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# The Impacts of Brownian Motion on the Solutions of Stochastic Kakutani–Matsuuchi Equation

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**Abstract.** In this study, the stochastic Kakutani–Matsuuchi equation (SKME) forced by a multiplicative stochastic term is considered. The Kakutani–Matsuuchi equation is a fundamental tool for analyzing internal gravity waves in fluid dynamics, particularly in geophysical contexts such as oceanography and meteorology. Internal gravity waves occur in stratified fluids where density variations are influenced by gravity, leading to fascinating phenomena among different layers of fluid. Due to the importance of the Kakutani–Matsuuchi equation in examining the waves of internal gravity in the oceans and atmosphere, the solutions of the SKME can help us comprehend a variety of exciting scientific phenomena. Utilizing two distinct techniques, namely the Sardar subequation technique and the Jacobi elliptic function method, we derive novel bright and dark solitons, periodic solitons, as well as kink and anti-kink soliton solutions for the SKME. Moreover, we show many 3D and 2D graphs illustrating the influence of noise on SKME solutions.

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**Key Words and Phrases**: Sardar subequation method, stability by noise, exact solutions, stochastic PDE

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## 1. Introduction

Stochastic partial differential equations (SPDEs) might sound complicated, but they are essential tools in science and mathematics. They are used to model situations where randomness plays a crucial role. Many real-world systems, like the weather, stock markets, and even the spread of diseases, involve random factors that cannot be predicted precisely. These systems can be better understood and predicted using SPDEs, which incorporate randomness into their equations. This makes them crucial in many scientific areas including finance, physics, biology, and environmental science [1–5], because they help create more accurate models that consider uncertainty.

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An example of these equations is the Kakutani-Matsuuchi equation (SKME). The SKME, in its classical form [6], models the propagation of internal gravity waves in a stratified fluid environment, considering the effects of buoyancy and the vertical density gradient. These waves are crucial for understanding phenomena such as ocean mixing and atmospheric instability. However, real-world systems are often subject to random fluctuations, such as turbulent wind currents or variable temperature gradients. This brings us to the need for incorporating stochastic processes into the equation, leading to the formulation of the Stochastic Kakutani-Matsuuchi equation. This adaptation allows scientists to model the uncertainties inherent in natural systems, providing a more comprehensive view of wave dynamics.

In recent years, advances in computational power and numerical methods have further highlighted the importance of solving SPDEs. The complexity involved in analytical solutions often requires significant computational resources and sophisticated algorithms. With the increasing availability of powerful computers and advanced software, obtaining solutions to SPDEs has become more feasible, leading to breakthroughs in various scientific and engineering disciplines. Recently, there are many papers such as [7–14], and the references therein to find the analytical solutions for SPDEs.

In this study, we are looking for the exact solutions to the SKME [6, 15]:

$$\mathcal{Y}_t + \mathcal{Y}_x - \mathcal{Y}_{txx} + \mathcal{Y}\mathcal{Y}_x = \mu(\mathcal{Y} - \mathcal{Y}_{xx})\mathcal{B}_t, \tag{1}$$

where $\mathcal{Y} = \mathcal{Y}(x,t)$ represents the velocity of the fluid; $\mathcal{Y}_t$ is the time evolution term; $\mathcal{Y}_x$ is the lowest-order dispersive dispersion of the waves; $\mathcal{Y}_{txx}$ is the dispersive term; $\mathcal{Y}\mathcal{Y}_x$ is the advective nonlinear waves steepening; $\mathcal{B}(t)$ is the Brownian motion; $\mu$ is the noise intensity.

Due to the importance of SKME (1), some authors have acquired their solutions by applying various methods, for instance, the reliable implicit finite difference method [16], shifted Jacobi-Gauss-Lobatto collocation [17], finite difference schemes [15], auxiliary subequation and Khater II methods [18], and the extended tanh function method and the mapping method [19].

Our novelty of this paper is to discover the exact stochastic solutions of the SKME (1) by applying two different methods, including the sardar subequation and Jacobi elliptic function methods. The SKME (1) possesses essential uses in investigating the waves of internal gravity in the oceans and atmosphere; then the acquired solutions can be implemented to address several significant natural phenomena. Moreover, we provide multiple 3D and 2D graphical representations utilizing the MATLAB software (R2022b) to analyze the effects of noise on the exact solutions of the SKME (1).

The structure of the paper is as follows: In Sec. 2, the wave transformation is used to obtain the wave equation for the SKME (1), while the solutions of the SKME (1) is presented in Sec. 3. In Sec. 4, we introduce a discussion on the impacts of Brownian motion on the SKME (1) solutions. The conclusions of this paper are eventually presented.

## 2. Traveling Wave Equation for SKME

To generate the wave equation for the SKME (1), the following wave transformation is applied:

$$\mathcal{Y}(x,t) = \mathcal{U}(\xi)e^{[\mu\mathcal{B}(t)-\frac{1}{2}\mu^2 t]}, \quad \xi = \xi_1 x + \xi_2 t, \tag{2}$$

where $\mathcal{U}$ is a real deterministic function, $\xi_1$ and $\xi_2$ are unidentified constants. We note that

$$\begin{aligned}\mathcal{Y}_t &= [\xi_2\mathcal{U}' + \mu\mathcal{U}\mathcal{B}_t + \frac{1}{2}\mu^2\mathcal{U} - \frac{1}{2}\mu^2\mathcal{U}]e^{[\mu\mathcal{B}(t)-\frac{1}{2}\mu^2 t]} \\ &= [\xi_2\mathcal{U}' + \mu\mathcal{U}\mathcal{B}_t]e^{[\mu\mathcal{B}(t)-\frac{1}{2}\mu^2 t]},\end{aligned} \tag{3}$$

and

$$\mathcal{Y}_{txx} = [\xi_1^2\xi_2\mathcal{U}''' + \mu\xi_1^2\mathcal{U}''\mathcal{B}_t]e^{[\mu\mathcal{B}(t)-\frac{1}{2}\mu^2 t]}, \ \mathcal{Y}_x = \xi_1\mathcal{U}'e^{[\mu\mathcal{B}(t)-\frac{1}{2}\mu^2 t]}. \tag{4}$$

Inserting Eq. (2) into Eq. (1) and utilizing (3-4), we obtain

$$-\xi_1^2\xi_2\mathcal{U}''' + (\xi_2 + \xi_1)\mathcal{U}' + \xi_1\mathcal{U}\mathcal{U}'e^{[\mu\mathcal{B}(t)-\frac{1}{2}\mu^2 t]} = 0. \tag{5}$$

Taking the expectations, we have

$$-\xi_1^2\xi_2\mathcal{U}''' + (\xi_2 + \xi_1)\mathcal{U}' + \xi_1\mathcal{U}\mathcal{U}'e^{-\frac{1}{2}\mu^2 t}\mathbb{E}e^{[\mu\mathcal{B}(t)]} = 0. \tag{6}$$

Since $\mathcal{B}(t)$ is a normal process, then $\mathbb{E}(e^{\mu\mathcal{B}(t)}) = e^{\frac{1}{2}\mu^2 t}$. Hence, Eq. (6) becomes

$$-\xi_1^2\xi_2\mathcal{U}''' + (\xi_2 + \xi_1)\mathcal{U}' + \xi_1\mathcal{U}\mathcal{U}' = 0. \tag{7}$$

After integrating Eq. (7) once and omitting the integration constant, we have

$$\mathcal{U}'' + \mathcal{H}_1\mathcal{U} + \mathcal{H}_2\mathcal{U}^2 = 0, \tag{8}$$

where

$$\mathcal{H}_1 = \frac{-\xi_2 - \xi_1}{\xi_1^2\xi_2} \ \text{ and } \mathcal{H}_2 = \frac{-1}{2\xi_1\xi_2}.$$

## 3. Exact Solutions of SKME

The JEF and Sardar subequation methods are used to solve the wave equation (8). The Jacobi elliptic function method provides a unified framework for generating periodic, solitary, and kink-type solutions, depending on the choice of the elliptic modulus. Meanwhile, the Sardar subequation method is flexible, easy to implement, and capable of capturing multiple solution families within a unified framework, making it a powerful tool for analyzing nonlinear wave models in physics and applied mathematics. After that, by applying the transformation (2), we obtain the solutions of the SKME (1).

### 3.1. JEF-method

First, let us use the JEF-method (see [20]). Considering that the solutions of Eq. (8) take the following form:

$$\mathcal{U}(\xi) = \sum_{j=0}^{m} a_j [\chi(\xi)]^j, \tag{9}$$

where $\chi(\xi) = sn(\xi, \tilde{n})$, for $0 < \tilde{n} < 1$, is Jacobi elliptic sine function; $a_0,\ a_1,\ ....,\ a_M$ are unknown constants and $a_M \neq 0$. To determine $m$, we balance $\mathcal{U}^2$ with $\mathcal{U}''$ in Eq. (8) to have

$$2m = m + 2,$$

hence

$$m = 2. \tag{10}$$

With $m = 2$, we can write Eq. (9) as

$$\mathcal{U}(\xi) = a_0 + a_1\chi(\xi) + a_2\chi^2(\xi). \tag{11}$$

Differentiating Eq. (11) twice

$$\mathcal{U}''(\xi) = 2a_2 - a_1(\tilde{n}^2 + 1)\chi - 4a_2(\tilde{n}^2 + 1)\chi^2 + 2a_1\tilde{n}^2\chi^3 + 6a_2\tilde{n}^2\chi^4. \tag{12}$$

Substituting Eqs (11) and (12) into Eq. (8), we have:

$$(6\tilde{n}^2 a_2 + \mathcal{H}_2 a_2^2)\chi^4 + (2\tilde{n}^2 a_1 + 2\mathcal{H}_2 a_1 a_2)\chi^3 + (2a_0\mathcal{H}_2 a_2 - 4a_2(\tilde{n}^2 + 1) + \mathcal{H}_1 a_2 + \mathcal{H}_2 a_1^2)\chi^2$$
$$-[(\tilde{n}^2 + 1)a_1 - \mathcal{H}_1 a_1 - 2\mathcal{H}_2 a_0 a_1]\chi + (2a_2 + \mathcal{H}_1 a_0 + \mathcal{H}_2 a_0^2) = 0.$$

For $n = 4,\ 3,\ 2,\ 1,\ 0$, we put the coefficient of $\chi^n$ equal to zero:

$$6\tilde{n}^2 a_2 + \mathcal{H}_2 a_2^2 = 0,$$

$$2\tilde{n}^2 a_1 + 2\mathcal{H}_2 a_1 a_2 = 0,$$

$$2a_0\mathcal{H}_2 a_2 - 4a_2(\tilde{n}^2 + 1) + \mathcal{H}_1 a_2 + \mathcal{H}_2 a_1^2 = 0,$$

$$(\tilde{n}^2 + 1)a_1 - \mathcal{H}_1 a_1 - 2\mathcal{H}_2 a_0 a_1 = 0,$$

and

$$2a_2 + \mathcal{H}_1 a_0 + \mathcal{H}_2 a_0^2 = 0.$$

The above equations are solved to get the following two sets:

**First set:**

$$\begin{cases} a_0 = \frac{2(\tilde{n}^2+1) - 2\sqrt{\tilde{n}^4 - \tilde{n}^2 + 1}}{\mathcal{H}_2}, \\ a_1 = 0, \\ a_2 = \frac{-6\tilde{n}^2}{\mathcal{H}_2}, \\ \xi_2 = \frac{-\xi_1}{1 + 4\xi_1^2\sqrt{\tilde{n}^4 - \tilde{n}^2 + 1}}. \end{cases}$$

**Second set:**

$$\begin{cases} a_0 = \frac{2(\tilde{n}^2+1)+2\sqrt{\tilde{n}^4-\tilde{n}^2+1}}{\mathcal{H}_2}, \\ a_1 = 0, \\ a_2 = \frac{-6\tilde{n}^2}{\mathcal{H}_2}, \\ \xi_2 = \frac{-\xi_1}{1-4\xi_1^2\sqrt{\tilde{n}^4-\tilde{n}^2+1}}. \end{cases}$$

*For the first set*: The solutions of SKME (1), utilizing (11), is

$$\mathcal{Y}(x,t) = [\frac{2(\tilde{n}^2+1)-2\sqrt{\tilde{n}^4-\tilde{n}^2+1}}{\mathcal{H}_2} - \frac{6\tilde{n}^2}{\mathcal{H}_2}sn^2(\xi,\tilde{n})]e^{(\mu\mathcal{B}(t)-\frac{1}{2}\mu^2 t)}, \tag{13}$$

where $\xi = \xi_1 x - \frac{\xi_1}{1+4\xi_1^2\sqrt{\tilde{n}^4-\tilde{n}^2+1}}t$. If $\tilde{n} \to 1$, then Eq. (13) tends to:

$$\mathcal{Y}(x,t) = [\frac{2}{\mathcal{H}_2} - \frac{6}{\mathcal{H}_2}\tanh^2(\xi)]e^{(\mu\mathcal{B}(t)-\frac{1}{2}\mu^2 t)}, \tag{14}$$

where $\xi = \xi_1 x - \frac{\xi_1}{1+4\xi_1^2}t$.

*For the second set*: The solutions of SKME (1), utilizing (11), is

$$\mathcal{Y}(x,t) = [\frac{2(\tilde{n}^2+1)+2\sqrt{\tilde{n}^4-\tilde{n}^2+1}}{\mathcal{H}_2} - \frac{6\tilde{n}^2}{\mathcal{H}_2}sn^2(\xi,\tilde{n})]e^{(\mu\mathcal{B}(t)-\frac{1}{2}\mu^2 t)}, \tag{15}$$

where $\xi = \xi_1 x - \frac{\xi_1}{1-4\xi_1^2\sqrt{\tilde{n}^4-\tilde{n}^2+1}}t$.

The solution (15) tends (if $\tilde{n} \to 1$) to

$$\mathcal{Y}(x,t) = [\frac{6}{\mathcal{H}_2} - \frac{6}{\mathcal{H}_2}\tanh^2(\xi)]e^{(\mu\mathcal{B}(t)-\frac{1}{2}\mu^2 t)} = \frac{6}{\mathcal{H}_2}\text{sech}^2(\xi)e^{(\mu\mathcal{B}(t)-\frac{1}{2}\mu^2 t)}, \tag{16}$$

where $\xi = \xi_1 x - \frac{\xi_1}{1-4\xi_1^2}t$.

In the same way, we can transfer $sn(\xi,\tilde{n})$ in (11) by $cn(\xi,\tilde{n})$ to obtain the SKME (1) solutions as follows:

$$\mathcal{Y}(x,t) = [\frac{(2-4\tilde{n}^2)-2\sqrt{\tilde{n}^4-\tilde{n}^2+1}}{\mathcal{H}_2} + \frac{6\tilde{n}^2}{\mathcal{H}_2}cn^2(\xi,\tilde{n})]e^{(\mu\mathcal{B}(t)-\frac{1}{2}\mu^2 t)}, \tag{17}$$

or

$$\mathcal{Y}(x,t) = [\frac{(2-4\tilde{n}^2)+2\sqrt{\tilde{n}^4-\tilde{n}^2+1}}{\mathcal{H}_2} + \frac{6\tilde{n}^2}{\mathcal{H}_2}cn^2(\xi,\tilde{n})]e^{(\mu\mathcal{B}(t)-\frac{1}{2}\mu^2 t)}. \tag{18}$$

The solutions (17) and (18) tend (if $\tilde{n} \to 1$) to

$$\mathcal{Y}(x,t) = [\frac{-4}{\mathcal{H}_2} + \frac{6}{\mathcal{H}_2}\text{sech}^2(\xi)]e^{(\mu\mathcal{B}(t)-\frac{1}{2}\mu^2 t)}, \tag{19}$$

or

$$\mathcal{Y}(x,t) = \frac{6}{\mathcal{H}_2}\text{sech}^2(\xi)e^{(\mu\mathcal{B}(t)-\frac{1}{2}\mu^2 t)}. \tag{20}$$

### 3.2. Sardar subequation method

Consider the solution for Eq. (8) with $m = 2$ as follows:

$$\mathcal{U}(\xi) = \ell_0 + \ell_1\mathcal{P}+\ell_2\mathcal{P}^2, \tag{21}$$

where $\mathcal{P}$ solves

$$\mathcal{P}' = \sqrt{\mathcal{P}^4 + \hbar_1\mathcal{P}^2 + \hbar_2}, \tag{22}$$

where $\hbar_1$, and $\hbar_2$ are real constants. There are many cases that depend on $\hbar_1$, and $\hbar_2$ for the solutions of Eq. (22) as follows:

**Case 1:** If $\hbar_1 > 0$, and $\hbar_2 = 0$, then

$$\mathcal{P}_1(\xi) = \pm\sqrt{pq\hbar_1}\text{sech}_{pq}(\sqrt{\hbar_1}\xi), \tag{23}$$

and

$$\mathcal{P}_2(\xi) = \pm\sqrt{pq\hbar_1}\text{csch}_{pq}(\sqrt{\hbar_1}\xi), \tag{24}$$

where

$$\text{sech}_{pq}(\theta) = \frac{2}{pe^{\theta} + qe^{-\theta}} \quad \text{and } \text{csch}_{pq}(\theta) = \frac{2}{pe^{\theta} - qe^{-\theta}}.$$

**Case 2:** If $\hbar_1 < 0$, and $\hbar_2 = 0$, then

$$\mathcal{P}_3(\xi) = \pm\sqrt{-pq\hbar_1}\,\sec_{pq}(\sqrt{-\hbar_1}\xi), \tag{25}$$

and

$$\mathcal{P}_4(\xi) = \pm\sqrt{-pq\hbar_1}\,\csc_{pq}(\sqrt{-\hbar_1}\xi), \tag{26}$$

where

$$\sec_{pq}(\theta) = \frac{2}{pe^{\theta i} + qe^{-\theta i}} \quad \text{and } \csc_{pq}(\theta) = \frac{2}{pe^{\theta i} - qe^{-\theta i}}.$$

**Case 3:** If $\hbar_1 < 0$, and $\hbar_2 = \frac{\hbar_1^2}{4}$, then

$$\mathcal{P}_5(\xi) = \pm\sqrt{\frac{-\hbar_1}{2}}\,\tanh_{pq}(\sqrt{\frac{-\hbar_1}{2}}\xi), \tag{27}$$

$$\mathcal{P}_6(\xi) = \pm\sqrt{\frac{-\hbar_1}{2}}\,\coth_{pq}(\sqrt{\frac{-\hbar_1}{2}}\xi), \tag{28}$$

$$\mathcal{P}_7(\xi) = \pm\sqrt{\frac{-\hbar_1}{2}}[\coth_{pq}(\sqrt{-2\hbar_1}\xi) \pm \sqrt{pq}\text{csch}_{pq}(\sqrt{-2\hbar_1}\xi)], \tag{29}$$

and

$$\mathcal{P}_8(\xi) = \pm\sqrt{\frac{-\hbar_1}{8}}[\tanh_{pq}(\sqrt{\frac{-\hbar_1}{8}}\xi) + \coth_{pq}(\sqrt{\frac{-\hbar_1}{8}}\xi)], \tag{30}$$

where

$$\coth_{pq}(\theta) = \frac{pe^{\theta} + qe^{-\theta}}{pe^{\theta} - qe^{-\theta}} \quad \text{and } \tanh_{pq}(\theta) = \frac{pe^{\theta} - qe^{-\theta}}{pe^{\theta} + qe^{-\theta}}.$$

**Case 4:** If $\hbar_1 > 0$, and $\hbar_2 = \frac{\hbar_1^2}{4}$, then

$$\mathcal{P}_9(\xi) = \pm\sqrt{\frac{\hbar_1}{2}}\tan_{pq}(\sqrt{\frac{\hbar_1}{2}}\xi), \tag{31}$$

$$\mathcal{P}_{10}(\xi) = \pm\sqrt{\frac{\hbar_1}{2}}\cot_{pq}(\sqrt{\frac{\hbar_1}{2}}\xi), \tag{32}$$

$$\mathcal{P}_{11}(\xi) = \pm\sqrt{\frac{\hbar_1}{2}}[\tan_{pq}(\sqrt{2\hbar_1}\xi) \pm \sqrt{pq}\sec_{pq}(\sqrt{2\hbar_1}\xi)], \tag{33}$$

$$\mathcal{P}_{12}(\xi) = \pm\sqrt{\frac{\hbar_1}{2}}[\cot_{pq}(\sqrt{2\hbar_1}\xi) \pm \sqrt{pq}\csc_{pq}(\sqrt{2\hbar_1}\xi)], \tag{34}$$

and

$$\mathcal{P}_{13}(\xi) = \pm\sqrt{\frac{\hbar_1}{8}}[\tan_{pq}(\sqrt{\frac{\hbar_1}{8}}\xi) + \cot_{pq}(\sqrt{\frac{\hbar_1}{8}}\xi)], \tag{35}$$

where

$$\tan_{pq}(\theta) = -i\frac{pe^{\theta i} - qe^{-\theta i}}{pe^{\theta i} + qe^{-\theta i}} \text{ and } \cot_{pq}(\theta) = i\frac{pe^{\theta i} + qe^{-\theta i}}{pe^{\theta i} - qe^{-\theta i}}.$$

Now, we attempt to solve the Eq. (8). Using (22) and twice differentiating Eq. (21), we obtain

$$\mathcal{U}'' = \ell_1(\hbar_1\mathcal{P} + 2\mathcal{P}^3) + 2\ell_2(\hbar_2 + 2\hbar_1\mathcal{P}^2 + 3\mathcal{P}^4). \tag{36}$$

Setting Eq. (21) and Eq. (36) into Eq. (8) we have

$$(6\ell_2 + \mathcal{H}_2\ell_2^2)\mathcal{P}^4 + (2\ell_1 + 2\ell_1\ell_2\mathcal{H}_2)\mathcal{P}^3 + (4\ell_2\hbar_1 + 2\mathcal{H}_2\ell_0\ell_2 + \ell_1^2$$

$$+\ell_2\mathcal{H}_1)\mathcal{P}^2 + (\ell_1\hbar_1 + 2\mathcal{H}_2\ell_0\ell_1 + \mathcal{H}_1\ell_1)\mathcal{P} + (2\hbar_2\ell_2 + \mathcal{H}_1\ell_0 + \mathcal{H}_2\ell_0^2) = 0.$$

Setting all coefficients of $\mathcal{P}^k$ to zero yields an algebraic system of equations. Solving this system for $\hbar_1^2 - 3\hbar_2 > 0$ yields the next sets:

**Set I:**

$$\ell_0 = (\frac{-2\hbar_1 - 2\sqrt{(\hbar_1^2 - 3\hbar_2)}}{\mathcal{H}_2}), \quad \ell_1 = 0, \; \ell_2 = \frac{-6}{\mathcal{H}_2}, \quad \mathcal{H}_1 = 4\sqrt{(\hbar_1^2 - 3\hbar_2)}. \tag{37}$$

**Set II:**

$$\ell_0 = (\frac{-2\hbar_1 + 2\sqrt{(\hbar_1^2 - 3\hbar_2)}}{\mathcal{H}_2}), \quad \ell_1 = 0, \; \ell_2 = \frac{-6}{\mathcal{H}_2}, \quad \mathcal{H}_1 = -4\sqrt{(\hbar_1^2 - 3\hbar_2)}. \tag{38}$$

**For the Set I:** By using (37), the solution of Eq. (8) is

$$\mathcal{U}(\xi) = \frac{-2\hbar_1 - 2\sqrt{(\hbar_1^2 - 3\hbar_2)}}{\mathcal{H}_2} - \frac{6}{\mathcal{H}_2}\mathcal{P}^2(\xi). \tag{39}$$

Therefore, the solution of the SKME (1) is

$$\mathcal{Y}(x,t) = [\frac{-2\hbar_1 - 2\sqrt{(\hbar_1^2 - 3\hbar_2)}}{\mathcal{H}_2} - \frac{6}{\mathcal{H}_2}\mathcal{P}^2(\xi)]e^{(\mu\mathcal{B}(t)-\frac{1}{2}\mu^2 t)}. \tag{40}$$

Now by using Eqs (23)-(35), we acquire:

**Case I-1:** If $\hbar_1 > 0$, and $\hbar_2 = 0$, then

$$\mathcal{Y}(x,t) = (\frac{-4\hbar_1}{\mathcal{H}_2} - \frac{6pq\hbar_1}{\mathcal{H}_2}\text{sech}_{pq}^2(\sqrt{\hbar_1}\xi))e^{(\mu\mathcal{B}(t)-\frac{1}{2}\mu^2 t)}, \tag{41}$$

and

$$\mathcal{Y}(x,t) = (\frac{-4\hbar_1}{\mathcal{H}_2} - \frac{6pq\hbar_1}{\mathcal{H}_2}\text{csch}_{pq}^2(\sqrt{\hbar_1}\xi))e^{(\mu\mathcal{B}(t)-\frac{1}{2}\mu^2 t)}. \tag{42}$$

**Case I-2:** If $\hbar_1 < 0$, and $\hbar_2 = 0$, then

$$\mathcal{Y}(x,t) = \frac{6pq\hbar_1}{\mathcal{H}_2}\sec_{pq}^2(\sqrt{-\hbar_1}\xi)e^{(\mu\mathcal{B}(t)-\frac{1}{2}\mu^2 t)}, \tag{43}$$

and

$$\mathcal{Y}(x,t) = \frac{6pq\hbar_1}{\mathcal{H}_2}\csc_{pq}(\sqrt{-\hbar_1}\xi)e^{(\mu\mathcal{B}(t)-\frac{1}{2}\mu^2 t)}. \tag{44}$$

**Case I-3:** If $\hbar_1 < 0$, and $\hbar_2 = \frac{\hbar_1^2}{4}$, then

$$\mathcal{Y}(x,t) = (\frac{-\hbar_1}{\mathcal{H}_2} + \frac{3\hbar_1}{\mathcal{H}_2}\tanh_{pq}^2(\sqrt{\frac{-\hbar_1}{2}}\xi))e^{(\mu\mathcal{B}(t)-\frac{1}{2}\mu^2 t)}, \tag{45}$$

$$\mathcal{Y}(x,t) = (\frac{-\hbar_1}{\mathcal{H}_2} + \frac{3\hbar_1}{\mathcal{H}_2}\coth_{pq}^2(\sqrt{\frac{-\hbar_1}{2}}\xi))e^{(\mu\mathcal{B}(t)-\frac{1}{2}\mu^2 t)}, \tag{46}$$

$$\mathcal{Y}(x,t) = (\frac{-\hbar_1}{\mathcal{H}_2} + \frac{3\hbar_1}{\mathcal{H}_2}(\coth_{pq}(\sqrt{-2\hbar_1}\xi) \pm \sqrt{pq}\text{csch}_{pq}(\sqrt{-2\hbar_1}\xi))^2)e^{(\mu\mathcal{B}(t)-\frac{1}{2}\mu^2 t)}, \tag{47}$$

and

$$\mathcal{Y}(x,t) = (\frac{-\hbar_1}{\mathcal{H}_2} + \frac{3\hbar_1}{4\mathcal{H}_2}(\tanh_{pq}(\sqrt{\frac{-\hbar_1}{8}}\xi) + \coth_{pq}(\sqrt{\frac{-\hbar_1}{8}}\xi))^2)e^{(\mu\mathcal{B}(t)-\frac{1}{2}\mu^2 t)}. \tag{48}$$

**Case I-4:** If $\hbar_1 > 0$, and $\hbar_2 = \frac{\hbar_1^2}{4}$, then

$$\mathcal{Y}(x,t) = (\frac{-3\hbar_1}{\mathcal{H}_2} - \frac{3\hbar_1}{\mathcal{H}_2}\tan_{pq}^2(\sqrt{\frac{\hbar_1}{2}}\xi))e^{(\mu\mathcal{B}(t)-\frac{1}{2}\mu^2 t)} = \frac{-3\hbar_1}{\mathcal{H}_2}\sec_{pq}^2(\sqrt{\frac{\hbar_1}{2}}\xi))e^{(\mu\mathcal{B}(t)-\frac{1}{2}\mu^2 t)}, \tag{49}$$

$$\mathcal{Y}(x,t) = (\frac{-3\hbar_1}{\mathcal{H}_2} - \frac{3\hbar_1}{\mathcal{H}_2}\cot_{pq}^2(\sqrt{\frac{\hbar_1}{2}}\xi))e^{(\mu\mathcal{B}(t)-\frac{1}{2}\mu^2 t)} = \frac{-3\hbar_1}{\mathcal{H}_2}\csc_{pq}^2(\sqrt{\frac{\hbar_1}{2}}\xi))e^{(\mu\mathcal{B}(t)-\frac{1}{2}\mu^2 t)}, \tag{50}$$

$$\mathcal{Y}(x,t) = (\frac{-3\hbar_1}{\mathcal{H}_2} - \frac{3\hbar_1}{\mathcal{H}_2}(\tan_{pq}(\sqrt{2\hbar_1}\xi) \pm \sqrt{pq}\sec_{pq}(\sqrt{2\hbar_1}\xi))^2)e^{(\mu\mathcal{B}(t)-\frac{1}{2}\mu^2 t)}, \tag{51}$$

$$\mathcal{Y}(x,t)=(\frac{-3\hbar_1}{\mathcal{H}_2}-\frac{3\hbar_1}{\mathcal{H}_2}(\cot_{pq}(\sqrt{2\hbar_1}\xi)\pm\sqrt{pq}\csc_{pq}(\sqrt{2\hbar_1}\xi))^2)e^{(\mu\mathcal{B}(t)-\frac{1}{2}\mu^2t)}, \tag{52}$$

and

$$\mathcal{Y}(x,t)=(\frac{-3\hbar_1}{\mathcal{H}_2}-\frac{3\hbar_1}{4\mathcal{H}_2}(\tan_{pq}(\sqrt{\frac{\hbar_1}{8}}\xi)+\cot_{pq}(\sqrt{\frac{\hbar_1}{8}}\xi))^2)e^{(\mu\mathcal{B}(t)-\frac{1}{2}\mu^2t)}. \tag{53}$$

**For the Set II:** By using (38), the solution of Eq. (8) is

$$\mathcal{U}(\xi)=\frac{-2\hbar_1+2\sqrt{(\hbar_1^2-3\hbar_2)}}{\mathcal{H}_2}-\frac{6}{\mathcal{H}_2}\mathcal{P}^2(\xi). \tag{54}$$

Therefore, the solution of the SKME (1) is

$$\mathcal{Y}(x,t)=[\frac{-2\hbar_1+2\sqrt{(\hbar_1^2-3\hbar_2)}}{\mathcal{H}_2}-\frac{6}{\mathcal{H}_2}\mathcal{P}^2(\xi)]e^{(\mu\mathcal{B}(t)-\frac{1}{2}\mu^2t)}. \tag{55}$$

Now by using Eqs (23)-(35), we obtain:

**Case II-1:** If $\hbar_1>0$, and $\hbar_2=0$, then

$$\mathcal{Y}(x,t)=-\frac{6pq\hbar_1}{\mathcal{H}_2}\text{sech}^2_{pq}(\sqrt{\hbar_1}\xi)e^{(\mu\mathcal{B}(t)-\frac{1}{2}\mu^2t)}, \tag{56}$$

and

$$\mathcal{Y}(x,t)=-\frac{6pq\hbar_1}{\mathcal{H}_2}\text{csch}^2_{pq}(\sqrt{\hbar_1}\xi)e^{(\mu\mathcal{B}(t)-\frac{1}{2}\mu^2t)}. \tag{57}$$

**Case II-2:** If $\hbar_1<0$, and $\hbar_2=0$, then

$$\mathcal{Y}(x,t)=(\frac{-4\hbar_1}{\mathcal{H}_2}+\frac{6pq\hbar_1}{\mathcal{H}_2}\sec^2_{pq}(\sqrt{-\hbar_1}\xi))e^{(\mu\mathcal{B}(t)-\frac{1}{2}\mu^2t)}, \tag{58}$$

and

$$\mathcal{Y}(x,t)=(\frac{-4\hbar_1}{\mathcal{H}_2}+\frac{6pq\hbar_1}{\mathcal{H}_2}\csc_{pq}(\sqrt{-\hbar_1}\xi))e^{(\mu\mathcal{B}(t)-\frac{1}{2}\mu^2t)}. \tag{59}$$

**Case II-3:** If $\hbar_1<0$, and $\hbar_2=\frac{\hbar_1^2}{4}$, then

$$\mathcal{Y}(x,t)=(\frac{-\hbar_1}{\mathcal{H}_2}+\frac{3\hbar_1}{\mathcal{H}_2}\tanh^2_{pq}(\sqrt{\frac{-\hbar_1}{2}}\xi))e^{(\mu\mathcal{B}(t)-\frac{1}{2}\mu^2t)}, \tag{60}$$

$$\mathcal{Y}(x,t)=(\frac{-\hbar_1}{\mathcal{H}_2}+\frac{3\hbar_1}{\mathcal{H}_2}\coth^2_{pq}(\sqrt{\frac{-\hbar_1}{2}}\xi))e^{(\mu\mathcal{B}(t)-\frac{1}{2}\mu^2t)}, \tag{61}$$

$$\mathcal{Y}(x,t)=(\frac{-\hbar_1}{\mathcal{H}_2}+\frac{3\hbar_1}{\mathcal{H}_2}(\coth_{pq}(\sqrt{-2\hbar_1}\xi)\pm\sqrt{pq}\text{csch}_{pq}(\sqrt{-2\hbar_1}\xi))^2)e^{(\mu\mathcal{B}(t)-\frac{1}{2}\mu^2t)}, \tag{62}$$

and

$$\mathcal{Y}(x,t)=(\frac{-\hbar_1}{\mathcal{H}_2}+\frac{3\hbar_1}{4\mathcal{H}_2}(\tanh_{pq}(\sqrt{\frac{-\hbar_1}{8}}\xi)+\coth_{pq}(\sqrt{\frac{-\hbar_1}{8}}\xi))^2)e^{(\mu\mathcal{B}(t)-\frac{1}{2}\mu^2t)}. \tag{63}$$

**Case II-4:** If $\hbar_1 > 0$, and $\hbar_2 = \frac{\hbar_1^2}{4}$, then

$$\mathcal{Y}(x,t) = (\frac{-\hbar_1}{\mathcal{H}_2} - \frac{3\hbar_1}{\mathcal{H}_2}\tan^2_{pq}(\sqrt{\frac{\hbar_1}{2}}\xi))e^{(\mu\mathcal{B}(t)-\frac{1}{2}\mu^2 t)}, \tag{64}$$

$$\mathcal{Y}(x,t) = (\frac{-\hbar_1}{\mathcal{H}_2} - \frac{3\hbar_1}{\mathcal{H}_2}\cot^2_{pq}(\sqrt{\frac{\hbar_1}{2}}\xi))e^{(\mu\mathcal{B}(t)-\frac{1}{2}\mu^2 t)}, \tag{65}$$

$$\mathcal{Y}(x,t) = (\frac{-\hbar_1}{\mathcal{H}_2} - \frac{3\hbar_1}{\mathcal{H}_2}(\tan_{pq}(\sqrt{2\hbar_1}\xi) \pm \sqrt{pq}\sec_{pq}(\sqrt{2\hbar_1}\xi))^2)e^{(\mu\mathcal{B}(t)-\frac{1}{2}\mu^2 t)}, \tag{66}$$

$$\mathcal{Y}(x,t) = (\frac{-\hbar_1}{\mathcal{H}_2} - \frac{3\hbar_1}{\mathcal{H}_2}(\cot_{pq}(\sqrt{2\hbar_1}\xi) \pm \sqrt{pq}\csc_{pq}(\sqrt{2\hbar_1}\xi))^2)e^{(\mu\mathcal{B}(t)-\frac{1}{2}\mu^2 t)}, \tag{67}$$

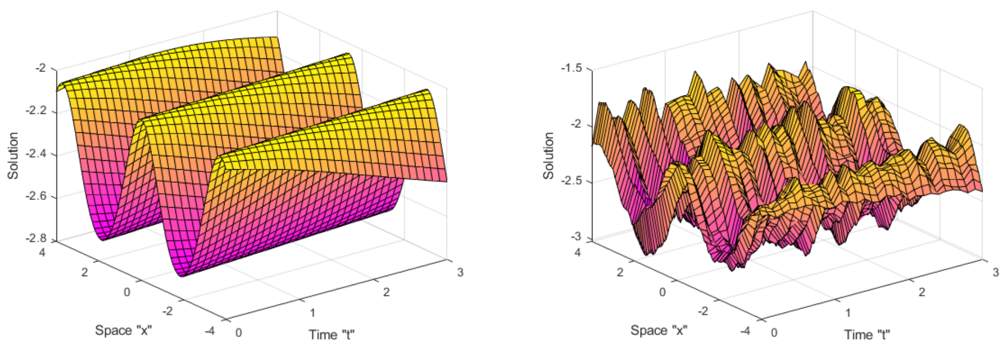
and

$$\mathcal{Y}(x,t) = (\frac{-\hbar_1}{\mathcal{H}_2} - \frac{3\hbar_1}{4\mathcal{H}_2}(\tan_{pq}(\sqrt{\frac{\hbar_1}{8}}\xi) + \cot_{pq}(\sqrt{\frac{\hbar_1}{8}}\xi))^2)e^{(\mu\mathcal{B}(t)-\frac{1}{2}\mu^2 t)}. \tag{68}$$

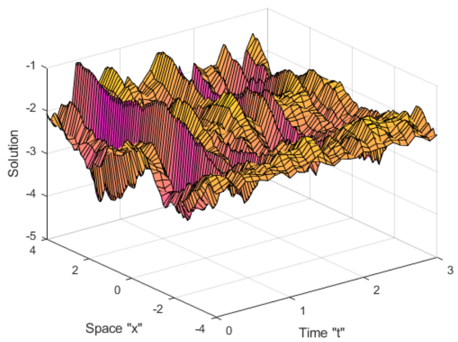
## 4. Impacts of noise

The stochastic Kakutani-Matsuuchi equation represents a significant advancement in the study of the waves of internal gravity within the context of fluid dynamics. By introducing stochastic elements into the classical equation, researchers have deepened their understanding of wave behavior under uncertain conditions, revealing new insights into natural phenomena characterized by randomness. As modeling techniques continue to evolve, the integration of stochastic processes into established equations will likely become a standard practice, paving the way for more accurate predictions in atmospheric and oceanic sciences. Such developments will undoubtedly enhance our ability to address pressing environmental challenges, further emphasizing the importance of stochastic analysis in fluid dynamics.

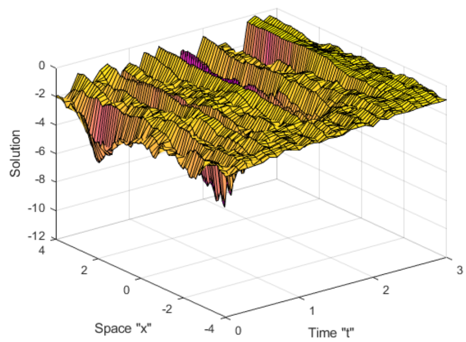
Let us now demonstrate how noise affects on the attained solution of the SKME (1). A number of figures for the solutions (13), (16) and (56) are provided to illustrate how some of the discovered solutions behave as follows:



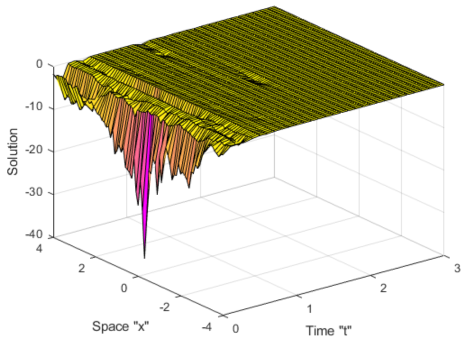

(a) $\mu = 0$ (b) $\mu = 0.1$



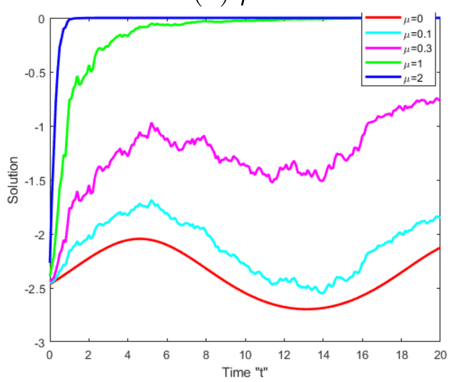

(c) $\mu = 0.3$




(d) $\mu = 1$

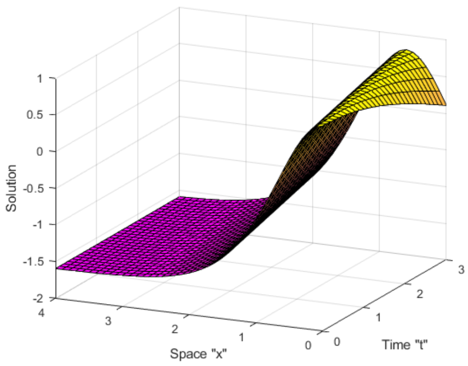



(e) $\mu = 2$

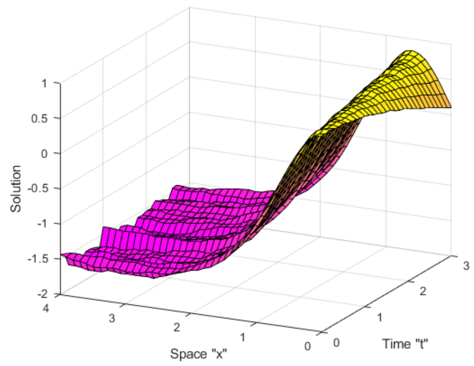



(f) $\mu = 0,\ 0.1,\ 0.3,\ 1,\ 2$

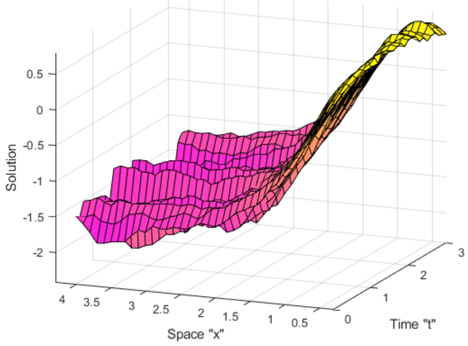
Figure 1. (a-c) give a 3D-shape of the Eq (13) with distinct $\mu$ and with $\xi_1 = 1,\ \xi_2 = -2,\ x \in [-4, 4],\ t \in [0, 3],\ \check{n} = 0.5$, (d) gives a 2D-shape of Eq. (13) with $x = 1$ and distinct $\mu$



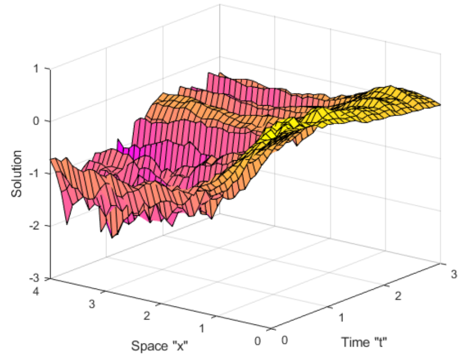

(a) $\mu = 0$



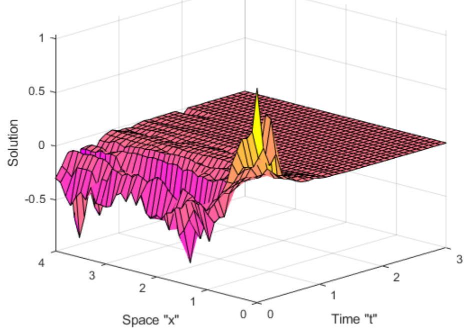

(b) $\mu = 0.1$



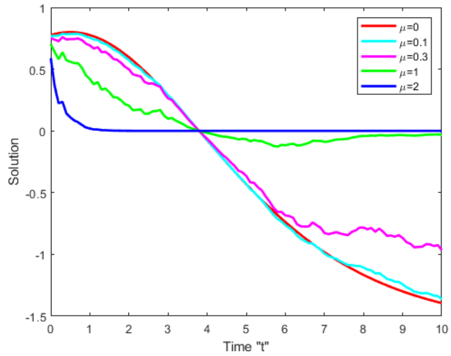

(c) $\mu = 0.3$




(d) $\mu = 1$

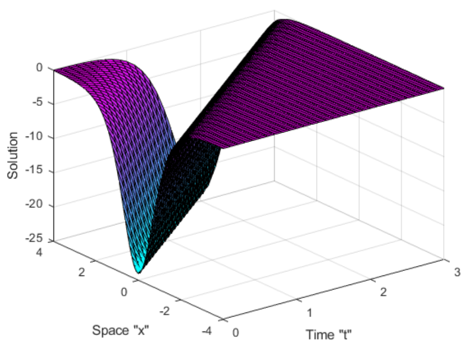



(e) $\mu = 2$

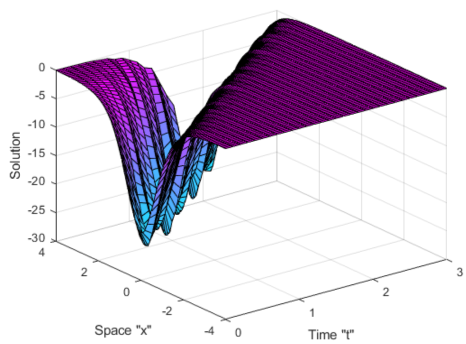



(f) $\mu = 0,\ 0.1,\ 0.3,\ 1,\ 2$

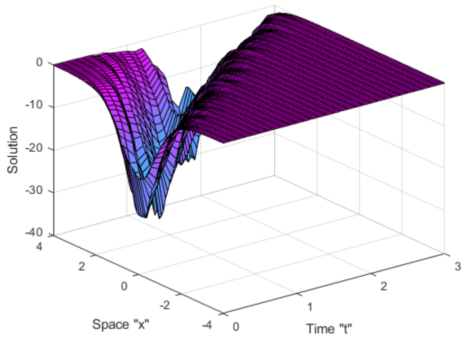
Figure 2. (a-e) give a 3D-shape of the Eq. (16) with distinct $\mu$ and with $\xi_1 = 1,\ \xi_2 = -2,\ x \in [0, 4],\ t \in [0, 3]$ (f) gives a 2D-shape of Eq. (16) with $x = 0.8$ and, distinct $\mu$



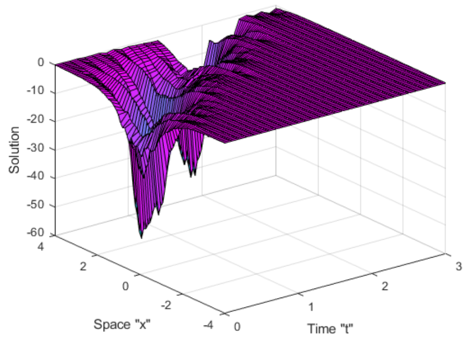

(a) $\mu = 0$



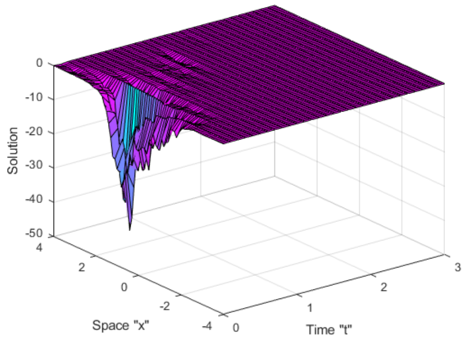

(b) $\mu = 0.1$



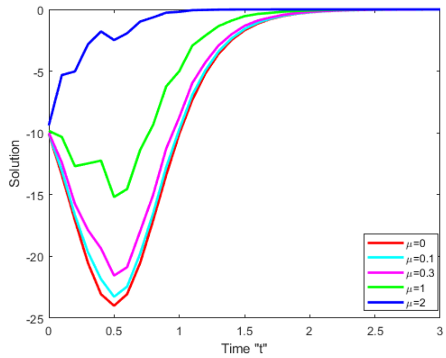

(c) $\mu = 0.3$




(d) $\mu = 1$




(e) $\mu = 2$




(f) $\mu = 0,\ 0.1,\ 0.3,\ 1,\ 2$

Figure 3. (a-e) give a 3D shape of the Eq. (56) with distinct $\mu$ and with $\xi_1 = 1,\ \xi_2 = -2,\ x \in [-4, 4],\ t \in [0, 3]$, (f) gives a 2D shape of Eq. (56) with $x = 1.4$ and, distinct $\mu$

Figures 1-3 show that when noise is eliminated ($\mu = 0$), there are several solutions, including periodic, bright, dark, soliton, and among others. When noise is introduced with $\mu =\ 0.1,\ 0.3,\ 1,\ 2$, the surface becomes much flatter and exhibits modest transit patterns, as the 2D graph illustrates. This shows that the solutions of the SKME (1) tend to converge toward zero when noise is present.

## 5. Conclusions

In this study, the stochastic Kakutani–Matsuuchi equation (SKME) (1) perturbed in the Itô sense by multiplicative stochastic term was examined. By implementing two distinct methods, including the Jacobi elliptic function method and the Sardar subequation method, we were able to find novel periodic, bright, kink, dark and anti-kink soliton solutions for SKME. The solutions of the SKME are helpful in interpreting a number of intriguing scientific phenomena, as the model is crucial for exploring internal gravity waves in the oceans and atmosphere. Using MATLAB, we provide many two- and three-dimensional graphs that demonstrate the influence of the Brownian motion on the analytical solutions of SKME (1). We concluded that the solutions remained close to zero due to the stochastic term. We may obtain analytical solutions for the Kakutani–Matsuuchi equation with fractional derivative operators in further study.

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