\in I} C_i$. Then, exist $i_0 \in I$ such that $\pi_1 \in C_{i_0}$,

i.e., $\mathcal{MN}_\iota \sqcap (C_{i_0})^c \notin \mathcal{L}$.

$\Rightarrow \mathcal{MN}_\iota \sqcap (\sqcup_{i\in I} C_i) \notin \mathcal{L}$

$\Rightarrow C_1 \sqcap C_2 \in \tau^{\mathcal{L}}_{\mathcal{MN}_\iota}$.

$\tau^{\mathcal{L}}_{\mathcal{MN}_\iota}$ indicates a $\mathcal{MN}^{\mathcal{L}}_\iota$-topology on Π with respect to the grill $\mathcal{L}$.

Definition 15. *Let $(\Pi, \times, \mathcal{MN}_\iota)$ be a $\mathcal{MN}_\iota$-app space, $\mathcal{L}$ be a grill on Π, and $C \sqsubseteq \Pi$. If $C \in \tau^{\mathcal{L}}_{\mathcal{MN}_\iota}$, then it is known as $\mathcal{L}_{\mathcal{MN}_\iota}$-open, and if $C^c \in \tau^{\mathcal{L}}_{\mathcal{MN}_\iota}$, that is closed $\mathcal{L}_{\mathcal{MN}_\iota}$-closed. All $\mathcal{L}_{\mathcal{MN}_\iota}$-closed subset of Π are revered by $\tau^{\mathcal{L}}_{\mathcal{MN}_\iota}$.*

Definition 16. *Let $(\Pi, \times, \mathcal{MN}_\iota)$ be a $\mathcal{MN}_\iota$-app space, and $\mathcal{L}$ be a grill on Π. The $\mathcal{L}_{\mathcal{MN}_\iota}$-interior and $\mathcal{L}_{\mathcal{MN}_\iota}$-closure of $C \sqsubseteq \Pi$ are*

$$Int^{\mathcal{L}}_{\mathcal{MN}_\iota}(C) = \sqcup\{G \in \tau^{\mathcal{L}}_{\mathcal{MN}_\iota} : G \sqsubseteq C\}, \text{ and}$$

$$Cl^{\mathcal{L}}_{\mathcal{MN}_\iota}(C) = \sqcap\{Y : Y^C \in \tau^{\mathcal{L}}_{\mathcal{MN}_\iota} : C \sqsubseteq Y\}.$$

The prior topologies in [34, 56] are weaker than the present ones, as shown by Theorem 3.4.

Theorem 5. *If $(\Pi, \times, \mathcal{MN}_\iota)$ is a $\mathcal{MN}_\iota$-NS, and $\mathcal{L}$ is a grill on Π. Then, $\tau_{\mathcal{MN}_\iota} \sqsubseteq \tau^{\mathcal{L}}_{\mathcal{MN}_\iota}$.*

Proof. Consider $C \in \tau_{\mathcal{MN}_\iota}$. Thus, $\mathcal{MN}_\iota(\pi_1) \sqsubseteq C$ $\forall \pi_1 \in C$. Therefore, $\mathcal{MN}_\iota(\pi_1) \sqcap C^c \notin \mathcal{L}$. So, $C \in \tau^{\mathcal{L}}_{\mathcal{MN}_\iota}$. Therefore, $\tau_{\mathcal{MN}_\iota} \sqsubseteq \tau^{\mathcal{L}}_{\mathcal{MN}_\iota}$.

Remark 1. *The following should be understood:*
(1) At $\mathcal{L} = \{\Pi\}$ in an $\tau_{\mathcal{MN}_\iota}$-topology, then $\tau_{\mathcal{MN}_\iota} = \tau^{\mathcal{L}}_{\mathcal{MN}_\iota}$.

(2) $\tau_{\mathcal{MN}_\iota} \sqsubseteq \tau^{\mathcal{L}}_{\mathcal{MN}_\iota}$, as Observed in Example 3.9.

Example 1. *Let $\times = \{\{\pi_1, \pi_1\}, \{\pi_1, \pi_2\}, \{\pi_2, \pi_2\}, \{\pi_3, \pi_3\}, \{\pi_2, \pi_3\}, \{\pi_3, \pi_2\}\}$*

be a br on $\Pi = \{\pi_1, \pi_2, \pi_3, \pi_4\}$. Table 1 contains all $\mathcal{MN}_\iota$ - nbds.

Table 1: $\mathcal{MN}_\iota$ - *nbds*

	π_1	π_2	π_3	π_4.
$\mathcal{N}_r$	$\{\pi_1,\pi_2\}$	$\{\pi_2,\pi_3\}$	$\{\pi_2,\pi_3\}$	ϕ
$\mathcal{N}_l$	$\{\pi_1\}$	$\{\pi_1,\pi_2,\pi_3\}$	$\{\pi_2,\pi_3\}$	ϕ
$\mathcal{N}_u$	$\{\pi_1,\pi_2\}$	$\{\pi_1,\pi_2,\pi_3\}$	$\{\pi_2,\pi_3\}$	ϕ
$\mathcal{N}_i$	$\{\pi_1\}$	$\{\pi_2,\pi_3\}$	$\{\pi_2,\pi_3\}$	ϕ
$\mathcal{MN}_r$	$\{\pi_1,\pi_2\}$	$\{\pi_2\}$	$\{\pi_2,\pi_3\}$	ϕ
$\mathcal{MN}_l$	$\{\pi_1\}$	$\{\pi_2,\pi_3\}$	$\{\pi_2,\pi_3\}$	ϕ
$\mathcal{MN}_u$	$\{\pi_1,\pi_2\}$	$\{\pi_2,\pi_3\}$	$\{\pi_2,\pi_3\}$	ϕ
$\mathcal{MN}_i$	$\{\pi_1\}$	$\{\pi_2,\pi_3\}$	$\{\pi_2,\pi_3\}$	ϕ
$\mathcal{MN}_{\langle r\rangle}$	$\{\pi_1,\pi_2\}$	$\{\pi_2\}$	$\{\pi_2,\pi_3\}$	ϕ
$\mathcal{MN}_{\langle l\rangle}$	$\{\pi_1\}$	$\{\pi_2,\pi_3\}$	$\{\pi_2,\pi_3\}$	ϕ
$\mathcal{MN}_{\langle u\rangle}$	$\{\pi_1,\pi_2\}$	$\{\pi_2,\pi_3\}$	$\{\pi_2,\pi_3\}$	ϕ
$\mathcal{MN}_{\langle i\rangle}$	$\{\pi_1\}$	$\{\pi_2\}$	$\{\pi_2,\pi_3\}$	ϕ

Let $\mathcal{L} = \{\{\pi_2\},\{\pi_1,\pi_2\},\{\pi_2,\pi_3\},\{\pi_2,\pi_4\},\{\pi_1,\pi_2,\pi_3\},\{\pi_1,\pi_2,\pi_4\},\{\pi_2,\pi_3,\pi_4\},\Pi\}$.

As a result, the following claims are accurate:

(1) $\tau_{\mathcal{MN}_r} = \{\Pi,\phi,\{\pi_2\},\{\pi_4\},\{\pi_1,\pi_2\},\{\pi_2,\pi_3\},\{\pi_2,\pi_4\},\{\pi_1,\pi_2,\pi_3\},$

$\{\pi_1,\pi_2,\pi_4\},\{\pi_2,\pi_3,\pi_4\}\}$, *and*

$\tau^{\mathcal{L}}_{\mathcal{MN}_r} = \{\{\pi_2\},\{\pi_4\},\{\pi_1,\pi_2\},\{\pi_2,\pi_3\},\{\pi_2,\pi_4\},\{\pi_1,\pi_2,\pi_3\},\{\pi_1,\pi_2,\pi_4\},\{\pi_2,\pi_3,\pi_4\},$

$\Pi,\phi\}$.

(2) $\tau_{\mathcal{MN}_l} = \{\Pi,\phi,\{\pi_1\},\{\pi_4\},\{\pi_1,\pi_4\},\{\pi_2,\pi_3\},\{\pi_1,\pi_2,\pi_3\},\{\pi_2,\pi_3,\pi_4\}\}$, *and*

$\tau^{\mathcal{L}}_{\mathcal{MN}_l} = \{\{\pi_1\},\{\pi_2\},\{\pi_4\},\{\pi_1,\pi_2\},\{\pi_1,\pi_4\},\{\pi_2,\pi_3\},\{\pi_2,\pi_4\},\{\pi_1,\pi_2,\pi_3\},\{\pi_1,\pi_2,\pi_4\},$

$\{\pi_2,\pi_3,\pi_4\},\Pi,\phi\}$.

(3) $\tau_{\mathcal{MN}_u} = \{\Pi,\phi,\{\pi_4\},\{\pi_2,\pi_3\},\{\pi_1,\pi_2,\pi_3\},\{\pi_2,\pi_3,\pi_4\}\}$, *and*

$\tau^{\mathcal{L}}_{\mathcal{MN}_u} = \{\{\pi_2\},\{\pi_4\},\{\pi_1,\pi_2\},\{\pi_2,\pi_3\},\{\pi_2,\pi_4\},\{\pi_1,\pi_2,\pi_3\},\{\pi_1,\pi_2,\pi_4\},\{\pi_2,\pi_3,\pi_4\},$

$\Pi,\phi\}$.

(4) $\tau_{\mathcal{MN}_i} = \{\Pi,\phi,\{\pi_1\},\{\pi_2\},\{\pi_4\},\{\pi_1,\pi_2\},\{\pi_1,\pi_4\},\{\pi_2,\pi_3\},\{\pi_2,\pi_4\},\{\pi_1,\pi_2,\pi_3\},$

$\{\pi_1,\pi_2,\pi_4\},\{\pi_2,\pi_3,\pi_4\}\}$, *and*

$\tau^{\mathcal{L}}_{\mathcal{MN}_i} = \{\{\pi_1\}, \{\pi_2\}, \{\pi_4\}, \{\pi_1, \pi_2\}, \{\pi_1, \pi_4\}, \{\pi_2, \pi_3\}, \{\pi_2, \pi_4\}, \{\pi_1, \pi_2, \pi_3\},$

$\{\pi_1, \pi_2, \pi_4\}, \{\pi_2, \pi_3, \pi_4\}, \Pi, \phi\}.$

(5) $\tau_{\mathcal{MN}_{\langle r\rangle}} = \{\Pi, \phi, \{\pi_2\}, \{\pi_4\}, \{\pi_1, \pi_2\}, \{\pi_2, \pi_3\}, \{\pi_2, \pi_4\}, \{\pi_1, \pi_2, \pi_3\},$

$\{\pi_1, \pi_2, \pi_4\}, \{\pi_2, \pi_3, \pi_4\}\}$, *and*

$\tau^{\mathcal{L}}_{\mathcal{MN}_{\langle r\rangle}} = \{\{\pi_2\}, \{\pi_4\}, \{\pi_1, \pi_2\}, \{\pi_2, \pi_3\}, \{\pi_2, \pi_4\}, \{\pi_1, \pi_2, \pi_3\}, \{\pi_1, \pi_2, \pi_4\}, \{\pi_2, \pi_3, \pi_4\},$

$\Pi, \phi\}.$

(6) $\tau_{\mathcal{MN}_{\langle l\rangle}} = \{\Pi, \phi, \{\pi_1\}, \{\pi_4\}, \{\pi_1, \pi_4\}, \{\pi_2, \pi_3\}, \{\pi_1, \pi_2, \pi_3\}, \{\pi_2, \pi_3, \pi_4\}\}$, *and*

$\tau^{\mathcal{L}}_{\mathcal{MN}_{\langle l\rangle}} = \{\{\pi_1\}, \{\pi_2\}, \{\pi_4\}, \{\pi_1, \pi_2\}, \{\pi_1, \pi_4\}, \{\pi_2, \pi_3\}, \{\pi_2, \pi_4\}, \{\pi_1, \pi_2, \pi_3\}, \{\pi_1, \pi_2, \pi_4\},$

$\{\pi_2, \pi_3, \pi_4\}, \Pi, \phi\}.$

(7) $\tau_{\mathcal{MN}_{\langle u\rangle}} = \{\Pi, \phi, \{\pi_4\}, \{\pi_2, \pi_3\}, \{\pi_1, \pi_2, \pi_3\}, \{\pi_2, \pi_3, \pi_4\}\}$, *and*

$\tau^{\mathcal{L}}_{\mathcal{MN}_{\langle u\rangle}} = \{\{\pi_2\}, \{\pi_4\}, \{\pi_1, \pi_2\}, \{\pi_2, \pi_3\}, \{\pi_2, \pi_4\}, \{\pi_1, \pi_2, \pi_3\}, \{\pi_1, \pi_2, \pi_4\}, \{\pi_2, \pi_3, \pi_4\},$

$\Pi, \phi\}.$

(8) $\tau_{\mathcal{MN}_{\langle i\rangle}} = \{\Pi, \phi, \{\pi_1\}, \{\pi_2\}, \{\pi_4\}, \{\pi_1, \pi_2\}, \{\pi_1, \pi_4\}, \{\pi_2, \pi_3\}, \{\pi_2, \pi_4\}, \{\pi_1, \pi_2, \pi_3\},$

$\{\pi_1, \pi_2, \pi_4\}, \{\pi_2, \pi_3, \pi_4\}\}$, *and*

$\tau^{\mathcal{L}}_{\mathcal{MN}_{\langle i\rangle}} = \{\{\pi_1\}, \{\pi_2\}, \{\pi_4\}, \{\pi_1, \pi_2\}, \{\pi_1, \pi_4\}, \{\pi_2, \pi_3\}, \{\pi_2, \pi_4\}, \{\pi_1, \pi_2, \pi_3\},$

$\{\pi_1, \pi_2, \pi_4\}, \{\pi_2, \pi_3, \pi_4\}, \Pi, \phi\}.$

Remark 2. *Example 3.9 indicates that the method described in this section in distinct from those in [16, 19, 52].*

Proposition 2. *While* $\mathcal{L}$ *is a grill on* Π, $(\Pi, \times, \mathcal{MN}_\iota)$ *is a* $\mathcal{MN}_\iota$*-NS. Consequently, the following claims are accurate:*

(1) $\tau^{\mathcal{L}}_{\mathcal{MN}_u} \sqsubseteq \tau^{\mathcal{L}}_{\mathcal{MN}_r}$;

(2) $\tau^{\mathcal{L}}_{\mathcal{MN}_u} \sqsubseteq \tau^{\mathcal{L}}_{\mathcal{MN}_l}$;

(3) $\tau^{\mathcal{L}}_{\mathcal{MN}_r} \sqsubseteq \tau^{\mathcal{L}}_{\mathcal{MN}_i}$;

(4) $\tau^{\mathcal{L}}_{\mathcal{MN}_l} \sqsubseteq \tau^{\mathcal{L}}_{\mathcal{MN}_i}$;

(5) $\tau^{\mathcal{L}}_{\mathcal{MN}_{\langle u\rangle}} \sqsubseteq \tau^{\mathcal{L}}_{\mathcal{MN}_{\langle r\rangle}}$;

(6) $\tau^{\mathcal{L}}_{\mathcal{MN}_{\langle u\rangle}} \sqsubseteq \tau^{\mathcal{L}}_{\mathcal{MN}_{\langle l\rangle}}$;

(7) $\tau^{\mathcal{L}}_{\mathcal{MN}_{\langle r\rangle}} \sqsubseteq \tau^{\mathcal{L}}_{\mathcal{MN}_{\langle i\rangle}}$;

(8) $\tau^{\mathcal{L}}_{\mathcal{MN}_{\langle l\rangle}} \sqsubseteq \tau^{\mathcal{L}}_{\mathcal{MN}_{\langle i\rangle}}$.

Proof.

Let $C \in \tau^{\mathcal{L}}_{\mathcal{MN}_u}$. Consequently, $\mathcal{MN}_u(\pi_1) \sqcap C^c \notin \mathcal{L}$ for every $\pi_1 \in C$. Consequently, $(\mathcal{MN}_r(\pi_1) \sqcup \mathcal{MN}_l(\pi_1)) \sqcap C^c \notin \mathcal{L}, \forall \pi_1 \in C$. Therefore, For every $\pi_1 \in C$, $\mathcal{MN}_r(\pi_1) \sqcap C^c \notin \mathcal{L}$, and $\mathcal{MN}_l(\pi_1) \sqcap C^c \notin \mathcal{L}$. Consequently, $C \in \mathcal{MN}_r$ for all $\pi_1 \in C$, and $C \in \mathcal{MN}_l$ for all $\pi_1 \in C$. Therefore, $\tau^{\mathcal{L}}_{\mathcal{MN}_u} \sqsubseteq \tau^{\mathcal{L}}_{\mathcal{MN}_r}$ and this demonstrates (1), (2), (3) and (4). Similar evidence can be used to support statements (5), (6), (7), and (8).

Corollary 1. *Let* $(\Pi, \times, \zeta_\iota)$ *denote a* ι*-NS, and let* $\mathcal{L}$ *represent a grill on* Π*. Subsequently, the ensuing statements are accurate:*

(1) $\tau^{\mathcal{L}}_{\mathcal{MN}_u} \sqsubseteq \tau^{\mathcal{L}}_{\mathcal{MN}_r} \sqsubseteq \tau^{\mathcal{L}}_{\mathcal{MN}_i}$;

(2) $\tau^{\mathcal{L}}_{\mathcal{MN}_u} \sqsubseteq \tau^{\mathcal{L}}_{\mathcal{MN}_l} \sqsubseteq \tau^{\mathcal{L}}_{\mathcal{MN}_i}$;

(3) $\tau^{\mathcal{L}}_{\mathcal{MN}_{\langle u\rangle}} \sqsubseteq \tau^{\mathcal{L}}_{\mathcal{MN}_{\langle r\rangle}} \sqsubseteq \tau^{\langle\mathcal{L}\rangle}_{\mathcal{MN}_{\langle i\rangle}}$;

(4) $\tau^{\mathcal{L}}_{\mathcal{MN}_{\langle u\rangle}} \sqsubseteq \tau^{\mathcal{L}}_{\mathcal{MN}_{\langle l\rangle}} \sqsubseteq \tau^{\mathcal{L}}_{\mathcal{MN}_{\langle i\rangle}}$.

Theorem 3.13 offers a distinctive characterisation of the suggested topologies by contrasting $\tau^{\mathcal{L}}_{\mathcal{MN}_\iota}$ and $\tau^{\mathcal{L}}_{\mathcal{MN}_{\langle\iota\rangle}}$. Thus, Corollary 3.14 delineates both the smallest $\tau^{\mathcal{L}}_{\mathcal{MN}_r}$ and the greatest $\tau^{\mathcal{L}}_{\mathcal{MN}_{\langle l\rangle}}$.

Theorem 6. *If* $(\Pi, \times, \zeta_\iota)$ *constitutes a* ι*-NS, and* $\mathcal{L}$ *represents a grill on* Π*. Consequently,* $\tau^{\mathcal{L}}_{\mathcal{MN}_\iota} \sqsubseteq \tau^{\mathcal{L}}_{\mathcal{MN}_{\langle\iota\rangle}}$*, where* $\iota \in \{r, l, i, u\}$*.*

Proof. Let $C \in \tau^{\mathcal{L}}_{\mathcal{MN}_r}$. Then, for every $\pi_1 \in C$, it holds that $\mathcal{MN}_r(\pi_1) \sqcap C^c \notin \mathcal{L}$, and consequently, $\mathcal{MN}_{\langle r\rangle}(\pi_1) \sqcap C^c \notin \mathcal{L}$ for all $\pi_1 \in C$. Consequently, $C \in \tau^{\mathcal{L}}_{\mathcal{MN}_{\langle r\rangle}}$. Therefore, $\tau^{\mathcal{L}}_{\mathcal{MN}_r} \sqsubseteq \tau^{\mathcal{L}}_{\mathcal{MN}_{\langle r\rangle}}$ The remaining assertions can be substantiated in a same way.

Corollary 2. *If* $(\Pi, \times, \zeta_\iota)$ *constitutes a* ι*-NS, and* $\mathcal{L}$ *represents a grill on* Π*. Consequently,* $\tau^{\mathcal{L}}_{\mathcal{MN}_u} \sqsubseteq \tau^{\mathcal{L}}_{\mathcal{MN}_\iota} \sqsubseteq \tau^{\mathcal{L}}_{\mathcal{MN}_{\langle i\rangle}}$*, where* $\iota \in \{r, l, i, \langle r\rangle, \langle l\rangle, \langle i\rangle\}$*.*

Remark 3. *Example 3.9 illustrates that the notable differences between the current methodology and those presented in [16, 18] are that* $\tau^{\mathcal{L}}_{\mathcal{MN}_\iota} \sqsubseteq \tau^{\mathcal{L}}_{\mathcal{MN}_{\langle\iota\rangle}}$*, where* $\iota \in \{r, l, i, u\}$*, despite the fact that* $\tau^{\mathcal{L}}_{\mathcal{N}_\iota}$ *and* $\tau^{\mathcal{L}}_{\mathcal{N}_{\langle\iota\rangle}}$ *are not comparable. Moreover, it confirms that the converses of Proposition 3.11 and Corollary 3.12 are not universally valid.*

(1) $\tau^{\mathcal{L}}_{\mathcal{MN}_i} \nsubseteq \tau^{\mathcal{L}}_{\mathcal{MN}_r}$*;*

(2) $\tau^{\mathcal{L}}_{\mathcal{MN}_i} \nsubseteq \tau^{\mathcal{L}}_{\mathcal{MN}_u}$*;*

(3) $\tau^{\mathcal{L}}_{\mathcal{MN}_l} \nsubseteq \tau^{\mathcal{L}}_{\mathcal{MN}_u}$*;*

(4) $\tau^{\mathcal{L}}_{\mathcal{MN}_l} \nsubseteq \tau^{\mathcal{L}}_{\mathcal{MN}_r}$*;*

(5) $\tau^{\mathcal{L}}_{\mathcal{MN}_{\langle i\rangle}} \nsubseteq \tau^{\mathcal{L}}_{\mathcal{MN}_{\langle r\rangle}}$*;*

(6) $\tau^{\mathcal{L}}_{\mathcal{MN}_{\langle i\rangle}} \nsubseteq \tau^{\mathcal{L}}_{\mathcal{MN}_{\langle u\rangle}}$*;*

(7) $\tau^{\mathcal{L}}_{\mathcal{MN}_{\langle l\rangle}} \nsubseteq \tau^{\mathcal{L}}_{\mathcal{MN}_{\langle u\rangle}}$*;*

(8) $\tau^{\mathcal{L}}_{\mathcal{MN}_{\langle l\rangle}} \nsubseteq \tau^{\mathcal{L}}_{\mathcal{MN}_{\langle r\rangle}}$*.*

Theorem 3.17 specifies the necessary criteria to determine equivalents among the suggested topologies.

Theorem 7. *If* $(\Pi, \times, \zeta_\iota)$ *constitutes a* ι*-NS, and* $\mathcal{L}$ *represents a grill on* Π*. Then,*

(1) $\tau^{\mathcal{L}}_{\mathcal{MN}_r} = \tau^{\mathcal{L}}_{\mathcal{MN}_i} = \tau^{\mathcal{L}}_{\mathcal{MN}_l} = \tau^{\mathcal{L}}_{\mathcal{MN}_u}$ *and* $\tau^{\mathcal{L}}_{\mathcal{MN}_{\langle l\rangle}} = \tau^{\mathcal{L}}_{\mathcal{MN}_{\langle r\rangle}} = \tau^{\mathcal{L}}_{\mathcal{MN}_{\langle i\rangle}} = \tau^{\mathcal{L}}_{\mathcal{MN}_{\langle u\rangle}}$ *at* $\times$ *is symmetric;*

(2) $\tau^{\mathcal{L}}_{\mathcal{MN}_r} = \tau^{\mathcal{L}}_{\mathcal{MN}_i} = \tau^{\mathcal{L}}_{\mathcal{MN}_l} = \tau^{\mathcal{L}}_{\mathcal{MN}_u} = \tau^{\mathcal{L}}_{\mathcal{MN}_{\langle l\rangle}} = \tau^{\mathcal{L}}_{\mathcal{MN}_{\langle r\rangle}} = \tau^{\mathcal{L}}_{\mathcal{MN}_{\langle i\rangle}} = \tau^{\mathcal{L}}_{\mathcal{MN}_{\langle u\rangle}}$ *at* $\times$ *is equivalence.*

Proof. (1) Let $C \in \tau^{\mathcal{L}}_{\mathcal{MN}_r}$. Consequently, $\mathcal{MN}_r(\pi_1) \sqsubseteq C$, $\forall \pi_1 \in C$.

$\Leftrightarrow \mathcal{MN}_i(\pi_1) \sqsubseteq C$, $\mathcal{MN}_l(\pi_1) \sqsubseteq C$, $\mathcal{MN}_u(\pi_1) \sqsubseteq C$, $\forall \pi_1 \in C$ (From Theorem 3.4).

Consequently $\tau^{\mathcal{L}}_{\mathcal{MN}_r} = \tau^{\mathcal{L}}_{\mathcal{MN}_l} = \tau^{\mathcal{L}}_{\mathcal{MN}_i} = \tau^{\mathcal{L}}_{\mathcal{MN}_u}$, and $\tau^{\mathcal{L}}_{\mathcal{MN}_{\langle r\rangle}} = \tau^{\mathcal{L}}_{\mathcal{MN}_{\langle l\rangle}} = \tau^{\mathcal{L}}_{\mathcal{MN}_{\langle i\rangle}} = \tau^{\mathcal{L}}_{\mathcal{MN}_{\langle u\rangle}}$.

(2) The evidence is simple.

To investigate the characteristic of monotonicity in the fourth section, which is essential, we must initially analyze the relationship between the two topologies produced by the two subset relations. This is discussed in the following major proposition.

Proposition 3. *Let* $(\Pi, \times., \zeta_{1\iota})$*, and* $(\Pi, \times.., \zeta_{2\iota})$ *be two* ι*-NS,* $\mathcal{L}$ *be a grill on* Π*, and* $\times. \sqsubseteq \times..$*. Then,* $\tau^{\mathcal{MN}}_{2\iota} \sqsubseteq \tau^{\mathcal{MN}}_{1\iota}, \iota \in \{r, l, i, u\}$*.*

Proof. Let $C \in \tau_{2r}^{\mathcal{MN}}$. Then, $\mathcal{MN}_{2r}(\pi_1) \sqcap C^c \notin \mathcal{L}$, $\forall \pi_1 \in C$. Thus, $\mathcal{MN}_{1r}(\pi_1) \sqcap C^c \notin \mathcal{L}$, $\forall \pi_1 \in C$. Consequently, $C \in \mathcal{MN}_{1r}$, and hence, $\tau_{2r}^{\mathcal{MN}}(\pi_1) \sqsubseteq \tau_{1r}^{\mathcal{MN}}(\pi_1)$; The remaining cases can be demonstrated in a comparable way.

4. Approximate Models Grill

This part presents preliminary models employing $\tau^{\mathcal{L}}_{\mathcal{MN}_\iota}$-topologies and elucidates their principal characteristics. These mathematical representations preserve the characteristic of monotonicity unencumbered by limitations. Conversely, it may be either forfeited or maintained under stringent terms in certain prior methodologies [16, 52].
Definition 4.1 employs the topologies established in Section 3 to furnish *apps*.

Definition 17. *If* $(\Pi, \times, \zeta_\iota)$ *constitutes a* ι*-NS, and* $\mathcal{L}$ *represents a grill on* Π*. The* $\mathcal{L}$*-*$\mathcal{MN}_\iota$*-lw-app* $\underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}$*, and* $\mathcal{L}$*-*$\mathcal{MN}_\iota$*-up-app* $\overline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}$ *of* $C \sqsubseteq \Pi$ *are*

$$\underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C) = \sqcup\{G \in \tau^{\mathcal{L}}_{\mathcal{MN}_\iota} : G \sqsubseteq C\} = Int^{\mathcal{L}}_{\mathcal{MN}_\iota},$$

$$\overline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C) = \sqcap\{Y \in \tau^{\mathcal{L}}_{\mathcal{MN}_\iota} : C \sqsubseteq Y\} = Cl^{\mathcal{L}}_{\mathcal{MN}_\iota}.$$

The principal characteristics of the proposed *apps* are addressed in the exposition of the ensuing outcomes.

Proposition 4. *If* $(\Pi, \times, \zeta_\iota)$ *forms a* ι*-NS,* $\mathcal{L}$ *denotes a grill on* Π*, and* $C, G \in \Pi$*. Subsequently, the ensuing assertions are accurate:*

(1) $\underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C) \sqsubseteq C$*;*

(2) $\underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(\phi) = \phi$*;*

(3) $\underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(\Pi) = \Pi$*;*

(4) If $C \sqsubseteq G$*, then* $\underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C) \sqsubseteq \underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(G)$*;*

(5) $\underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C \sqcap G) = \underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C) \sqcap \underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(G)$*;*

(6) $\underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C^c) = (\overline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C))^c$*;*

(7) $\underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(\underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C)) = \underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C)$*.*

Proof. Given that 1, 2, and 3 are readily demonstrable, we shall commence with the proof of 4.

(4) Let $C \sqsubseteq G$. Then, $\sqcup\{G \in \tau^{\mathcal{L}}_{\mathcal{MN}_\iota} : G \sqsubseteq C\} \sqsubseteq \sqcup\{G \in \tau^{\mathcal{L}}_{\mathcal{MN}_\iota} : G \sqsubseteq G\}$, and so $\underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C) \sqsubseteq \underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(G)$.

(5) From (4), $\underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C \sqcap G) \sqsubseteq \underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C) \sqcap \underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(G)$. Since $\underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C) \sqsubseteq C$, and $\underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(G) \sqsubseteq G$, it follows that $\underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C) \sqcap \underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(G) \sqsubseteq C \sqcap G$. Consequently, $\underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(\underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C) \sqcap \underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(G)) \sqsubseteq \underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C \sqcap G)$. Then, $\underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C) \sqcap \underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(G) \sqsubseteq \underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C \sqcap G)$. Thus, $\underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C \sqcap G) = \underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C) \sqcap \underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(G)$.

(6) Let $\pi_1 \in \underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C^c)$. At hence, $\exists\ G \in \tau^{\mathcal{L}}_{\mathcal{MN}_\iota}$ such that $\pi_1 \in G \sqsubseteq C^c$, and so $G \sqcap C = \phi$.
Therefore, $\pi_1 \notin \overline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C)$. Thus, $\pi_1 \in (\overline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C))^c$. Conversely, let $\pi_1 \in (\overline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C))^c$. Then, $\pi_1 \notin \overline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C)$, and so $\exists \mathcal{O} \in \tau^{\mathcal{L}}_{\mathcal{MN}_\iota}$ that $\pi_1 \in G$, and $C \sqcap G = \phi$. So, $\pi_1 \in G \sqsubseteq C^c$. Hence, $\pi_1 \in \underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C^c)$.

(7) $\underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(\underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C)) \sqsubseteq \underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C)$ by (1). On the other hand, let $\pi_1 \in \underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C)$. Then, $\exists G \in \tau^{\mathcal{L}}_{\mathcal{MN}_\iota}$ that $\pi_1 \in G \sqsubseteq C$, $\underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(G) \sqsubseteq \underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C)$ (from(4)). It is observed that $G = \underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(G)$ from Definition 4.1. So $\{\pi_1\} \in \underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(G) \sqsubseteq \underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(\underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C))$. Consequently, $\underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(\underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C)) \sqsupseteq \underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C)$.

Corollary 3. *If $(\Pi, \times, \zeta_\iota)$ forms a ι-NS, $\mathcal{L}$ denotes a grill on Π, and $C, G \in \Pi$. Then,* $\underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C) \sqcup \underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(G) \sqsubseteq \underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C \sqcup G)$.

Proof. This can be directly deduced from (5) of Proposition 4.2.

Remark 4. *In Example 3.10, the subsequent propositions are accurate:*
(1) $\underline{\times}^{\mathcal{L}}_{\mathcal{MN}_r}(\{\pi_3\}) = \phi \sqsubseteq \{\pi_3\}$;

(2) $\underline{\times}^{\mathcal{L}}_{\mathcal{MN}_r}(\{\pi_3\}) = \phi \sqsubseteq \{\pi_4\} = \underline{\times}^{\mathcal{L}}_{\mathcal{MN}_r}(\{\pi_4\})$ *but* $\{\pi_3\} \not\sqsubseteq \{\pi_4\}$;

(3) $\underline{\times}^{\mathcal{L}}_{\mathcal{MN}_r}(\{\pi_2\}) \sqcup \underline{\times}^{\mathcal{L}}_{\mathcal{MN}_r}(\{\pi_1, \pi_2, \pi_3\}) = \{\pi_1, \pi_2, \pi_3\}) = \underline{\times}^{\mathcal{L}}_{\mathcal{MN}_r}(\{\pi_2\} \sqcup \underline{\times}^{\mathcal{L}}_{\mathcal{MN}_r}\{\pi_1, \pi_2, \pi_3\})$

Proposition 5. *If $(\Pi, \times, \zeta_\iota)$ forms a ι-NS, $\mathcal{L}$ denotes a grill on Π, and $C, G \in \Pi$. Subsequently, the ensuing assertions are accurate:*

(1) $C \sqsubseteq \overline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C)$;

(2) $\overline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(\phi) = \phi$;

(3) $\overline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C) = C$;

(4) If $C \sqsubseteq G$, *then* $\overline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C) \sqsubseteq \overline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(G)$;

(5) $\overline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C \sqcup G) = \overline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C) \sqcup \underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(G)$;

(6) $\overline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C^c) = (\underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C))^c$;

(7) $\overline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(\overline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C)) = \overline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C)$.

Proof. The demonstration parallels that of Proposition 4.2.

Corollary 4. *If* $(\Pi, \times, \zeta_\iota)$ *forms a* ι*-NS,* $\mathcal{L}$ *denotes a grill on* Π*, and* $C, G \in \Pi$*. Then,* $\overline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C \sqcap G) \sqsubseteq \overline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C) \sqcap \overline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(G)$

Proof. It is directly deducible from Proposition 4.2 (4).

Remark 5. *In Example 3.10, the subsequent propositions are accurate:*

(1) $\overline{\times}^{\mathcal{L}}_{\mathcal{MN}_r}(\{\pi_1 \sqcap \pi_2\}) = \phi \sqsubseteq \{\pi_2\} = \overline{\times}^{\mathcal{L}}_{\mathcal{MN}_r}(\{\pi_1\}) \sqcap \overline{\times}^{\mathcal{L}}_{\mathcal{MN}_r}(\{\pi_2)\})$;

(2) $\overline{\times}^{\mathcal{L}}_{\mathcal{MN}_r}(\{\pi_2\}) = \{\pi_2\} \sqsubseteq \{\pi_2, \pi_4\} = \overline{\times}^{\mathcal{L}}_{\mathcal{MN}_r}(\{\pi_2, \pi_4\})$;

(3) $\{\pi_1, \pi_3\} \sqsubseteq \overline{\times}^{\mathcal{L}}_{\mathcal{MN}_r}(\{\pi_1, \pi_3\}) = \{\pi_1, \pi_2, \pi_3\}$.

Definition 18. *If* $(\Pi, \times, \zeta_\iota)$ *forms a* ι*-NS,* $\mathcal{L}$ *denotes a grill on* Π*. The* $\mathcal{L}$*-*$\mathcal{MN}_\iota$*-accuracy is*

$$\mathcal{A}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C) = \frac{|\underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C)|}{|\overline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C)|}, \text{ where } C \neq \phi,$$

Proposition 4.9 ensures that when more data is added, the *lo-app* won't get smaller. Similarly, there is no drop in the *up-app*. Thus, monotonicity is a characteristic of the suggested model. This characteristic guarantees that when additional information becomes available, the *apps* either increase in accuracy or remain constant, never declining.

Proposition 6. *Let* $(\Pi, \times_1, \zeta_{1\iota})$ *and* $(\Pi, \times_2, \zeta_{2\iota})$ *be two* ι*-NS,* $\mathcal{L}$ *represent a grill on* Π*, and* $\times. \sqsubseteq \times..$*. For all* $\iota \in \{r, l, i, u\}, C$ *is a subclass of* Π*, and the subsequent propositions are accurate:*

(1) $\overline{\times}^{\mathcal{L}}_{\mathcal{MN}_{1\iota}}(C) \sqsubseteq \overline{\times}^{\mathcal{L}}_{\mathcal{MN}_{2\iota}}(C)$;

(2) $\underline{\times}^{\mathcal{L}}_{\mathcal{MN}_{2\iota}}(C) \sqsubseteq \underline{\times}^{\mathcal{L}}_{\mathcal{MN}_{1\iota}}(C)$;

(3) $\mathcal{A}^{\mathcal{L}}_{\mathcal{MN}_{2\iota}}(C) \leq \mathcal{A}^{\mathcal{L}}_{\mathcal{MN}_{1\iota}}(C)$.

Proof. (1) $\overline{\times}^{\mathcal{L}}_{\mathcal{MN}_{1\iota}}(C) = \sqcap\{Y \in \tau^{\mathcal{L}}_{\mathcal{MN}_{1\iota}} : C \sqsubseteq Y\} \sqsubseteq \sqcap\{Y \in \tau^{\mathcal{L}}_{\mathcal{MN}_{2\iota}} : C \sqsubseteq Y\} = \overline{\times}^{\mathcal{L}}_{\mathcal{MN}_{2\iota}}(C)$ (from proposition 3.18). So, $\overline{\times}^{\mathcal{L}}_{\mathcal{MN}_{1\iota}}(C) \sqsubseteq \overline{\times}^{\mathcal{L}}_{\mathcal{MN}_{2\iota}}(C)$.

(2) Let $\underline{\times}^{\mathcal{L}}_{\mathcal{MN}_{2\iota}}(C) = \sqcup\{G \in \tau^{\mathcal{L}}_{\mathcal{MN}_{2\iota}} : G \sqsubseteq C\} \sqsubseteq \sqcup\{G \in \tau^{\mathcal{L}}_{\mathcal{MN}_{1\iota}} : G \sqsubseteq C\} = \underline{\times}^{\mathcal{L}}_{\mathcal{MN}_{1\iota}}(C)$ (from proposition 3.18). So, $\underline{\times}^{\mathcal{L}}_{\mathcal{MN}_{2\iota}}(C) \sqsubseteq \underline{\times}^{\mathcal{L}}_{\mathcal{MN}_{1\iota}}(C)$.

(3) $\mathcal{A}^{\mathcal{L}}_{\mathcal{MN}_{1\iota}}(C) = \frac{|\underline{\times}^{\mathcal{L}}_{\mathcal{MN}_{1\iota}}(C)|}{|\overline{\times}^{\mathcal{L}}_{\mathcal{MN}_{1\iota}}(C)|} \leq \frac{|\underline{\times}^{\mathcal{L}}_{\mathcal{MN}_{2\iota}}(C)|}{|\overline{\times}^{\mathcal{L}}_{\mathcal{MN}_{2\iota}}(C)|} = \mathcal{A}^{\mathcal{L}}_{\mathcal{MN}_{2\iota}}(C)$

Definition 19. *If* $(\Pi, \times, \zeta_\iota)$ *forms a* ι*-NS,* $\mathcal{L}$ *denotes a grill on* Π*. The* $\mathcal{L}$-$\mathcal{MN}_\iota$*-positive,* $\mathcal{L}$-$\mathcal{MN}_\iota$*-boundary, and* $\mathcal{L}$-$\mathcal{MN}_\iota$*-negative regions of* $C \sqsubseteq \Pi$ *are*

$$\times^{\mathcal{L}+}_{\mathcal{MN}_\iota}(C) = \underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C),$$

$$\times^{\mathcal{L}-}_{\mathcal{MN}_\iota}(C) = \Pi \backslash \overline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C),$$

$$\mathcal{B}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C) = \overline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C) \backslash \underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C).$$

Proposition 7. *Let* $(\Pi, \times_1, \zeta_{1\iota})$ *and* $(\Pi, \times_2, \zeta_{2\iota})$ *be two* ι*-NS,* $\mathcal{L}$ *represent a grill on* Π*, and* $\times. \sqsubseteq \times...$ *For all* $\iota \in \{r, l, i, u\}, C$ *is a subclass of* Π*, and the subsequent propositions are accurate:*

(1) $\mathcal{B}^{\mathcal{L}}_{\mathcal{MN}_{1\iota}}(C) \sqsubseteq \mathcal{B}^{\mathcal{L}}_{\mathcal{MN}_{2\iota}}(C)$*;*

(2) $\times^{\mathcal{L}-}_{\mathcal{MN}_{2\iota}}(C) \sqsubseteq \times^{\mathcal{L}-}_{\mathcal{MN}_{1\iota}}(C)$*.*

Proof.

(1) Let $\pi_1 \in \mathcal{B}^{\mathcal{L}}_{\mathcal{MN}_{1\iota}}(C)$. Then, $\pi_1 \in \overline{\times}^{\mathcal{L}}_{\mathcal{MN}_{1\iota}}(C) \backslash \underline{\times}^{\mathcal{L}}_{\mathcal{MN}_{1\iota}}(C)$. Therefore, $\pi_1 \in \overline{\times}^{\mathcal{L}}_{\mathcal{MN}_{1\iota}}(C)$, and $\pi_1 \in (\underline{\times}^{\mathcal{L}}_{\mathcal{MN}_{1\iota}}(C))^c$. So, $\pi_1 \in \overline{\times}^{\mathcal{L}}_{\mathcal{MN}_{2\iota}}(C)$, and $\pi_1 \in (\underline{\times}^{\mathcal{L}}_{\mathcal{MN}_{2\iota}}(C))^c$. Therefore, $\pi_1 \in \mathcal{B}^{\mathcal{L}}_{\mathcal{MN}_{2\iota}}(C)$, $\mathcal{B}^{\mathcal{L}}_{\mathcal{MN}_{1\iota}}(C) \sqsubseteq \mathcal{B}^{\mathcal{L}}_{\mathcal{MN}_{2\iota}}(C)$.

(2) This is derived from Proposition 4.9.

Definition 20. *If* $(\Pi, \times, \zeta_\iota)$ *forms a* ι*-NS,* $\mathcal{L}$ *denotes a grill on* Π*.* $C \sqsubseteq \Pi$ *is*

$\mathcal{L}$-$\mathcal{MN}_\iota$*-exact if* $\underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C)) = \overline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C)) = C$*; otherwise, it is* $\mathcal{L}$-$\mathcal{MN}_\iota$*-rough.*

Proposition 8. *If* $(\Pi, \times, \zeta_\iota)$ *forms a* ι*-NS,* $\mathcal{L}$ *denotes a grill on* Π*.* $C \sqsubseteq \Pi$ *is*

$\mathcal{L}$-$\mathcal{MN}_\iota$*-exact if and only if* $\mathcal{B}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C) = \phi$ *.*

Proof.

Let C be any $\mathcal{L}$-$\mathcal{MN}_\iota$-exact. Then, $\mathcal{B}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C) = \overline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C) \backslash \underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C) = \phi$. Conversely, $\mathcal{B}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C) = \phi$; hence, $\overline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C) \backslash \underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C) = \phi$, and so $\overline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C) \sqsubseteq \underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C)$. However, $\underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C) \sqsubseteq \overline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C)$. Thus, $\overline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C) = \underline{\times}^{\mathcal{L}}_{\mathcal{MN}_\iota}(C)$, and C is $\mathcal{L}$-$\mathcal{MN}_\iota$-exact.

5. An Application of the Suggested Approach for Heart Failure

This paragraph presents the experimental results of a preparation study conducted on five symptoms of heart disease, as outlined by Dickstein et al. [57], including seven patients. The research was carried out in the cardiology department of Al-Azhar University [58]. The quantity of training data utilized Twenty-five records were analyzed,while the other data were forwarded to this institution displaying similar presented symptoms, comprehensive history, physical examination, complete laboratory tests, resting electrocardiogram, and conventional echocardiographic assessment were conducted. The information system contains data for just seven patients with similar characteristics, as discussed in Table 2, regarding the heart failure issue. The columns denote the symptoms, where 'E' indicates the presence of symptoms and 'NE' signifies their absence, pertaining to the diagnosis of heart failure [57] (condition characteristics). where BS is the breathlessness, OA is the orthopnea, PA is the paroxysmal nocturnal dyspnea, RE reduced exercise tolerance, AG is the ankle swelling. The attribute $\mathcal{D}$ represents the determination of heart failure. The rows in Table 2, $\Pi = \{\pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_8, \pi_9\}$ represents the patients. Let us consider the expert of the system who has offered the subsequent relation $\times$ on the set of patients Π to delineate the connections among them based on their symptoms:
$\pi_i \times \pi_j \Leftrightarrow f(\pi_i) \sqsubseteq f(\pi_j)$, where the function is defined by $f(\pi_1) = \{BS, OA, PA, RE\}$, $f(\pi_2) = \{RE, AG\}$, $f(\pi_3) = \{BS, OA, PA, RE, AG\}$, $f(\pi_4) = \{RE\}$, $f(\pi_5) = \{BS, RE, AG\}$, $f(\pi_8) = \{BS, OA, RE, AG\}$, $f(\pi_9) = \{BS, PA, RE\}$.

Then, $\times = \{(\pi_1, \pi_1), (\pi_1, \pi_3), (\pi_2, \pi_2), (\pi_2, \pi_3), (\pi_2, \pi_5), (\pi_2, \pi_8), (\pi_3, \pi_3), (\pi_4, \pi_1)$ $(\pi_4, \pi_2), (\pi_4, \pi_3), (\pi_4, \pi_4), (\pi_4, \pi_5), (\pi_4, \pi_8), (\pi_4, \pi_9), (\pi_5, \pi_3), (\pi_5, \pi_5), (\pi_5, \pi_8)$ $(\pi_8, \pi_3), (\pi_8, \pi_8), (\pi_9, \pi_1), (\pi_9, \pi_3), (\pi_9, \pi_9)\}$.

Hence, $\mathcal{MN}_r(\pi_1) = \{\pi_1, \pi_3\}$, $\mathcal{MN}_r(\pi_2) = \{\pi_2, \pi_3, \pi_5, \pi_8\}$, $\mathcal{MN}_r(\pi_3) = \{\pi_3\}$, $\mathcal{MN}_r(\pi_4) = \Pi$, $\mathcal{MN}_r(\pi_5) = \{\pi_3, \pi_5, \pi_8\}$, $\mathcal{MN}_r(\pi_8) = \{\pi_3, \pi_8\}$, and $\mathcal{MN}_r(\pi_9) = \{\pi_1, \pi_3, \pi_9\}$.

Let $\mathcal{L} = \{\Pi\}$, Then $\tau^{\mathcal{L}}_{\mathcal{MN}_r} = 2^{\Pi}$(the set of all subsets of Π). $I_1 = \{\pi_1, \pi_3, \pi_8, \pi_9\}$ are the afflicted patients, In contrast, the uninfected patients are represented by $I_2 = \{\pi_2, \pi_4, \pi_5\}$. Consequently, the *lw-app*, the *up-app*, and the accuracy of I_1 are computed as
Case(i) The the afflicted patients $I_1 = \{\pi_1, \pi_3, \pi_8, \pi_9\}$

(1) Ismail's method [52] in Definition 17:
$\underline{\times}_{\mathcal{MN}_r}(I_1) = I_1$;

$\overline{\times}_{\mathcal{MN}_r}(I_1) = \Pi$;

$\mathcal{A}_{\mathcal{MN}_r}(I_1) = \frac{4}{7}$;

$\mathcal{B}^{\mathcal{L}}_{\mathcal{MN}_r}(I_1) = \{\pi_2, \pi_4, \pi_5\}$

(2) The proposed Definitions 4.1, 4.8, and 4.10.

$\underline{\times}^{\mathcal{L}}_{\mathcal{MN}_r}(I_1) = I_1;$

$\overline{\times}^{\mathcal{L}}_{\mathcal{MN}_r}(I_1) = I_1;$

$\mathcal{A}^{\mathcal{L}}_{\mathcal{MN}_r}(I_1) = 1;$

$\mathcal{B}^{\mathcal{L}}_{\mathcal{MN}_r}(I_2) = \phi$

Case (ii) The uninfected patients $I_2 = \{\pi_2, \pi_4, \pi_5\}$.(1) Ismail's method [52] in Definition 17:
$\underline{\times}_{\mathcal{MN}_r}(I_2) = \phi;$

$\overline{\times}_{\mathcal{MN}_r}(I_2) = \{\pi_2, \pi_4, \pi_5\};$

$\mathcal{A}_{\mathcal{MN}_r}(I_1) = 0;$

$\mathcal{B}^{\mathcal{L}}_{\mathcal{MN}_r}(I_2) = \{\pi_2, \pi_4, \pi_5\}$

(2) The proposed Definitions 4.1, 4.8, and 4.10.

$\underline{\times}^{\mathcal{L}}_{\mathcal{MN}_r}(I_2) = I_2;$

$\overline{\times}^{\mathcal{L}}_{\mathcal{MN}_r}(I_2) = I_2;$

$\mathcal{A}^{\mathcal{L}}_{\mathcal{MN}_r}(I_2) = 1;$

$\mathcal{B}^{\mathcal{L}}_{\mathcal{MN}_r}(I_2) = \phi.$

Consequently, Ismail's boundaries [52] for infected and uninfected persons are $\{\pi_2, \pi_4, \pi_5\}$, resulting in ambiguity and diminished decision precision. Conversely, the current methodology produces a null border, so diminishing ambiguity and improving precision.

6. conclusion

Rough set theory is a mathematical framework designed to address uncertainty. Grills can enhance this idea, serving as an effective instrument for diminishing ambiguity by enabling a wider approximation. A primary emphasis in the study of rough sets is the minimization of boundaries to improve precision. Grills is one of the most efficient methods for accomplishing this. Consequently, numerous techniques employing grills for the construction of diverse topologies have been suggested. The previous topologies utilizing

Table 2: Information system for heart failure

Patients	BS	OA	PA	RE	AG	$\mathcal{D}.$
π_1	E	E	E	E	NE	E
π_2	NE	NE	NE	E	E	NE
π_3	E	E	E	E	E	E
π_4	NE	NE	NE	E	NE	NE
π_5	E	NE	NE	E	E	NE
π_8	E	E	NE	E	E	E
π_9	E	NE	E	E	NE	E
π_{10}	NE	NE	NE	E	E	NE

grills were less refined than the current ones. The existing topologies are more extensive and provide a significant amount of information that is beneficial for the analysis of rough sets. We utilized the neighborhood relation with the grill, which is among the most effective relations and has broadened the topological structures. This increases its relevance in domains necessitating extensive samples, such as worldwide epidemics. The characteristics of these topologies were examined, including comparative analyses among them. The smallest and largest topologies were determined, a task not accomplished in prior studies. Furthermore, novel approximations were proposed, employing these suggested topologies as an extension of the previous model. This increases its relevance in domains necessitating extensive samples, such as worldwide epidemics. A promising avenue for future study will include the following:
(1) Introducing two grills instead of one to generalize the current method;
(2) Developing a soft-topology to enhance the existing work;
(3) Extending the present paper to encompass fuzzy and picture sets;
(4) Comparison between grills and ideals to generalize the prevailing methodology .

Conflict of interest

There are no conflicting interests, according to the authors.

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