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Optimization-Based Modified Laplace Transform Techniques for Addressing Fuzzy Fractional Advection-Diffusion Problems

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Abstract. This paper presents an optimization–enhanced Laplace transform homotopy perturbation framework for solving fuzzy time–fractional advection–diffusion equations with memory effects and parametric uncertainty. The model is formulated in a double–parametric fuzzy setting, allowing uncertainty in system parameters to be consistently propagated through the fractional dynamics. The Atangana–Baleanu–Caputo fractional derivative is extended to fuzzy–valued functions via the generalized Hukuhara difference.To improve the convergence and accuracy of the classical LT–HPM, two systematic parameter identification strategies are incorporated. The first employs a residual error collocation point (RECP) technique, while the second introduces a least–squares optimization of the fuzzy residual, leading to an optimized variant referred to as OMLT–HPM. Both strategies preserve the original Laplace–homotopy structure while enabling robust and reproducible calibration of auxiliary parameters.Rigorous theoretical results are established, including existence, uniqueness, stability, and convergence of the fuzzy solution in the double–parametric framework. Numerical experiments for a benchmark fuzzy fractional advection–diffusion problem demonstrate that the proposed methods significantly outperform the standard LT–HPM and classical finite difference schemes, with the optimized approach achieving high accuracy using only a small number of perturbation terms. The results highlight the effectiveness and computational efficiency of the proposed framework for applied fractional models under uncertainty.

2020 Mathematics Subject Classifications: 26A33, 35R11, 03E72, 65M06, 90C30

Key Words and Phrases: Fuzzy fractional advection-diffusion, Laplace transform, homotopy perturbation method, Atangana-Baleanu operator, least squares optimization

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1. Introduction

Fractional differential equations (FDEs) have become essential tools for modeling complex physical and engineering phenomena characterized by nonlocal interactions and memory effects [1–3]. In particular, fractional advection–diffusion equations play a central role

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in describing transport processes such as heat conduction, mass diffusion, and fluid flow, where anomalous and history-dependent behaviors frequently arise.

Among the available fractional operators, the Atangana–Baleanu–Caputo (ABC) derivative has attracted considerable attention due to its nonsingular Mittag–Leffler kernel, which avoids the singularities associated with classical fractional derivatives while providing a realistic representation of fading memory effects [4–6]. These features make the ABC operator especially suitable for diffusion-type problems requiring numerical stability and physical consistency.

In practical applications, model parameters such as diffusion coefficients, advection velocities, source terms, and initial conditions are often affected by uncertainty originating from measurement errors, environmental variability, and incomplete information. To address this issue, fuzzy set theory has been combined with fractional calculus, leading to fuzzy fractional differential equations (fuzzy FDEs). In this framework, uncertain parameters are represented by fuzzy numbers, allowing imprecision to be incorporated directly into the mathematical model alongside fractional memory effects [7–10].

Fuzzy fractional models have been successfully applied in various scientific and engineering contexts, including fuzzy diffusion and reaction–diffusion systems governed by the ABC operator [8, 10], decision-making problems based on fuzzy aggregation operators [11], and semi-analytical treatments of fuzzy fractional partial differential equations [12–14]. Moreover, several numerical strategies have been developed to improve accuracy and convergence, such as reproducing kernel algorithms for fuzzy integrodifferential equations [15] and Galerkin-type methods based on Lucas polynomials for fuzzy differential models with engineering applications [16].

Despite these advances, the efficient numerical treatment of fuzzy fractional models remains challenging. Many semi-analytical methods exhibit slow convergence or strong dependence on the initial approximation, particularly in the presence of the ABC operator. The Homotopy Perturbation Method (HPM) and its Laplace-based variant, the Laplace Transform Homotopy Perturbation Method (LT-HPM), are widely used but remain sensitive to the selection of auxiliary functions and initial guesses [17–19]. Although modified versions of LT-HPM introduce problem-dependent trial functions to enhance flexibility, the associated auxiliary parameters are often chosen heuristically, which may limit convergence and accuracy for certain fractional orders and fuzzy parameter configurations [20].

Motivated by these limitations, this paper investigates three solution strategies for fuzzy time-fractional advection–diffusion equations governed by the ABC derivative: the standard LT-HPM, the Modified LT-HPM (MLT-HPM), and an optimized variant referred to as OMLT-HPM. In the optimized approach, the auxiliary constants in the trial functions are determined automatically by minimizing the governing-equation residual in a least-squares sense over a prescribed collocation grid. This residual-driven optimization eliminates heuristic parameter tuning, reduces sensitivity to the initial approximation, and significantly enhances convergence and accuracy.

The main contributions of this work are summarized as follows: (i) formulation of a fuzzy time-fractional advection–diffusion model using the ABC derivative within a double-parametric fuzzy framework; (ii) systematic development and comparison of LT-HPM, MLT-HPM, and OMLT-HPM; (iii) rigorous analysis of convergence, uniqueness, and stability; and (iv) comprehensive numerical validation using benchmark problems with known exact solutions, including quantitative comparisons with classical BTCS and Crank–Nicolson schemes.

The remainder of the paper is organized as follows. Section 2 presents the mathematical preliminaries and model formulation. Section 3 describes the three solution methods. Section 4 provides the theoretical analysis, Section 5 discusses numerical results, and Section 6 concludes the paper.

2. Preliminaries

This section presents the fundamental concepts and mathematical tools required for the development of the proposed solution strategies. Specifically, we recall basic notions of fuzzy numbers, r-level sets, and double-parametric representations, followed by generalized Hukuhara differentiability. We then introduce the Atangana–Baleanu–Caputo (ABC) fractional derivative for fuzzy-valued functions and formulate the fuzzy time-fractional advection–diffusion equation in double-parametric form. These preliminaries provide the theoretical foundation for the LT-HPM, MLT-HPM, and OMLT-HPM developed in subsequent sections.

This section introduces several theorems and definitions that will be instrumental in the subsequent sections of this paper.

Definition 2.1. (r-level set based on [21]) The r-level set of a fuzzy set D, denoted as D_r, is a classical set containing all elements $x \in X$ for which the membership degree $\mu_{\mathsf{D}}(x)$ is no less than r. In other words,

$$\mathsf{D}_r = \{x \in X \mid \mu_{\mathsf{D}}(x) \geq r\}, \quad r \in [0,1].$$

Definition 2.2. (Fuzzy numbers according to [22]) A fuzzy number represents an uncertain real value, where membership is defined by a function. If a fuzzy number μ is triangular, it is described by three values $h < j < k$, forming a triangular membership function:

$$\mu(x) = \begin{cases} 0, & \text{for } x < h, \\ \frac{x-h}{j-h}, & \text{for } h \leq x \leq j, \\ \frac{k-x}{k-j}, & \text{for } j \leq x \leq k, \\ 0, & \text{for } x > k. \end{cases}$$

The r-level set for a triangular fuzzy number is given by

$$[\mu]_r = [h + r(j-h),\, k - r(k-j)], \quad r \in [0,1].$$

Definition 2.3. (Double-parametric representation of fuzzy numbers, as in [23]) Using a single parametric form, a fuzzy number $\tilde{U}$ can be written as $\tilde{U} = [\underline{u}(r), \overline{u}(r)]$. The double-parametric form expresses it as

$$\tilde{U}(r,\beta) = \beta \cdot [\overline{u}(r) - \underline{u}(r)] + \underline{u}(r),$$

where $r, \beta \in [0,1]$. Here, β represents the fuzzy weighting parameter in the double-parametric form [24], and the r-level denotes the α-cut confidence level used in fuzzy set modeling [21].

Notation. Throughout this work, the parameter $r \in [0,1]$ represents the confidence level of the r-cut, while $\eta \in [0,1]$ denotes the fuzzy weighting parameter used in the double-parametric representation. Unless otherwise stated, β is used interchangeably with η following common conventions in fuzzy modeling.

Definition 2.4. For two fuzzy numbers w and v expressed in their parametric form, the arithmetic operations are defined as:
Addition:

$$(w+v)[r] = [\underline{w}(r) + \underline{v}(r),\, \overline{w}(r) + \overline{v}(r)],$$

Scalar multiplication:

$$(\eta \cdot w)[r] = \begin{cases} [\eta \underline{w}(r),\, \eta \overline{w}(r)], & \text{if } \eta \geq 0, \\ [\eta \overline{w}(r),\, \eta \underline{w}(r)], & \text{if } \eta < 0, \end{cases}$$

where $r \in [0,1]$.

Definition 2.5. (Generalized Hukuhara derivative) [25] The gH-derivative of a function U can be defined as

$$g_H U'(\tau) = \lim_{h \to 0^+} \frac{U(\tau + h) \ominus_{gH} U(\tau)}{h} = \lim_{h \to 0^+} \frac{U(\tau) \ominus_{gH} U(\tau + h)}{h},$$

where $\frac{dU(\tau)}{dt}\big|_{gH} \in C^F(I) \cap L^F(I)$, and $C^F(I) \cap L^F(I)$ denotes the space of fuzzy-valued functions whose level-set representations are continuous and Lebesgue integrable on the interval I. Moreover,

$$U(\tau + h) \ominus_{gH} U(\tau) = z(\tau) \quad \Leftrightarrow \quad \begin{cases} \text{i)} \quad U(\tau + h) = U(\tau) \oplus z(\tau), \\ \text{ii)} \quad U(\tau) = U(\tau + h) \oplus (-z(\tau)). \end{cases}$$

This translates to the Hukuhara difference $U(\tau+h) \ominus U(\tau)$ in the first case. Applying the definition of the gH-difference and taking into account the r-cut of both sides yields the derivative in interval form.

Case 1. (i-Differentiability)

$${}^{i-gH}[U']_r = [\underline{U}'(t,r),\, \overline{U}'(t,r)], \quad r \in [0,1].$$

Case 2. (ii-Differentiability)

$${}^{ii-gH}[U']_r = [\overline{U}'(t,r),\, \underline{U}'(t,r)], \quad r \in [0,1].$$

To define the ABC derivative in the interval parametric form, the Lebesgue integral of $U(t)$ can be expressed as

$$\left[\int_0^t U'(\tau)\, d\tau\right]_r = \int_0^t [U'(\tau)]_r\, d\tau = \begin{cases} \left[\int_0^t \underline{U}'(\tau,r)\, d\tau,\, \int_0^t \overline{U}'(\tau,r)\, d\tau\right], & \text{for Case 1,} \\ \left[\int_0^t \overline{U}'(\tau,r)\, d\tau,\, \int_0^t \underline{U}'(\tau,r)\, d\tau\right], & \text{for Case 2.} \end{cases}$$

Component-wise fractional differentiation. For fuzzy-valued functions expressed in r-level or double-parametric form, fractional differentiation is performed component-wise. That is, the ABC fractional derivative of a fuzzy-valued function is obtained by applying the classical ABC operator separately to its lower and upper bounding functions, consistent with the generalized Hukuhara differentiability.

2.1. ABC-type fuzzy fractional derivatives in Double Parametric form.

In this section, we present our definition of Atangana–Baleanu-type fuzzy fractional derivatives based on the generalized Hukuhara difference. This definition parallels the ABC-type derivative in the non-fuzzy case and serves as a direct extension of strongly generalized H-differentiability into the fractional domain.

Definition 2.1.1. [26] Let $y \in C^{\mathbb{F}}(I) \cap L^{\mathbb{F}}(I)$ be a fuzzy-valued function. Then the Atangana–Baleanu derivative in the Caputo sense of y is given by:

$$^{ABC}D^{\alpha}y(t) = \frac{B(\alpha)}{1-\alpha}\int_0^t \left(g_H y'(s)\right) E_{\alpha}\left(-\frac{\alpha}{1-\alpha}(t-s)^{\alpha}\right) ds,$$

where $B(\alpha)$ is a normalized function defined by $B(\alpha) = 1-\alpha+\frac{\alpha}{\Gamma(\alpha)}$ and satisfies $B(0) = B(1) = 1$.

Definition 2.1.2. The fuzzy Laplace transform of the Atangana–Baleanu derivative in the Caputo sense is described as [27]:

$$\mathcal{L}\left\{\, ^{ABC}D_t^{\alpha}\tilde{f}(t)\right\}(s) = \frac{B(\alpha)}{1-\alpha}\,\frac{s^{\alpha}\mathcal{L}\{\tilde{f}(t)\}(s) - s^{\alpha-1}\tilde{f}(0)}{s^{\alpha}+\frac{\alpha}{1-\alpha}}.$$

Theorem 2.1. Let $y(t)$ be a fuzzy-valued function expressed in the double-parametric form as:

$$y(t,r,\eta) = \eta[\overline{y}(t,r) - \underline{y}(t,r)] + \underline{y}(t,r),$$

where $r \in [0,1]$ and $\eta \in [0,1]$. The ABC fractional derivative in the Caputo sense of $y(t)$ is given by:

- **Case 1: i-Differentiability**

$$^{ABC}D^{\alpha}y(t,r,\eta) = \frac{B(\alpha)}{1-\alpha}\int_0^t \left[\eta(\overline{y}'(s,r) - \underline{y}'(s,r)) + \underline{y}'(s,r)\right] E_{\alpha}\left(-\frac{\alpha}{1-\alpha}(t-s)^{\alpha}\right) ds$$

- **Case 2: ii-Differentiability**

$$^{ABC}D^{\alpha}y(t,r,\eta) = \frac{B(\alpha)}{1-\alpha}\int_0^t \left[\eta(\underline{y}'(s,r) - \overline{y}'(s,r)) + \overline{y}'(s,r)\right] E_{\alpha}\left(-\frac{\alpha}{1-\alpha}(t-s)^{\alpha}\right) ds$$

where $\underline{y}(t,r)$ and $\overline{y}(t,r)$ are the lower and upper bounds of the fuzzy function, respectively.

Proof. **Case 1: i-Differentiability.** **gH-Derivative:** The gH-derivative for i-differentiability is given by:

$$g_H^i y'(t,r,\eta) = \lim_{h\to 0^+} \frac{y(t+h,r,\eta) \ominus_{gH} y(t,r,\eta)}{h}.$$

Substituting the double-parametric form yields

$$g_H^i y'(t,r,\eta) = \eta(\overline{y}'(t,r) - \underline{y}'(t,r)) + \underline{y}'(t,r).$$

ABC Fractional Derivative: Hence,

$${}^{ABC}D^\alpha y(t,r,\eta) = \frac{B(\alpha)}{1-\alpha}\int_0^t \left[\eta(\overline{y}'(s,r) - \underline{y}'(s,r)) + \underline{y}'(s,r)\right] E_\alpha\left(-\frac{\alpha}{1-\alpha}(t-s)^\alpha\right) ds.$$

Effect of η:

- When $\eta = 0$: The derivative reduces to the lower bound's derivative.
- When $\eta = 1$: The derivative becomes the upper bound's derivative.
- For $0 < \eta < 1$: The derivative is a linear combination of the lower and upper bounds' derivatives.

Case 2: ii-Differentiability. **gH-Derivative:** The gH-derivative for ii-differentiability is given by:

$$g_H^{ii} y'(t,r,\eta) = \lim_{h\to 0^+} \frac{y(t,r,\eta) \ominus_{gH} y(t+h,r,\eta)}{h}.$$

Substituting the double-parametric form yields

$$g_H^{ii} y'(t,r,\eta) = \eta(\underline{y}'(t,r) - \overline{y}'(t,r)) + \overline{y}'(t,r).$$

ABC Fractional Derivative: Hence,

$${}^{ABC}D^\alpha y(t,r,\eta) = \frac{B(\alpha)}{1-\alpha}\int_0^t \left[\eta(\underline{y}'(s,r) - \overline{y}'(s,r)) + \overline{y}'(s,r)\right] E_\alpha\left(-\frac{\alpha}{1-\alpha}(t-s)^\alpha\right) ds.$$

Effect of η:

- When $\eta = 0$: The derivative reduces to the upper bound's derivative.
- When $\eta = 1$: The derivative becomes the lower bound's derivative.
- For $0 < \eta < 1$: The derivative is a linear combination, but the order of the contributions is reversed compared to i-differentiability.

So the proof is complete.

2.2. Double Parametric Form of Fuzzy Fractional Advection-Diffusion Equation

Using the above fuzzy fractional operators, we now formulate the fuzzy time-fractional advection–diffusion equation in double-parametric form.

In 2019, the study presented in [20] explored the fuzzy time fractional advection-diffusion equation (FTFADE) in the double-parametric form of a fuzzy number, characterized by:

$$\begin{aligned} \frac{\partial^\gamma \hat{u}(x,t,\eta)}{\partial t^\gamma} &= -\hat{v}(x)\frac{\partial \hat{u}(x,t,\eta)}{\partial x} + \hat{a}(x)\frac{\partial^2 \hat{u}(x,t,\eta)}{\partial x^2} + \hat{b}(x,t,\eta), \\ & 0 < x < L,\ t > 0,\ \eta \in [0,1], \\ \hat{u}(x,0,\eta) &= \hat{f}(x,r,\eta), \\ \hat{u}(0,t,\eta) &= \hat{g}, \\ \hat{u}(L,t,\eta) &= \hat{z}. \end{aligned} \tag{1}$$

We obtain the lower and upper limits of the solution (single parametric form) by setting $\eta = 0$ and $\eta = 1$, respectively:

$$\hat{u}(x,t;r,0) = \underline{u}(x,t;r), \quad \text{and} \quad \hat{u}(x,t;r,1) = \overline{u}(x,t;r).$$

According to this equation, the fuzzy time fractional derivative of order γ is represented by $\hat{u}(x,t,\eta)$, and the fuzzy concentration of a quantity (such as mass or energy) for the crisp variables t and x is described by $\frac{\partial^\gamma \hat{u}(x,t,\eta)}{\partial t^\gamma}$. The symbol γ represents the fractional order.

The average velocity of the quantity in question is $\hat{v}(x)$, and the diffusion coefficient (diffusivity) is $\hat{a}(x)$. Fuzzy functions linked to the crisp variable x are denoted by $\hat{b}(x,t,\eta)$. Furthermore, the fuzzy boundary data are $\hat{g}$ and $\hat{z}$.

Remark 2.1. In this work, the fuzzy time-fractional advection–diffusion equation is treated under the *i-differentiability* case of Theorem 2.1. This choice ensures a consistent and physically meaningful interpretation of the fuzzy solution and allows the lower and upper bounds to be obtained directly by setting $\eta = 0$ and $\eta = 1$, respectively. All subsequent theoretical analysis and numerical implementations are carried out under this differentiability assumption.

3. Overview of the Suggested Techniques

This section outlines the standard Laplace Transform Homotopy Perturbation Method (LT-HPM) and its modified variant (MLT-HPM) for solving fuzzy time-fractional advection–diffusion equations (FTFADEs) formulated using the Atangana–Baleanu–Caputo (ABC) fractional derivative. Throughout the analysis, fuzzy-valued functions are treated in the double-parametric sense under i-differentiability (Remark 2.1).

3.1. Standard Laplace Transform Homotopy Perturbation Method with the Atangana–Baleanu–Caputo fractional derivative

Consider the fuzzy time-fractional advection–diffusion equation

$$^{ABC}D^{\alpha}_{a,t}\tilde{u}(x,t,\beta) = -\tilde{v}(x)\frac{\partial \tilde{u}}{\partial x} + \tilde{a}(x)\frac{\partial^2 \tilde{u}}{\partial x^2} + \tilde{b}(x,t,\beta), \qquad 0 < x < l,\ t > 0, \tag{2}$$

subject to the initial condition

$$D^j \tilde{u}(x,0,\beta) = \tilde{h}(x), \qquad j = 0,1,\ldots,n-1, \tag{3}$$

where $^{ABC}D^{\alpha}_{a,t}(\cdot)$ denotes the ABC fractional derivative.

Following He's homotopy framework, the LT-HPM constructs the homotopy

$$(1-p)\left[^{ABC}D^{\alpha}_{a,t}\tilde{u} - {}^{ABC}D^{\alpha}_{a,t}\tilde{u}_0\right] + p\left[^{ABC}D^{\alpha}_{a,t}\tilde{u} - \mathsf{L}(\tilde{u}) - \mathsf{N}(\tilde{u})\right] = 0, \tag{4}$$

where $\tilde{u}_0$ is a fixed initial approximation determined directly from the initial data, L denotes the linear advection–diffusion operator, and N represents the remaining source terms.

Applying the Laplace transform with respect to t and using the ABC–Laplace identity yields an integral formulation. After applying the inverse Laplace transform, a recursive relation is obtained. Assuming the series expansion

$$\tilde{u}(x,t) = \sum_{n=0}^{\infty} p^n \tilde{u}_n(x,t), \tag{5}$$

and equating like powers of p, the sequence $\{\tilde{u}_n\}$ is generated iteratively. Taking $p \to 1$, the truncated LT-HPM approximation reads

$$\tilde{u}_{\text{LT-HPM}}(x,t,\beta) \approx \sum_{n=0}^{N} \tilde{u}_n(x,t,\beta).$$

3.2. Modified Laplace Transform Homotopy Perturbation Method

The Modified Laplace Transform Homotopy Perturbation Method (MLT-HPM) follows the same analytical procedure as LT-HPM, but replaces the fixed initial approximation $\tilde{u}_0$ with a trial function

$$z(x,t;\mathbf{C}), \qquad \mathbf{C} = (C_0, C_1, \ldots, C_m),$$

containing unknown auxiliary constants. This modification enhances convergence flexibility while preserving the underlying homotopy structure.

The modified homotopy is defined as

$$(1-p)\left[^{ABC}D^{\alpha}_{a,t}\tilde{u} - z(x,t;\mathbf{C})\right] + p\left[^{ABC}D^{\alpha}_{a,t}\tilde{u} - \mathsf{L}(\tilde{u}) - \mathsf{N}(\tilde{u})\right] = 0, \tag{6}$$

or equivalently,

$$^{ABC}D^{\alpha}_{a,t}\tilde{u} = z(x,t;\mathbf{C}) + p[-z(x,t;\mathbf{C}) - \mathsf{L}(\tilde{u}) - \mathsf{N}(\tilde{u})]\,. \tag{7}$$

Applying the Laplace transform and its inverse leads to a recursive scheme analogous to that of LT-HPM. Assuming

$$\tilde{u}(x,t) = \sum_{n=0}^{\infty} p^n \tilde{u}_n(x,t),$$

the truncated MLT-HPM approximation is given by

$$\tilde{u}_{\text{MLT-HPM}}(x,t,\beta) \approx \sum_{n=0}^{N} \tilde{u}_n(x,t,\beta).$$

Determination of auxiliary constants. The auxiliary constants $\mathbf{C}$ introduced through the trial function $z(x,t;\mathbf{C})$ can be determined using one of the following two strategies:

- **Residual error cancellation procedure (RECP).** The constants are obtained by enforcing cancellation of the governing equation residual at selected collocation points $\{(x_j,t_j)\} \subset [0,l] \times [0,T]$, i.e.,

$$\mathcal{R}(x_j,t_j;\mathbf{C}) = 0,$$

  where $\mathcal{R}$ denotes the PDE residual. This strategy has been widely used in semi-analytical methods (see, [28]).

- **Least-squares residual minimization.** Alternatively, the auxiliary constants can be determined through a least-squares minimization of the governing equation residual over a finite set of collocation points $\{(x_i,t_i)\}_{i=1}^{M} \subset [0,L] \times [0,T]$. After constructing the truncated MLT-HPM approximation $\tilde{u}_{\text{app}}(x,t,\beta;\mathbf{C})$, the associated PDE residual $\mathcal{R}(x,t,\beta;\mathbf{C})$ is evaluated and the optimal parameter vector $\mathbf{C}^*$ is obtained by minimizing the sum of squared residuals [29–31]. This optimization-based strategy enhances global accuracy and robustness, while preserving the original MLT-HPM iterative structure.

Algorithm 1 Unified Laplace Transform Homotopy Perturbation Method, Modified Laplace Transform Homotopy Perturbation Method (Residual Error Collocation), and Optimized Modified Laplace Transform Homotopy Perturbation Method (Least-Squares) for fuzzy time-fractional advection–diffusion equations with the Atangana–Baleanu–Caputo fractional derivative

Require: Domain $\Omega = [0, L] \times [0, T]$, fractional order $\alpha \in (0, 1]$; fuzzy coefficients $\tilde{v}(x)$, $\tilde{a}(x)$, $\tilde{b}(x, t, \beta)$; initial/boundary data in double-parametric form; truncation order N; method flag $\mathcal{M} \in$ {Laplace Transform Homotopy Perturbation Method, Modified Laplace Transform Homotopy Perturbation Method, Optimized Modified Laplace Transform Homotopy Perturbation Method}.

Ensure: Approximate fuzzy solution $\tilde{u}_{\mathcal{M}}(x, t, \beta) \approx \sum_{n=0}^{N} \tilde{u}_n(x, t, \beta)$.

Fix $(r, \beta) \in [0, 1]^2$ and evaluate the double-parametric representation.
if $\mathcal{M}$ = Laplace Transform Homotopy Perturbation Method **then**
  Set the initial approximation $u_0(x, t)$ from the prescribed initial condition.
else
  Select a trial function $z(x, t; \mathbf{C})$ with unknown auxiliary parameters $\mathbf{C}$.
end if
Construct the corresponding homotopy equation.
Apply the Laplace transform in t using the Atangana–Baleanu–Caputo fractional identity.
Assume the homotopy series

$$u(x, t) = \sum_{n=0}^{\infty} p^n u_n(x, t).$$

Apply the inverse Laplace transform and equate like powers of p to compute $u_0, u_1, \ldots, u_N$.
Set $p \to 1$ and obtain the truncated approximation

$$u_{\text{app}}(x, t) = \sum_{n=0}^{N} u_n(x, t).$$

if $\mathcal{M}$ = Modified Laplace Transform Homotopy Perturbation Method **then**
  Determine $\mathbf{C}$ by enforcing the initial/boundary conditions and cancelling the governing-equation residual at selected collocation points using the residual error collocation procedure.
end if
if $\mathcal{M}$ = Optimized Modified Laplace Transform Homotopy Perturbation Method **then**
  Determine $\mathbf{C}$ by minimizing the squared governing-equation residual over a finite set of collocation points in a least-squares sense.
end if
Reconstruct the fuzzy solution $\tilde{u}_{\mathcal{M}}(x, t, \beta)$ by sweeping $(r, \beta) \in [0, 1]^2$.

4. Theoretical Results for the Modified Laplace Transform Homotopy Perturbation Method in the Double-Parametric Fuzzy Atangana–Baleanu–Caputo Setting

Throughout this section, we analyze the convergence, uniqueness, and stability of the Modified Laplace Transform Homotopy Perturbation Method (MLT-HPM) applied to the fuzzy time-fractional advection–diffusion equation formulated in double-parametric form. All fuzzy-valued functions are assumed to satisfy the *i-differentiability* condition stated in Remark 2.1, and all fractional operators are interpreted in the Atangana–Baleanu–Caputo (ABC) sense. The analysis is carried out for each fixed $(r, \beta) \in [0, 1]^2$, and convergence is understood in the uniform (supremum) norm.

Definition (The space $\mathcal{C}_{DPF}[0, T]$). Let $T > 0$ be fixed. The fuzzy Banach space $\mathcal{C}_{DPF}[0, T]$ is defined as

$$\mathcal{C}_{DPF}[0, T] := \left\{ \begin{matrix} \tilde{u} : [0, T] \times (0, 1]^2 \to \mathbb{R} \\ \tilde{u}(t; r, \beta) \text{ is continuous in } t \text{ and fuzzy in } (r, \beta) \end{matrix} \right\},$$

equipped with the norm

$$\|\tilde{u}\| := \max_{t\in[0,T]} \max_{(r,\beta)\in(0,1]^2} d_H(\tilde{u}(t; r, \beta), 0),$$

where d_H denotes the Hausdorff distance between the interval-valued r-level sets of fuzzy numbers.

In what follows, the linear operator $\mathcal{L}$ represents the diffusion and advection components of the governing equation, while $\mathcal{N}$ denotes the remaining nonlinear and source terms. Both operators are assumed to act component-wise on the double-parametric representation.

Theorem 1 (Convergence). *Let $\mathcal{L}$ and $\mathcal{N}$ be Lipschitz continuous operators with constants $\ell_1, \ell_2 > 0$, respectively. Suppose that $\tilde{u}_0 \in \mathcal{C}_{DPF}[0, T]$, and that the ABC kernel*

$$K_\alpha(t) := E_\alpha\left(-\frac{\alpha}{1-\alpha} t^\alpha\right)$$

is bounded on $[0, T]$. Then, the sequence $\{\tilde{u}_n\}$ generated by the MLT-HPM converges in $\mathcal{C}_{DPF}[0, T]$ to a unique fuzzy solution $\tilde{u}(t; r, \beta)$, provided that

$$\gamma := B(\alpha)(\ell_1 + \ell_2) \max_{t\in[0,T]} \int_0^t K_\alpha(t - \tau)\, d\tau < 1.$$

Proof. Define the error $e_n := \tilde{u}_{n+1} - \tilde{u}_n$. Using the convolution form associated with the ABC derivative, we obtain

$$e_n(t; r, \beta) = B(\alpha) \int_0^t K_\alpha(t - \tau)\, [\mathcal{L}(e_{n-1}) + \mathcal{N}(e_{n-1})]\, d\tau.$$

Taking the fuzzy norm and applying the Lipschitz bounds yields

$$\|e_n\| \leq B(\alpha)(\ell_1 + \ell_2) \max_{t\in[0,T]} \int_0^t K_\alpha(t-\tau)\, d\tau \cdot \|e_{n-1}\| = \gamma \|e_{n-1}\|.$$

By recursion,

$$\|e_n\| \leq \gamma^n \|e_0\|.$$

Hence, for $n > m$,

$$\|\tilde{u}_n - \tilde{u}_m\| \leq \sum_{k=m}^{n-1} \|e_{k+1}\| \leq \frac{\gamma^m}{1-\gamma}\|e_0\|.$$

Since $\gamma < 1$, the sequence $\{\tilde{u}_n\}$ is Cauchy in $\mathcal{C}_{DPF}[0,T]$ and therefore convergent.

Theorem 2 (Uniqueness)**.** *Let $\tilde{u}, \tilde{v} \in \mathcal{C}_{DPF}[0,T]$ be two fuzzy solutions of the same problem generated by the MLT-HPM. If the convergence parameter $\gamma < 1$, then $\tilde{u} = \tilde{v}$.*

Proof. Define $\delta(t;r,\beta) = \tilde{u}(t;r,\beta) - \tilde{v}(t;r,\beta)$. Using the integral formulation of the ABC derivative, we have

$$\delta(t;r,\beta) = B(\alpha) \int_0^t K_\alpha(t-\tau) \left[\mathcal{L}(\delta) + \mathcal{N}(\tilde{u}) - \mathcal{N}(\tilde{v})\right] d\tau.$$

Taking norms and applying the Lipschitz condition gives

$$\|\delta\| \leq \gamma \|\delta\|.$$

Since $\gamma < 1$, it follows that $\|\delta\| = 0$, and hence $\tilde{u} = \tilde{v}$.

Theorem 3 (Stability)**.** *Let $\tilde{u}(t;r,\beta)$ and $\tilde{v}(t;r,\beta)$ be fuzzy solutions obtained by applying the MLT-HPM to the unperturbed and perturbed problems, respectively. Assume that perturbations occur in the initial condition $\tilde{u}_0$ and the trial function z, and that $\gamma < 1$. Then,*

$$\|\tilde{u} - \tilde{v}\| \leq \frac{C}{1-\gamma}\left(\|\delta\tilde{u}_0\| + \|\delta z\|\right),$$

where $\delta\tilde{u}_0 = \tilde{u}_0 - \tilde{v}_0$, $\delta z = z - z^*$, and $C > 0$ depends on the ABC kernel and operator bounds.

Proof. Define the iterative error

$$e_n(t;r,\beta) = \tilde{u}_n(t;r,\beta) - \tilde{v}_n(t;r,\beta).$$

From the MLT-HPM recurrence relations,

$$e_{n+1}(t) = \mathcal{L}^{-1}\big(\mathcal{K}\big(\delta z + \mathcal{L}(e_n) + \mathcal{N}(\tilde{u}_n) - \mathcal{N}(\tilde{v}_n)\big)\big).$$

Applying the norm and Lipschitz bounds yields

$$\|e_{n+1}\| \leq C_1 \left(\|\delta z\| + (\ell_1 + \ell_2)\|e_n\| \right).$$

Let $\gamma = C_1(\ell_1 + \ell_2) < 1$. Iterating,

$$\|e_n\| \leq \gamma^n \|\delta \tilde{u}_0\| + \frac{C_1}{1-\gamma}(1 - \gamma^n)\|\delta z\|.$$

Taking the limit as $n \to \infty$ completes the proof.

Remark 4.1. The above convergence, uniqueness, and stability results are derived for the MLT-HPM framework. When the auxiliary parameters are further optimized using least-squares (OMLT-HPM), the same theoretical bounds remain valid, since the optimization procedure minimizes the residual norm and does not alter the contraction properties of the underlying operator.

5. Numerical simulation and discussion

All numerical experiments were carried out using Wolfram *Mathematica* 10 and *Python*. We validate the proposed LT-HPM, MLT-HPM (RECP), and OMLT-HPM (least-squares) schemes on a benchmark fuzzy time-fractional advection–diffusion equation (FTFADE) formulated with the ABC fractional derivative.

5.1. Test problem and fuzzy setting

Consider

$$^{ABC}D_t^{\alpha}\tilde{u}(x,t,\beta) = -\frac{\partial \tilde{u}(x,t,\beta)}{\partial x} + \frac{\partial^2 \tilde{u}(x,t,\beta)}{\partial x^2}, \qquad 0 < x < l,\ t > 0,\ 0 < \alpha \leq 1, \tag{8}$$

subject to the boundary conditions

$$\tilde{u}(0,t,\beta) = \tilde{u}(1,t,\beta) = 0, \tag{9}$$

and the fuzzy initial condition

$$\tilde{u}(x,0,\beta) = \tilde{k}\, e^{-x}, \qquad 0 < x < 1. \tag{10}$$

Fuzzy parameters and solution. In this example, uncertainty is introduced through the fuzzy number $\tilde{k}$ only, while the advection and diffusion coefficients are crisp. Consequently, the solution $\tilde{u}(x,t,\beta)$ is fuzzy-valued.

Double-parametric representation. We adopt the double-parametric form. For each fixed $(r,\beta) \in [0,1]^2$, the fuzzy number $\tilde{k}$ is represented by $\tilde{k}(r,\beta)$ as

$$\tilde{k}(r,\beta) = \beta(0.2 - 0.2r) + 0.1r - 0.1, \qquad (r,\beta) \in [0,1]^2. \tag{11}$$

Under i-differentiability (Remark 2.1), the ABC derivative acts component-wise on the parametric representation; hence, for each fixed (r, β), Eq. (8) becomes a crisp fractional PDE, and the full fuzzy solution is recovered by sweeping (r, β) over $[0, 1]^2$.

Exact solution. The exact solution of Eq. (8)–(10) is given in [32] by

$$\tilde{u}_{\text{exact}}(x, t, \alpha; r, \beta) = \sum_{n=0}^{\infty} \frac{2^n t^{n\alpha}}{\Gamma(n\alpha + 1)} \tilde{k}(r, \beta) e^{-x}. \tag{12}$$

Error measures and reporting metrics

For each fixed (r, β), the pointwise absolute error is

$$[\tilde{Er}]_r = |\tilde{U}(t, x; r) - \tilde{u}(t, x; r)| = \begin{cases} [\underline{Er}]_r = |\underline{U}(t, x; r) - \underline{u}(t, x; r)| \\ [\overline{Er}]_r = |\overline{U}(t, x; r) - \overline{u}(t, x; r)| \end{cases}$$

5.2. Results obtained by the Laplace Transform Homotopy Perturbation Method

Applying LT-HPM as described in Subsection 3.1 to Eq. (8), and using the ABC–Laplace identity, we obtain the truncated LT-HPM approximation used in the computations:

$$\tilde{U}_{\text{LT-HPM}}(x, t) = \tilde{k} e^{-x} \frac{B(\alpha)^2 - 2B(\alpha)(-1 + \alpha) + 4(-1 + \alpha)^2 + 2t^{\alpha} \left(\frac{4 + B(\alpha) - 4\alpha}{\Gamma(\alpha)} + \frac{t^{\alpha}\alpha}{\Gamma(2\alpha)} \right)}{B(\alpha)^2}. \tag{13}$$

Table 1 reports the exact and LT-HPM approximations at $t = 0.005$ and $\alpha = 0.5$ for different (r, β) values, together with the absolute error. Figure 1 depicts the exact profile and the corresponding LT-HPM approximation.

Table 1: Exact and numerical approximation with LT-HPM of FTFADE at $t = 0.005$, $x = 5.4$, and $\alpha = 0.5$. Here β denotes the double-parametric fuzzy weighting parameter and r denotes the certainty level.

Beta	r-level	Exact	LT-HPM	Abs. Error
0	0	-5.3382512E-04	-1.935156E-03	1.40E-03
	0.2	-4.2706009E-04	-1.548125E-03	1.12E-03
	0.4	-3.2029507E-04	-1.161094E-03	8.41E-04
	0.6	-2.1353005E-04	-7.740624E-04	5.61E-04
	0.8	-1.0676502E-04	-3.870312E-04	2.80E-04
	1	0	0	0
0.2	0	-3.2029507E-04	-1.161094E-03	8.41E-04
	0.2	-2.5623606E-04	-9.288748E-04	6.73E-04
	0.4	-1.9217704E-04	-6.966561E-04	5.04E-04
	0.6	-1.2811803E-04	-4.644374E-04	3.36E-04
	0.8	-6.4059014E-05	-2.322187E-04	1.68E-04
	1	0	0	0
0.8	0	3.2029507E-04	1.161094E-03	8.41E-04
	0.2	2.5623606E-04	9.288748E-04	6.73E-04
	0.4	1.9217704E-04	6.966561E-04	5.04E-04
	0.6	1.2811803E-04	4.644374E-04	3.36E-04
	0.8	6.4059014E-05	2.322187E-04	1.68E-04
	1	0	0	0
1	0	5.3382512E-04	1.935156E-03	1.40E-03
	0.2	4.2706009E-04	1.548125E-03	1.12E-03
	0.4	3.2029507E-04	1.161094E-03	8.41E-04
	0.6	2.1353005E-04	7.740624E-04	5.61E-04
	0.8	1.0676502E-04	3.870312E-04	2.80E-04
	1	0	0	0

5.3. Modified Laplace Transform Homotopy Perturbation Method results based on the Residual Error Cancellation Procedure

We next apply the modified LT-HPM (MLT-HPM) procedure described in Subsection 3.2 to the FTFADE model. In MLT-HPM, the initial approximation is enriched by an auxiliary trial function with free coefficients,

$$z(x,t;\mathbf{C}) = C_0 + C_1 x + C_2 t + C_3 xt, \qquad \mathbf{C} = (C_0, C_1, C_2, C_3), \tag{14}$$

which enables additional flexibility beyond the classical LT-HPM initial guess. Since the FTFADE considered here is linear (i.e., it contains no nonlinear source term), the resulting truncated series form is identical in structure for LT-HPM and MLT-HPM; nevertheless, the numerical accuracy depends strongly on the chosen coefficients $\mathbf{C}$.

Residual formulation. Let $\tilde{u}_{\text{app}}(x,t,\beta;\mathbf{C})$ denote the truncated MLT-HPM approximation. The governing-equation residual is defined by

$$\mathcal{R}(x,t,\beta;\mathbf{C}) := {}^{ABC}D_t^{\alpha}\tilde{u}_{\text{app}} + \frac{\partial \tilde{u}_{\text{app}}}{\partial x} - \frac{\partial^2 \tilde{u}_{\text{app}}}{\partial x^2}, \tag{15}$$

where ${}^{ABC}D_t^{\alpha}(\cdot)$ denotes the Atangana–Baleanu–Caputo fractional derivative with respect to t applied in the fuzzy setting (via the r-level representation; see Section **??**).

RECP identification of C. In the residual error control procedure (RECP), the coefficients $\mathbf{C}$ are computed by enforcing the boundary conditions (9) together with a set of collocation constraints $\mathcal{R}(x_j,t_j,\beta;\mathbf{C}) = 0$ at selected points $\{(x_j,t_j)\}_{j=1}^{M} \subset [0,1]\times[0,T]$ (see [28]). In the present computations, the collocation points are chosen inside the computational domain and the resulting algebraic system is solved for $\mathbf{C}$ for each prescribed α. The obtained values of C_0, C_1, C_2, and C_3 are summarized in Table 2.

Table 2: Values of constants C_0, C_1, C_2, and C_3 obtained by RECP for different values of α.

C's value	$\alpha = 0.2$	$\alpha = 0.5$	$\alpha = 1$
C_0	0.92303015	0.86156222	1.04520297
C_1	-0.16946674	-0.1573304	-0.1921014
C_2	-0.2056521	-0.21519559	0.37365709
C_3	0.38905959	0.34290907	0.29897898

MLT-HPM approximation. With the coefficients $\mathbf{C}$ determined by RECP, the MLT-HPM approximation used in the numerical evaluation is

$$\begin{aligned}\tilde{U}(x,t,\alpha,c_0,c_1,c_2,c_3) = {} & \frac{1}{B(\alpha)^2}e^{-x}\tilde{k}\,\big[B(\alpha)^2 - 2B(\alpha)(-1+\alpha) \\ & + e^{x}B(\alpha)(c_0 + c_2 t + (c_1 + c_3 t)x)(-1+\alpha) + 4(-1+\alpha)^2 \\ & + \frac{2t^{2\alpha}\alpha}{\Gamma(2\alpha)} + \frac{2t^{\alpha}(4 + B(\alpha) - 4\alpha)\alpha(1+\alpha)}{\Gamma(2+\alpha)} \\ & - \frac{e^{x}t^{\alpha}B(\alpha)\alpha(c_0 + c_2 t + c_1 x + c_3 t x + c_0\alpha + c_1 x\alpha)}{\Gamma(2+\alpha)}\Bigg]\end{aligned} \tag{16}$$

For ease of interpretation, we also report the specialized expressions obtained from (16) by substituting the RECP coefficients in Table 2. In particular, at $\alpha = 1$,

$$\tilde{U}(x,t) = e^{-x}\tilde{k}\,(1 + 2t + 2t^2) - \frac{1}{2}\tilde{k}t\,(2.09041 + 0.373657t - 0.384203x + 0.298979tx)\,,$$

at $\alpha = 0.5$,

$$\tilde{U}(x,t) = e^{-x}\tilde{k}\,(1 + 2t + 2t^2) - \frac{1}{2}\tilde{k}t\,(1.72312 - 0.215196t - 0.314661x + 0.342909tx)\,,$$

and at $\alpha = 0.2$,

$$\tilde{U}(x,t) = e^{-x}\tilde{k}\,(1 + 2t + 2t^2) - \frac{1}{2}\tilde{k}t\,(1.84606 - 0.205652t - 0.338933x + 0.38906tx)\,.$$

Numerical evaluation. Table 3 reports the exact and MLT-HPM approximations at $t = 0.005$, $x = 5.4$, and $\alpha = 0.5$ for representative values of the fuzzy parameters (r, β). The corresponding solution profiles are illustrated in Figure 2, while the absolute-error behavior is compared against the alternative schemes in Figure 5.

Table 3 lists the exact and MLT-HPM results at $t = 0.005$, $x = 5.4$, and $\alpha = 0.5$, and Figure 2 illustrates the numerical profiles.

Table 3: Exact and numerical approximation with MLT-HPM of FTFADE at $t = 0.005$, $x = 5.4$, and $\alpha = 0.5$.

Beta	r-level	Exact	MLT-HPM	Abs. Error
	0	-5.3382512E-04	-5.5734513E-04	2.35E-05
	0.2	-4.2706009E-04	-4.4587610E-04	1.88E-05
	0.4	-3.2029507E-04	-3.3440708E-04	1.41E-05
0	0.6	-2.1353005E-04	-2.2293805E-04	9.41E-06
	0.8	-1.0676502E-04	-1.1146903E-04	4.70E-06
	1	0	0	0
	0	-3.2029507E-04	-3.3440708E-04	1.41E-05
	0.2	-2.5623606E-04	-2.6752566E-04	1.13E-05
	0.4	-1.9217704E-04	-2.0064425E-04	8.47E-06
0.2	0.6	-1.2811803E-04	-1.3376283E-04	5.64E-06
	0.8	-6.4059014E-05	-6.6881415E-05	2.82E-06
	1	0	0	0
	0	3.2029507E-04	3.3440708E-04	1.41E-05
	0.2	2.5623606E-04	2.6752566E-04	1.13E-05
	0.4	1.9217704E-04	2.0064425E-04	8.47E-06
0.8	0.6	1.2811803E-04	1.3376283E-04	5.64E-06
	0.8	6.4059014E-05	6.6881415E-05	2.82E-06
	1	0	0	0
	0	5.3382512E-04	5.5734513E-04	2.35E-05
	0.2	4.2706009E-04	4.4587610E-04	1.88E-05
	0.4	3.2029507E-04	3.3440708E-04	1.41E-05
1	0.6	2.1353005E-04	2.2293805E-04	9.41E-06
	0.8	1.0676502E-04	1.1146903E-04	4.70E-06
	1	0	0	0

5.4. Results obtained by the Optimized Modified Laplace Transform Homotopy Perturbation Method using the Least-Squares Residual Minimization Procedure

While the RECP strategy provides an effective means for determining the auxiliary constants $\mathbf{C} = (C_0, C_1, C_2, C_3)$, it generally leads to a nonlinear algebraic system whose solution may depend on the choice of collocation points and initial guesses. To enhance robustness and obtain a globally optimized approximation, we employ a least-squares residual minimization strategy, referred to here as the Optimized MLT-HPM (OMLT-HPM).

After constructing the truncated MLT-HPM approximation $\tilde{u}_{\text{app}}(x, t, \beta; \mathbf{C})$, we define the associated *fuzzy* PDE residual in the double-parametric sense as $\tilde{\mathcal{R}}(x, t, \beta; r; \mathbf{C})$, whose lower and upper r-level functions are denoted by $\underline{\mathcal{R}}$ and $\overline{\mathcal{R}}$, respectively. The auxiliary constants are then determined by minimizing the squared fuzzy residual over a finite set of collocation points $\{(x_i, t_i)\}_{i=1}^{M} \subset [0, 1] \times [0, T]$:

$$\min_{\mathbf{C}} J(\mathbf{C}) := \sum_{i=1}^{M} \Big(\big|\underline{\mathcal{R}}(x_i, t_i, \beta; r; \mathbf{C})\big|^2 + \big|\overline{\mathcal{R}}(x_i, t_i, \beta; r; \mathbf{C})\big|^2 \Big). \tag{17}$$

This formulation ensures that both the lower and upper bounds of the fuzzy solution satisfy the governing equation in a least-squares sense. The optimization process is terminated when the residual norm satisfies $\|\widehat{\mathcal{R}}\|_2 < 10^{-6}$ or when a prescribed maximum number of iterations (500 in the present simulations) is reached.

Using this strategy for $\alpha = 0.5$, the optimized coefficients obtained in the reported run are

$$C_0 = 0.07934992, \qquad C_1 = -0.01151047, \qquad C_2 = 0.09989929, \qquad C_3 = 0.09945618.$$

Table 4 reports the exact and OMLT-HPM approximations at $t = 0.005$, $x = 5.4$, and $\alpha = 0.5$, and Figure 3 shows the corresponding numerical profiles.

Table 4: Exact and numerical approximation with OMLT-HPM of FTFADE using the least-squares optimization at $t = 0.005$, $x = 5.4$, and $\alpha = 0.5$.

Beta	**r-level**	**Exact**	**OMLT-HPM**	**Abs. Error**
0	0	-5.3382512E-04	-5.33987537E-04	1.62E-07
	0.2	-4.2706009E-04	-4.27190030E-04	1.30E-07
	0.4	-3.2029507E-04	-3.20392522E-04	9.75E-08
	0.6	-2.1353005E-04	-2.13595015E-04	6.50E-08
	0.8	-1.0676502E-04	-1.06797507E-04	3.25E-08
	1	0	0	0
0.2	0	-3.2029507E-04	-3.20392522E-04	9.75E-08
	0.2	-2.5623606E-04	-2.56314018E-04	7.80E-08
	0.4	-1.9217704E-04	-1.92235513E-04	5.85E-08
	0.6	-1.2811803E-04	-1.28157009E-04	3.90E-08
	0.8	-6.4059014E-05	-6.40785045E-05	1.95E-08
	1	0	0	0
0.8	0	3.2029507E-04	3.20392522E-04	9.75E-08
	0.2	2.5623606E-04	2.56314018E-04	7.80E-08
	0.4	1.9217704E-04	1.92235513E-04	5.85E-08
	0.6	1.2811803E-04	1.28157009E-04	3.90E-08
	0.8	6.4059014E-05	6.40785045E-05	1.95E-08
	1	0	0	0
1	0	5.3382512E-04	5.33987537E-04	1.62E-07
	0.2	4.2706009E-04	4.27190030E-04	1.30E-07
	0.4	3.2029507E-04	3.20392522E-04	9.75E-08
	0.6	2.1353005E-04	2.13595015E-04	6.50E-08
	0.8	1.0676502E-04	1.06797507E-04	3.25E-08
	1	0	0	0

5.5. Graphical validation and comparative discussion

Figures 1–5 provide a graphical comparison between the exact solution and the numerical approximations obtained by LT-HPM, MLT-HPM, and OMLT-HPM, and they also display the absolute error behaviour. In particular, Tables 1–4 show that introducing the trial function reduces the truncation error compared to the standard LT-HPM, while the least-squares identification yields the smallest absolute errors at the tested points. This behaviour is consistent with the interpretation that least-squares minimization reduces the PDE residual globally over the collocation set rather than enforcing cancellation at a limited number of RECP points. In addition to the reported absolute errors, the relative error of the proposed methods follows the same qualitative decay trend across the computational domain and remains small wherever the exact solution is nonzero, further supporting the accuracy and stability of the optimized MLT-HPM framework.

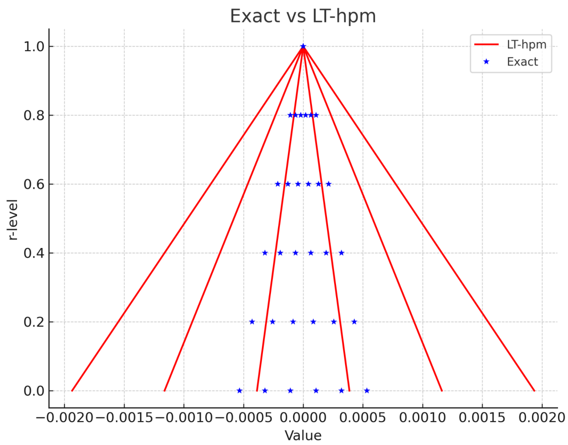

Figure 1: Exact and numerical approximation with LT-HPM at $t = 0.005$, $x = 5.4$, and $\alpha = 0.5$.

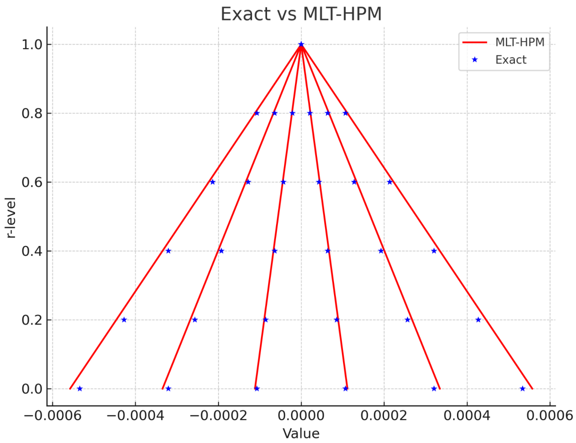

Figure 2: Exact and numerical approximation with MLT-HPM at $t = 0.005$, $x = 5.4$, and $\alpha = 0.5$.

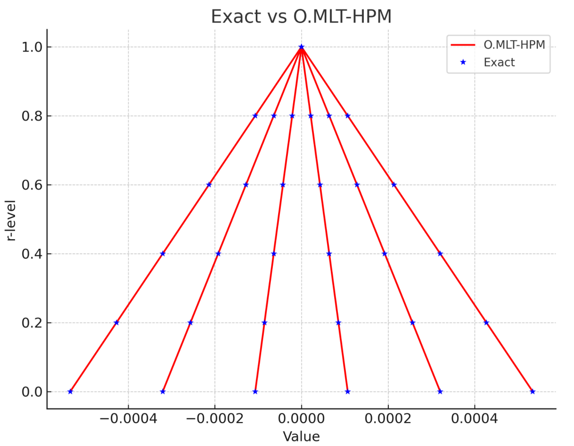

Figure 3: Exact and numerical approximation with OMLT-HPM at $t = 0.005$, $x = 5.4$, and $\alpha = 0.5$.

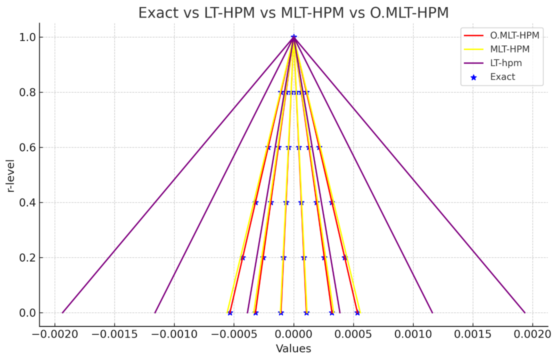

Figure 4: Exact vs. LT-HPM vs. MLT-HPM vs. OMLT-HPM at $t = 0.005$, $x = 5.4$, and $\alpha = 0.5$.

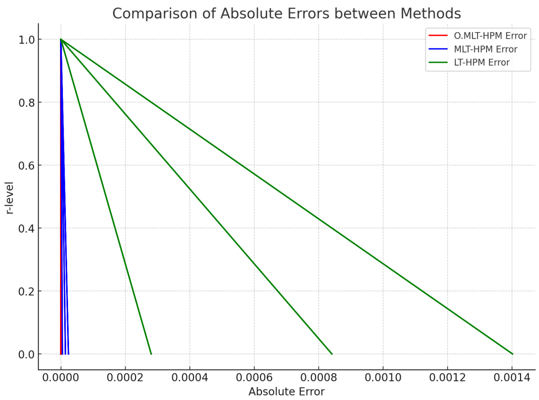

Figure 5: Comparison of absolute errors between methods.

5.6. Comparison with the Backward Time Central Space scheme and the Crank–Nicolson method

Table 5 compares the absolute errors of the proposed methods with BTCS and Crank–Nicolson values reported in [20] at the same evaluation point and for the same sampled (r, β) values. Since BTCS and Crank–Nicolson are classical schemes primarily designed for deterministic integer-order PDEs, the reported values are interpreted here as baseline benchmarks evaluated at fixed (r, β)-levels (crisp instances of the fuzzy parameters). A fully fuzzy–fractional discretization of BTCS/Crank–Nicolson is outside the scope of the present work; nevertheless, the comparison provides a useful reference on the same sampling configuration.

Table 5: Comparison of Absolute Errors between OMLT-HPM,MLT-HPM, Crank Nicholson, and BTCS methods of FTFADE at t =0.005, x=5.4, and $\alpha = 0.5$ in a Random Sample

Beta	r	E. OMLT-HPM	E. MLT-HPM	E. BTCS	E. Crank Nicholson
0	0	1.624190E-07	2.352001E-05	3.180245E-05	2.824154E-05
	0.2	1.299352E-07	1.881601E-05	2.544196E-05	2.259323E-05
	0.4	9.745141E-08	1.411201E-05	1.908147E-05	1.649493E-05
	0.8	3.248380E-08	4.704002E-06	6.360491E-06	5.648309E-06
0.4	0	3.248380E-08	4.704002E-06	6.360491E-05	5.648309E-05
	0.2	2.598704E-08	3.763201E-06	5.038583E-05	4.518647E-05
	0.4	1.949028E-08	2.822401E-06	3.816295E-05	3.388985E-05
	0.8	6.496761E-09	9.408003E-07	1.272098E-05	1.129662E-05
1	0	1.624190E-07	2.352001E-05	3.180245E-05	2.824154E-05
	0.2	1.299352E-07	1.881601E-05	2.544196E-05	2.259323E-05
	0.4	9.745141E-08	1.411201E-05	1.908147E-05	1.649493E-05
	0.8	3.248380E-08	4.704002E-06	6.360491E-06	5.648309E-06

5.7. Computational cost and reproducibility notes

The computational cost of LT-HPM is dominated by the construction of N homotopy terms and their evaluation at M collocation points, resulting in an overall complexity of $\mathcal{O}(NM)$. In the MLT-HPM (RECP) framework, an additional overhead arises from solving a small nonlinear algebraic system for the auxiliary constants $\mathbf{C}$ (four parameters in the present study), which remains negligible compared to the cost of series construction.

For OMLT-HPM, the least-squares identification introduces an optimization loop. This yields an additional cost of approximately $\mathcal{O}(I_{\max}M)$ residual evaluations, where $I_{\max}$ denotes the maximum number of optimization iterations. In practice, however, the improved accuracy and stability obtained through least-squares residual minimization significantly reduce the number of homotopy terms required to achieve a prescribed error tolerance. As a result, the total computational effort remains comparable to that of LT-HPM and RECP-based MLT-HPM.

Although OMLT-HPM incurs a modest optimization overhead, this cost is confined to the determination of a small number of auxiliary constants and does not affect the subsequent construction of the solution series. Consequently, the performance of the proposed method is more appropriately assessed in terms of convergence behavior and error reduction rather than hardware-dependent runtime measurements.

6. Conclusion

This study developed an optimization–enhanced Laplace transform homotopy perturbation framework for solving fuzzy time–fractional advection–diffusion equations with memory effects and parametric uncertainty. The governing model was formulated in a double–parametric fuzzy setting, allowing uncertainty in system parameters to be consistently propagated through the fractional dynamics. The Atangana–Baleanu–Caputo derivative was extended to fuzzy functions via the generalized Hukuhara difference, providing a mathematically consistent foundation for the proposed approach.

To address the slow convergence and sensitivity to initial approximations observed in the classical method, two parameter identification strategies were introduced. The residual error cancellation procedure determines auxiliary constants by enforcing fuzzy residual cancellation at selected points, while the optimized variant employs a least–squares minimization of the fuzzy residual to obtain globally calibrated parameters. Both strategies preserve the original Laplace–homotopy structure and provide systematic and reproducible mechanisms for improving accuracy.

Theoretical analysis established existence, uniqueness, stability, and convergence of the proposed methods in the double–parametric fuzzy framework, ensuring robustness with respect to uncertainty in model parameters and initial data. Numerical experiments on a benchmark fuzzy fractional advection–diffusion problem demonstrated that the modified methods significantly outperform the standard approach, with the optimized method achieving substantial accuracy improvements using only a small number of perturbation terms. Comparisons with classical finite difference schemes further confirmed the effectiveness of the proposed framework.

Although the optimization stage increases the per–iteration computational cost, the rapid convergence of the modified methods leads to a reduced overall computational effort for a prescribed accuracy. The resulting framework combines analytical transparency with numerical efficiency, making it well suited for applied and engineering problems involving fractional dynamics under uncertainty.

Future work will focus on extending the method to nonlinear and multidimensional fuzzy fractional systems, as well as data–driven and inverse problems arising in engineering and applied sciences.

Author contributions

This work was carried out through a coordinated effort among the authors. All authors contributed to the planning of the study and to the completion of the investigation. All authors read and approved the final version of the manuscript.

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Declarations

The authors declare that they have no affiliations with or involvement in any organization or entity with any financial or non-financial interest in the subject matter or materials discussed in this manuscript. During the preparation of this work, the authors used GPT-4 to assist with paraphrasing and improving the clarity and academic tone of the manuscript. After using this tool, the authors reviewed and edited the content as needed and take full responsibility for the content of the published article.

Code availability. The numerical algorithms presented in this study were implemented using standard scientific computing routines. The core implementation of the MLT-HPM and OMLT-HPM frameworks is available from the corresponding author upon reasonable request.

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