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On Soluble Intuitionistic Fuzzy Group and Its Properties

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Abstract. This article introduces the concept of soluble (or solvable) intuitionistic fuzzy groups as a novel intuitionistic fuzzy algebraic structure. Certain fundamental results concerning intuitionistic fuzzy normal subgroups and quotient intuitionistic fuzzy groups are established. Furthermore, a solvable series for an intuitionistic fuzzy group is constructed and illustrated through a detailed example. It is demonstrated that the family of intuitionistic fuzzy subgroups of a given intuitionistic fuzzy group shares the same support as the group itself. Finally, a biconditional relationship between a solvable intuitionistic fuzzy group and its support is proved.

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1. Introduction

The introduction of fuzzy sets by Zadeh [1] revolutionized decision-making in real-life problems and as well gave birth to the study of fuzzy algebra. By a direct application,

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Rosenfeld [2] introduced fuzzy groups and studied various group theoretic concepts in fuzzy group setting. In addition, the characterization of fuzzy subgroups was presented in [3], Frattini fuzzy subgroups were discussed in [4], and the concept of solvable groups was established in fuzzy multigroups [5].

Due to the limitation of fuzzy sets, the concept of intuitionistic fuzzy sets (IFSs) was introduced by Atanassov [6] and various properties of IFSs were presented in [7, 8]. Subsequently, Biswas [9, 10] applied IFSs to group theory by introducing intuitionistic fuzzy subgroups (IFSGs), Ahn [11] discussed the various forms of sublattice of lattice of IFSGs of a group, and proved the relationship of the sublattices of the lattice of IFGs. Some properties of IFSGs were discussed in [12], Fathi and Salleh [13] presented intuitionistic fuzzy groups (IFGs) from the notion of intuitionistic fuzzy space and discussed some of their properties, and Yuan et al. [14] shared some light on the description of IFSGs. Bal et al. [15] presented a brief note of kernel subgroups on IFGs and derived some properties of IFGs, Sharma [16] discussed the direct product of IFSGs, and the homomorphism of IFGs was presented in [17].

In addition, the α and β cuts of IFGs was presented in [18] and the t-IFSGs was introduced in [19]. Certain essential theorems of t-intuitionistic fuzzy isomorphism of t-IFSGs and the fuzzification of the famous Lagrange's theorem were presented in [20, 21]. Some algebraic characteristics of ε-IFSGs were discussed in [22], Shuaib et al. [23] presented the notion of η-IFSG based on η-IFSs and showed that every IFSG is an η-IFSG. Also, the study defined η-intuitionistic fuzzy cosets, η-intuitionistic fuzzy normal subgroups of a given group, and considered certain of their essential algebraic features. Gulzar et al. [24] discussed some properties of t-IFSGs, Rasuli [25] presented t- norn and s-norm of IFSGs, and Latif and Shuaib [26] applied t-IFSGs to Sylow theory.

Moreover, the idea of intuitionistic anti-fuzzy subgroups was discussed in [27] and Husban et al. [28] presented complex intuitionistic fuzzy group (CIFG) by allowing the membership and non-membership degrees to include complex numbers. Afterwards, Husban et al. [29] established the idea of normality in CIFGs with some properties, Al-Sharoa [30] presented $\alpha_{1,2}$ and $\beta_{1,2}$ cuts of CIFSGs with their algebraic structures, and Rasuli [31] presented t- norn and s-norm for CIFSGs. Recently, the renowned Jordan-Holder theorem has been instituted in IFG setting [32].

Although a lots of group theoretic properties have been established in IFGs, it is certain that solubility of IFGs has not been studied. In this study, the concept of soluble/solvable IFGs is presented and several of its properties are studied. In addition, the idea of solvable series for IFGs is considered and it is shown that the family of IFSGs of an IFG has the same support with the considered IFG

2. Preliminaries

Throughout the article, the symbols S and G represent a non-empty set and a group, respectively.

Definition 1 ([1])**.** A fuzzy subset ϱ of S is of the form:

$$\varrho = \{\langle s, \varrho_m(s)\rangle \mid s \in S\},$$

where $\varrho_m \colon X \to [0,1]$ is the membership grade of $s \in X$.

Definition 2 ([2])**.** A fuzzy subset ϱ of G is a fuzzy subgroup of G if

(i) $\varrho_m(gh) \geq \min\Big\{\varrho_m(g), \varrho_(h)\Big\}\ \forall\ g, h \in G$,

(ii) $\varrho_m(g^{-1}) = \varrho_m(g)\ \forall\ g \in G$.

In addition, $\varrho_m(e) = \varrho_m(gg^{-1}) \geq \min\Big\{\varrho_m(g), \varrho_m(g)\Big\} = \varrho_m(g)\ \forall\ g \in G$, where e is the unit element of G. Then, $\varrho_m(e)$ is the upper bound or tip of ϱ.

Definition 3 ([6])**.** An IFS β of S is of the form:

$$\beta = \{\langle s, \beta_m(s), \beta_n(s)\rangle \mid s \in S\},$$

where $\beta_m \colon X \to [0,1]$ and $\beta_n \colon X \to [0,1]$ are the membership and non-membership grades of $s \in S$ and $0 \leq \beta_m(s) + \beta_n(s) \leq 1$.

Definition 4 ([7])**.** Let β and γ be IFSs of S. Then,

(i) $\beta = \gamma \iff \beta_m(s) = \gamma_m(s)$ and $\beta_n(s) = \gamma_n(s)\ \forall s \in S$,

(ii) $\beta \subseteq \gamma \iff \beta_m(s) \leq \gamma_m(s)$ and $\beta_n(s) \geq \gamma_n(s)\ \forall s \in S$,

(iii) $\beta \cap \gamma = \Big\{\langle s, \min\{\beta_m(s), \gamma_m(s)\}, \max\{\beta_n(s), \gamma_n(s)\}\rangle \mid s \in S\Big\}$,

(iv) $\beta \cup \gamma = \Big\{\langle s, \max\{\beta_m(s), \gamma_m(s)\}, \min\{\beta_n(s), \gamma_n(s)\}\rangle \mid s \in S\Big\}$.

Definition 5 ([9])**.** An IFS β of G is an IFG/IFSG of G if

(i) $\beta_m(gh) \geq \min\Big\{\beta_m(g), \beta_m(h)\Big\}$ and $\beta_n(gh) \leq \max\Big\{\beta_n(g), \beta_n(h)\Big\}\ \forall\ g, h \in G$,

(ii) $\beta_m(g^{-1}) = \beta_m(g)$ and $\beta_n(g^{-1}) = \beta_n(g)\ \forall\ g \in G$.

In addition,

$$\beta_m(e) = \beta_m(gg^{-1}) \geq \min\Big\{\beta_m(g), \beta_m(g)\Big\} = \beta_m(g),$$
$$\beta_n(e) = \beta_n(gg^{-1}) \leq \max\Big\{\beta_n(g), \beta_n(g)\Big\} = \beta_n(g)$$

$\forall\ g \in G$, where e is the unit element of G. Then, $\beta_m(e)$ is the upper bound or tip of β and $\beta_n(e)$ is the lower bound of β, respectively.

Definition 6 ([9])**.** Let β and γ be IFGs of G. We say β is an IFSG of γ if $\beta \subseteq \gamma$. Again, β is a proper IFSG of γ if $\beta \subseteq \gamma$ and $\beta \neq \gamma$.

Definition 7 ([18])**.** Let β be an IFG of G. Then, the support of β is

$$\text{supp}(\beta) = \{g \in G \mid \beta_m(s) > 0 \text{ and } \beta_n(s) < 0\},$$

and it is a subgroup of G.

Definition 8 ([33])**.** Let β and γ be IFGs of G. Then, the product $\beta \circ \gamma$ is an IFS defined as:

$$(\beta \circ \gamma)(g) = \begin{cases} \bigvee_{g=hi} \min\left\{\beta_m(h), \gamma_m(i)\right\}, \bigwedge_{g=hi} \max\left\{\beta_n(h), \gamma_n(i)\right\}, \\ \text{if } \exists\, h, i \in G \text{ such that } g = hi \\ 0, \qquad \text{otherwise.} \end{cases}$$

Definition 9 ([33])**.** An IFG β of G is commutative if $\beta_m(gh) = \beta_m(hg)$ and $\beta_n(gh) = \beta_n(hg)$ $\forall g, h \in G$. Succinctly, an IFG β is commutative if G is a commutative group.

Definition 10 ([33])**.** Let β be an IFSG of an IFG γ of G. We say β is a normal IFSG of γ denoted as $\beta \lhd \gamma$ if $\beta_m(gh) = \beta_m(hg)$ and $\beta_n(gh) = \beta_n(hg) \iff \beta_m(h) = \beta_m(g^{-1}hg)$ and $\beta_n(h) = \beta_n(g^{-1}hg)$ $\forall\ g, h \in G$.

Definition 11 ([34])**.** If β is an IFSG of G. Then, an IFS $h\beta$ for $h \in G$ defined by $(h\beta)_m(g) = \beta_m(h^{-1}g)$ and $(h\beta)_n(g) = \beta_n(h^{-1}g)$ $\forall\ g \in G$ is called the left intuitionistic fuzzy coset of G. Similarly, an IFS βh for $h \in G$ defined by $(\beta h)_m(g) = \beta_m(gh^{-1})$ and $(\beta h)_n(g) = \beta_n(gh^{-1})$ $\forall\ g \in G$ is called the right intuitionistic fuzzy coset of G.

Definition 12 ([35])**.** If β and γ are IFGs of G and $\beta \lhd \gamma$. Then, the collection of the left intuitionistic fuzzy cosets/right intuitionistic fuzzy cosets of β such that $g\beta \circ h\beta = gh\beta$ $\forall\ g, h \in G$ is called an intuitionistic fuzzy quotient group (IFQG) of γ by β, denoted by γ/β.

3. Main Results

To begin with, the following results are presented before presenting the idea of soluble IFGs.

Theorem 1. *(i) Every commutative IFG is self-normal. (ii) If β and γ are IFGs in G where $\beta \lhd \gamma$, then β is self-normal.*

Proof. Assume β is a commutative IFG in G. Then, $\beta_m(gh) = \beta_m(hg)$ and $\beta_n(gh) = \beta_n(hg)$ $\forall\ g, h \in G$, and so $\beta_m(h) = \beta_m(g^{-1}hg)$ and $\beta_n(h) = \beta_n(g^{-1}hg)$. Hence $\beta \lhd \gamma$, which verifies (i).

Again, if $\beta \lhd \gamma$, then $\beta_m(g) \leq \gamma_m(g)$ and $\beta_n(g) \geq \gamma_n(g)$ $\forall\ x \in G$. Then, $\beta_m(g) > 0 < \beta_m(h) \implies \beta_m(gh) = \beta_m(hg)$ and $\beta_n(g) < 0$ and $\beta_n(h) < 0 \implies \beta_n(gh) = \beta_n(hg)$ $\forall$ $g, h \in G$, which establishes (ii).

Theorem 2. *Let β be an IFG in G. Then supp(β) is commutative $\Longleftrightarrow$ β is commutative.*

Proof. Let $g, h \in \text{supp}(\beta)$. If $\text{supp}(\beta)$ is commutative then $gh = hg$, and so $\beta_m(gh) = \beta_m(hg)$ and $\beta_n(gh) = \beta_n(hg)$ $\forall\ g, h \in G$, which defines commutative IFG β.

Conversely, suppose β is commutative then $\beta_m(gh) = \beta_m(hg)$ and $\beta_n(gh) = \beta_n(hg)$ $\forall\ g, h \in G$. Thus, $\text{supp}(\beta)$ is a commutative group because $\beta_m(gh) > 0 < \beta_m(hg)$ and $\beta_n(gh) < 0 > \beta_n(hg)$, then $gh = hg$ $\forall\ g, h \in \text{supp}(\beta)$.

Theorem 3. *Let β, γ and η be IFGs in G with the properties:*

(i) γ/β *and* η/β *are in canonical form,*

(ii) $\gamma/\beta \lhd \eta/\beta$,

(iii) $(\eta/\beta)/(\gamma/\beta)$ *is commutative.*

Then, $\gamma \lhd \eta$ *and* η/γ *is commutative.*

Proof. Suppose $\mathfrak{d}, \mathfrak{d}_1$ and $\mathfrak{d}_2$ are the supports of β, γ and η, respectively. Let $\mathfrak{d}'$ be the zone of β, η. Then γ/β and η/β are both IFGs of $\mathfrak{d}'/\mathfrak{d}$. If $g \in \mathfrak{d}_1$, then $\gamma_m(g) = (\gamma/\beta)_m(g\mathfrak{d}) \leq (\eta/\beta)_m(g\mathfrak{d}) = \eta_m(g)$ and $\gamma_n(g) = (\gamma/\beta)_n(g\mathfrak{d}) \geq (\eta/\beta)_n(g\mathfrak{d}) = \eta_n(g)$, and $\gamma \subseteq \eta$.

Thus $\eta_m(g) > 0 < \eta_m(h) \Longrightarrow (\eta/\beta)_m(g\mathfrak{d}) = \eta_m(g) > 0 < \eta_m(h) = (\eta/\beta)_m(h\mathfrak{d}) \Longrightarrow (\gamma/\beta)_m(gh\mathfrak{d}) = (\gamma/\beta)_m(hg\mathfrak{d}) \Longrightarrow \gamma_m(gh) = \gamma_m(hg)$ and $\eta_n(g) < 0 > \eta_n(h) \Longrightarrow (\eta/\beta)_n(g\mathfrak{d}) = \eta_n(g) < 0 > \eta_n(h) = (\eta/\beta)_n(h\mathfrak{d}) \Longrightarrow (\gamma/\beta)_n(gh\mathfrak{d}) = (\gamma/\beta)_n(hg\mathfrak{d}) \Longrightarrow \gamma_n(gh) = \gamma_n(hg)$ $\forall\ g, h \in G$, and so $\gamma \lhd \eta$.

In addition,

$$\text{supp}(\eta/\gamma) = \mathfrak{d}_2/\mathfrak{d}_1 \approx (\mathfrak{d}_2/\mathfrak{d})/(\mathfrak{d}_1/\mathfrak{d}) = \text{supp}(\eta/\beta)/\text{supp}(\gamma/\beta),$$

which is commutative. Hence, η/γ is commutative by Theorem 2.

Next, we define a solvable/soluble IFG as follows:

Definition 13. If β is an IFG in G, then a chain of successive IFSGs of β is:

$$\beta_0 \subseteq \beta_1 \subseteq \cdots \subseteq \beta_k = \beta, \tag{1}$$

such that $\text{supp}(\beta_0) = \text{supp}(\beta_1) = \cdots = \text{supp}(\beta_n) = \text{supp}(\beta)$.

Hence, (1) is of the form:

$$\left.\begin{aligned} \beta_{0_m}(g) \leq \beta_{1_m}(g) \leq \cdots \leq \beta_{k_m}(g) = \beta_m(g) \\ \beta_{0_n}(g) \geq \beta_{1_n}(g) \geq \cdots \geq \beta_{k_n}(g) = \beta_n(g) \end{aligned}\right\}, \tag{2}$$

$\forall\ g \in G$. But, if β is a trivial IFG, then $\beta_0 = \beta$.

Definition 14. An IFG β of G is soluble or solvable if it has a chain of successive IFSGs:

$$\left.\begin{array}{r}\beta_{0_m}(g) \leq \beta_{1_m}(g) \leq \cdots \leq \beta_{k_m}(g) = \beta_m(g) \\ \beta_{0_n}(g) \geq \beta_{1_n}(g) \geq \cdots \geq \beta_{k_n}(g) = \beta_n(g)\end{array}\right\},$$

where $\beta_i \triangleleft \beta_{i+1}$ and β_{i+1}/β_i is commutative for all $0 \leq i \leq k-1$.

Thus, the finite chain of successive IFSGs of β is a soluble series for β, represented by β_i and can be written as:

$$\beta_0 \triangleleft \beta_1 \triangleleft \cdots \triangleleft \beta_k = \beta. \tag{3}$$

Example 1. Let S_k be a symmetry group for $k = 3$, then $A_3 = \{\rho_0, \rho_1, \rho_2\} \subseteq S_3$ is a simple group ($\rho_1^{-1} = \rho_2$ and $\rho_2^{-1} = \rho_1$). Then, an IFG of A_3 is:

$$\beta = \{\langle\frac{0.6, 0.3}{\rho_0}\rangle, \langle\frac{0.4, 0.3}{\rho_1}\rangle, \langle\frac{0.4, 0.3}{\rho_2}\rangle\}.$$

Since $\rho_1.\rho_2 = \rho_2.\rho_1 = \rho_0$, then β is commutative. Then, the IFSGs of β are:

$$\begin{aligned}
\beta_1 &= \{\langle\frac{0.2, 0.7}{\rho_0}\rangle, \langle\frac{0.0, 0.7}{\rho_1}\rangle, \langle\frac{0.0, 0.7}{\rho_2}\rangle\}, \\
\beta_2 &= \{\langle\frac{0.3, 0.6}{\rho_0}\rangle, \langle\frac{0.1, 0.6}{\rho_1}\rangle, \langle\frac{0.1, 0.6}{\rho_2}\rangle\}, \\
\beta_3 &= \{\langle\frac{0.4, 0.5}{\rho_0}\rangle, \langle\frac{0.2, 0.5}{\rho_1}\rangle, \langle\frac{0.2, 0.5}{\rho_2}\rangle\}, \\
\beta_4 &= \{\langle\frac{0.5, 0.4}{\rho_0}\rangle, \langle\frac{0.3, 0.4}{\rho_1}\rangle, \langle\frac{0.3, 0.4}{\rho_2}\rangle\}, \\
\beta_5 &= \{\langle\frac{0.6, 0.3}{\rho_0}\rangle, \langle\frac{0.4, 0.3}{\rho_1}\rangle, \langle\frac{0.4, 0.3}{\rho_2}\rangle\}.
\end{aligned}$$

Then β_5 is a trivial IFSG of β because $\beta_5 = \beta$. Hence, the solvable series is:

$$\beta_1 \triangleleft \beta_2 \triangleleft \beta_3 \triangleleft \beta_4 \triangleleft \beta_5 = \beta,$$

because $\beta_i \triangleleft \beta_{i+1}$ and β_{i+1}/β_i is commutative for all $0 \leq i \leq k-1$. Thus, β is a solvable/soluble IFG of A_3.

Theorem 4. *If β is an IFG in G. Then, β is solvable iff supp(β) is solvable.*

Proof. Suppose β be a solvable IFG in G. Then, we have a solvable series of β:

$$\beta_0 \triangleleft \beta_1 \triangleleft \cdots \triangleleft \beta_k = \beta.$$

Set $\mathfrak{r} = \text{supp}(\beta)$, i.e. $\mathfrak{r} \subseteq G$. Then,

$$\{e\} = \mathfrak{r}_0 \triangleleft \mathfrak{r}_1 \triangleleft \cdots \triangleleft \mathfrak{r}_k = \mathfrak{r}$$

is a solvable series for $\mathfrak{r}$ since $\text{supp}(\beta) = \text{supp}(\beta_i)$. Thus, $\mathfrak{r}$ is a solvable group.

Conversely, let $\mathfrak{r} = \text{supp}(\beta)$ be solvable. Then,

$$\{e\} = \mathfrak{r}_0 \lhd \mathfrak{r}_1 \lhd \cdots \lhd \mathfrak{r}_k = \mathfrak{r}$$

is a solvable series for $\mathfrak{r}$. Consequently, we have

$$\beta_0 \lhd \beta_1 \lhd \cdots \lhd \beta_k = \beta,$$

which is a solvable series for β. Thus, β is a solvable IFG in G.

Theorem 5. *Let β and γ be IFGs in G with supp(β) = supp$(\gamma) = \mathfrak{r}$ such that $\beta \subseteq \gamma$ and β is self-normal. If β is solvable, then γ is a solvable IFG in G.*

Proof. Assume β is soluble, then $\beta_0 \lhd \beta_1 \lhd \cdots \lhd \beta_k = \beta$ is a solvable series for β. Because β is self-normal and $\text{supp}(\beta) = \text{supp}(\beta) = \mathfrak{r}$, then $\beta \lhd \beta$. Consequently, we have $\text{supp}(\gamma/\beta) = \mathfrak{r}/\mathfrak{r} = \mathfrak{r}$ and is commutative. Thus,

$$\beta_0 \lhd \beta_1 \lhd \cdots \lhd \beta_k = \beta \lhd \gamma$$

is a solvable series for γ. Hence, γ is solvable in G.

Theorem 6. *Let γ be a solvable IFG in G and let β be a self-normal IFSG of γ such that $\beta \subseteq \gamma_i$. Then, β is solvable.*

Proof. Let $\gamma_0 \lhd \gamma_1 \lhd \cdots \lhd \gamma_k = \gamma$ be a solvable series for γ because β is solvable. Since $\beta \subseteq \gamma_i$, we have

$$\gamma_0 \cap \beta \subseteq \gamma_1 \cap \beta \subseteq \cdots \subseteq \gamma_k \cap \beta = \beta.$$

Certainly, $(\gamma_1 \cap \beta)_m(g) > 0 < (\gamma_1 \cap \beta)_m(h) \Longrightarrow (\gamma_1)_m(g) > 0 < (\gamma_1)_m(h)$ and $(\beta)_m(g) > 0 < (\beta)_m(h) \Longrightarrow (\gamma_1)_m(gh) > 0 < (\gamma_1)_m(hg)$ and $(\beta)_m(gh) > 0 < (\beta)_m(hg) \Longrightarrow (\gamma_1 \cap \beta)_m(gh) = (\gamma_1 \cap \beta)_m(hg)\ \forall\ g, h \in G$.

Similarly, $(\gamma_1 \cap \beta)_n(g) < 0 > (\gamma_1 \cap \beta)_n(h) \Longrightarrow (\gamma_1)_n(g) < 0 > (\gamma_1)_n(h)$ and $(\beta)_n(g) < 0 > (\beta)_n(h) \Longrightarrow (\gamma_1)_n(gh) < 0 > (\gamma_1)_n(hg)$ and $(\beta)_n(gh) < 0 > (\beta)_n(hg) \Longrightarrow (\gamma_1 \cap \beta)_n(gh) = (\gamma_1 \cap \beta)_n(hg)\ \forall\ g, h \in G$.

Thus,

$$\gamma_0 \cap \beta \lhd \gamma_1 \cap \beta \lhd \cdots \lhd \gamma_k \cap \beta = \beta.$$

Again, let $\mathfrak{r}_i = \text{supp}(\gamma_i)$ and $\mathfrak{r} = \text{supp}(\beta)$. Then, we have $(\mathfrak{r}_2 \cap \mathfrak{r})/(\mathfrak{r}_1 \cap \mathfrak{r})$, which is commutative because $\mathfrak{r}_2/\mathfrak{r}_1$ is commutative. The similar logic follows for the other quotients, and so

$$\gamma_0 \cap \beta \subseteq \gamma_1 \cap \beta \subseteq \cdots \subseteq \gamma_k \cap \beta = \beta$$

is the solvable series for β. Hence, γ is solvable.

Theorem 7. *Suppose β is a normal IFSG of an IFG γ in G, and γ be self-normal. If β and γ/β are soluble, then γ is a soluble IFG in G.*

Proof. Assume the canonical form of γ/β is γ'/β'. Then, we have $\beta \subseteq \beta'$, $\gamma \subseteq \gamma'$, $\mathrm{supp}(\beta') = \mathrm{supp}(\beta) = \mathfrak{r}_1$ and $\mathrm{supp}(\gamma') = \mathrm{supp}(\gamma) = \mathfrak{r}_2$. Thus, we get a solvable series

$$\eta_0 \triangleleft \eta_1 \triangleleft \cdots \triangleleft \eta_k = \gamma'/\beta'.$$

Set $\beta' = \beta_l$ and $\gamma' = \gamma_k$. Suppose there is an IFG γ_i in G such that $\beta' \triangleleft \gamma_i \triangleleft \gamma_{i+1}$ and $\eta_i = \gamma_i/\beta'$ are the canonical form for $0 \leq i \leq k-1$. Thus,

$$\gamma_0/\beta' \triangleleft \gamma_1/\beta' \triangleleft \cdots \triangleleft \gamma_k/\beta' = \gamma'/\beta'$$

is a solvable series for γ'/β'. Because γ_0/β' is a trivial IFSG of γ'/β', then $\gamma_0 = \beta'$. By Theorem (3), we get

$$\beta' = \gamma_0 \triangleleft \gamma_1 \triangleleft \cdots \triangleleft \gamma_k = \gamma', \tag{4}$$

where γ_{i+1}/γ_i is commutative for $0 \leq i \leq k-1$.

Next, β is self-normal according to Theorem (1), and β' is solvable according to (5). Thus, there is a solvable series for β':

$$\beta_0 \triangleleft \beta_1 \triangleleft \cdots \triangleleft \beta_l = \beta'. \tag{5}$$

By putting (4) and (5) side by side, we get a solvable series for γ'. Hence, γ is a solvable IFG by Theorem (6).

4. Conclusion

In this paper, the concept of soluble (or solvable) IFG was introduced, and certain of its fundamental properties were established. A solvable series for an IFG was defined, and it was shown that the family of IFSGs of an IFG shares the same support as the IFG itself. To illustrate the theoretical development, an example of a soluble (solvable) IFG based on a permutation group was presented.

The findings of this study open avenues for further exploration. In particular, the results obtained here suggest the possibility of extending the framework to define and investigate nilpotent IFGs, which could provide deeper insights into the structure of intuitionistic fuzzy algebraic systems.

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