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Concomitant Extropy of Lai and Xie's Extensions under Generalized Order Statistics: Properties, Estimation, and Application to Saudi Arabia Industrial Data

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Abstract. In this work, we introduce and study the notion of concomitant extropy within the framework of generalized order statistics, extending the earlier contributions of Lai and Xie. The construction is supported with illustrative examples drawn from widely used probability distributions, highlighting the flexibility and applicability of the proposed measure. Several recurrence relations and important special cases are derived, providing further insights into the structure of the model. In addition, we explore the behavior of past and residual extropies associated with the model, and extend the analysis to include negative cumulative extropy as well as cumulative residual extropy. To complement the theoretical findings, a non-parametric estimation procedure is developed and applied to real-world Saudi Arabia Industrial data, demonstrating the practical utility and relevance of the proposed approach. The results indicate that concomitant extropy can serve as a valuable tool for modeling and analyzing uncertainty in diverse applied settings.

2020 Mathematics Subject Classifications: 94A17, 62E99, 62P30

Key Words and Phrases: Concomitants, extropy, generalized order statistics, non-parametric estimation, Saudi Arabia industrial data

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1. Introduction

In industrial data analysis, statistical entropy is essential for quantifying the degree of uncertainty, randomness, or disorder present within complex datasets generated by modern industrial systems. As industries increasingly rely on sensors, automation, and real time monitoring, the resulting data streams are often vast, multidimensional, and noisy. Traditional statistical measures such as variance or mean cannot fully capture the hidden patterns or unpredictability in such data. Statistical entropy provides a powerful metric to assess information content, detect anomalies, and evaluate system efficiency or stability. For instance, in manufacturing processes, higher entropy may indicate process instability or equipment malfunction, while lower entropy could reflect consistent and controlled operations. Thus, incorporating entropy based measures enables industries to enhance predictive maintenance, optimize resource allocation, and improve overall decision making in data driven industrial environments.

Shannon [1] was the first to introduce information-theoretic entropy, which has since been applied in a wide range of fields, including computer science, medical research, and financial analysis. Lad et al. [2] showed that "extropy" serves as a complementary dual functional to entropy. For a non-negative continuous random variable (r.v.) with probability density function (PDF) $g(y)$, the extropy is defined as:

$$\varepsilon(Y) = \frac{-1}{2} \int_0^\infty g^2(y)dy. \tag{1.1}$$

Extropy was explored for order statistics (OS), record values, and some features by Qiu [3]. Through the use of extropy, Qiu et al. [4] introduced a mixed systems lifetime and derived its properties and limit of it. For additional research on extropy, see Yang et al. [5], Noughabi and Jarrahiferiz [6], Raqab and Qiu [7], Lad et al. [8], and Qiu and Jia [9]. Qiu and Jia [10] also looked into the meaning of residual extropy for a non-negative r.v. as

$$\varepsilon^t(Y) = \varepsilon^R(Y;t) = \frac{-1}{2\bar{G}^2(t)} \int_t^\infty g^2(y)dy, t \geq 0, \tag{1.2}$$

where $\bar{G}(t) = 1 - G(t)$, $G(t)$ is the cumulative distribution function (CDF). For the past lifetime of r.v. $Y_t = [t - Y|Y \leq t]$, Krishnan et al. [11] provided the past extropy as follows

$$\varepsilon_t(Y) = \varepsilon_P(Y;t) = \frac{-1}{2G^2(t)} \int_0^t g^2(y)dy. \tag{1.3}$$

From any continuous distribution, Jose and Sathar [[12], [13]] used the k-records' residual and past extropies. Cumulative residual extropy (CREX) was proposed by Jahanshahi et al. [14]. The CREX is provided by a non-negative r.v. Y possessing a survival function $\bar{G}$ as follows

$$\xi(Y) = \frac{-1}{2}\int_0^{\infty} \bar{G}^2(y)dy. \tag{1.4}$$

Jahanshahi et al. [14] showed that $\zeta(Y)$ is always negative. Tahmasebi and Toomaj [15] recently proposed negative cumulative extropy (NCEX), which is comparable to (1.1) and is described as

$$\xi^*(Y) = \frac{1}{2}\int_0^{\infty} (1 - G^2(y))dy. \tag{1.5}$$

The Farlie-Gumbel-Morgenstern family (FGM) is described by a parameter δ, and the marginal distribution functions $G_X(x)$ and $G_Y(y)$, it was primarily derived by Morgenstern [16]. By adding further parameters, Lai and Xie [17] suggested the CDF as

$$G(x, y) = G_X(x)G_Y(y) + \delta\bar{G}_X(x)^{\alpha}\bar{G}_Y(y)^{\alpha}G_X(x)^{\lambda}G_Y(y)^{\lambda}, \alpha, \lambda \geq 1, \tag{1.6}$$

for 0$\leq \delta \leq 1$. The related PDF is

$$\begin{aligned} g(x, y) = g_X(x)g_Y(y)(1 + \delta\bar{G}_X(x)^{\alpha-1}\bar{G}_Y(y)^{\alpha-1}G_X(x)^{\lambda-1}G_Y(y)^{\lambda-1} \\ [\lambda - (\alpha + \lambda)G_X(x)][\lambda - (\alpha + \lambda)G_Y(y)]). \end{aligned} \tag{1.7}$$

According to Bairamov and Kotz [18], a bivariate copula for δ satisfies a broader range of conditions where:

$$\min\{\frac{1}{[K^+(\alpha, \lambda)]^2}, \frac{1}{[K^-(\alpha, \lambda)]^2}\} \leq \delta \leq \frac{1}{K^+(\alpha, \lambda)K^-(\alpha, \lambda)},$$

with noting that K^+ and K^- are functions of α and λ. Furthermore, the conditional CDF and PDF are:

$$G_{Y|X}(y \mid x) = G_Y(y) + \delta\bar{G}_X(x)^{\alpha}\bar{G}_Y(y)^{\alpha}G_X(x)^{\lambda-1}G_Y(y)^{\lambda}, \alpha, \lambda \geq 1, \tag{1.8}$$

$$\begin{aligned} g_{Y|X}(y \mid x) = g_Y(y)(1 + \delta\bar{G}_X(x)^{\alpha-1}\bar{G}_Y(y)^{\alpha-1}G_X(x)^{\lambda-1}G_Y(y)^{\lambda-1} \\ \times [\lambda - (\alpha + \lambda)G_X(x)][\lambda - (\alpha + \lambda)G_Y(y)]). \end{aligned} \tag{1.9}$$

Kamps [19] introduced the concept of generalized order statistics, with the special case of w-generalized order statistics (w-GOS), which provides a flexible framework for obtaining other ordered variables by choosing specific values of w. Gamal et al. [20] derived the

CDF and PDF of the concomitant $Y_{[r,n,w,l]}$ from the Lai and Xie extensions for the r-th w-GOS, respectively, as:

$$G_{[r,n,w,l]}(y) = G_Y(y)\left[1 + \delta\eta^*_{[r,n,w,l]}\bar{G}_Y(y)^{\alpha}G_Y^{\lambda-1}(y)\right], \tag{1.10}$$

$$g_{[r,n,w,l]}(y) = g_Y(y)\left[1 + \delta R^*_{[r,n,w,l]}\bar{G}_Y(y)^{\alpha-1}G_Y^{\lambda-1}(y)\left(\lambda - (\alpha+\lambda)G_Y(y)\right)\right], \tag{1.11}$$

and

$$\begin{aligned}\bar{G}_{[r,i,w,l]}(y) &= 1 - G_Y(y)\left[1 + \delta\eta^*_{[r,i,w,l]}\bar{G}_Y(y)^{\alpha}G_Y^{\lambda-1}(y)\right] \\ &= \bar{G}_Y(y)\left[1 + \delta\eta^*_{[r,i,w,l]}\bar{G}_Y(y)^{\alpha-1}G_Y^{\lambda}(y)\right].\end{aligned} \tag{1.12}$$

where

$$\eta^*_{[r,n,w,l]} = m_{r-1}\sum_{\epsilon=0}^{\lambda-1}\frac{\binom{\lambda-1}{\epsilon}(-1)^{\epsilon}}{\prod_{q=1}^{r}(\gamma_q+\alpha+\epsilon)}, \tag{1.13}$$

and

$$R^*_{[r,n,w,l]} = m_{r-1}[\sum_{\epsilon=0}^{\lambda-1}\frac{\lambda\binom{\lambda-1}{\epsilon}(-1)^{\epsilon}}{\prod_{q=1}^{r}(\gamma_q+\epsilon+\alpha-1)} - \sum_{j=0}^{\lambda}\frac{(\alpha+\lambda)\binom{\lambda}{j}(-1)^{j}}{\prod_{q=1}^{r}(\gamma_q+j+\alpha-1)}], \tag{1.14}$$

with parameters $l \geq 1$, $n \in \mathbb{N}$, $w_1 = w_2 = \ldots = w_{n-1} = w$, and $1 \leq r \leq n-1$, where $\gamma_q = l + (n-q)(w+1)$ and $m_{r-1} = \prod_{q=1}^{r}\gamma_q$. For more details on the concept of w-GOS, see Kamps [19].

Extropy is regarded as the dual counterpart of entropy in both information theory and thermodynamics. Whereas entropy quantifies disorder, randomness, or uncertainty within a system, extropy emphasizes order, structure, and information content. It is often linked to concepts such as progress, growth, and the natural tendency of systems to evolve toward higher levels of organization. Because of these properties, extropy has recently attracted considerable attention as a complementary measure to entropy in modeling uncertainty and information.

Within the FGM family, Almasoor et al. [21] investigated extropy measures for w-GOS concomitants, establishing beneficial properties and highlighting their flexibility. More recently, Mohamed et al. [[22], [23]] employed both real and simulated COVID-19 viral data to explore the non-parametric computation of the residual extropy measure for w-GOS concomitants, demonstrating the potential of extropy-based approaches in practical applications.

Entropy for w-GOS concomitants arising from the Lai and Xie extensions was studied by Gamal et al. [20], who provided several theoretical insights. In contrast, the present work focuses on the study of extropy in this setting, extending the analysis to include several new measures and estimation methods.

The main contributions and structure of this paper can be summarized as follows. In Section 2, we derive the expression for $\varepsilon(Y_{[r,n,w,l]})$ using the extropy measure. In the same section, we also present conclusions concerning the extropy of the concomitants of order statistics and record values for the uniform and exponential distributions. In addition, the Lai and Xie extensions are employed to study several related measures, including residual and past extropies, CREX, and NCEX for w-GOS concomitants. Section 3 introduces empirical estimators for CREX, supported by simulation studies that validate the accuracy and efficiency of the proposed estimators. Real-life data are further analyzed to illustrate the practical applicability of non-parametric CREX estimation under the Lai and Xie framework. Finally, Section 4 provides a summary of the key findings and discusses possible directions for future research.

2. Lai and Xie extensions via extropy

The measures of extropy, residual and past extropies, CREX, and NCEX for w-GOS concomitants will be derived in this section using the general framework of the Lai and Xie extensions. Furthermore, applications will be provided for the concomitants of OS and record values under the uniform and exponential distributions.

2.1. Extropy of concomitants for w-generalized order statistics

In this subsection, we will derive the measure of extropy of concomitants for w-GOS, $Y_{[r,n,w,l]}$, from Lai and Xie extensions through the application of the theorem.

Theorem 2.1.1. *Assume (X, Y) is a bivariate r.v. from the Lai and Xie extensions that is non-negative. Based on the concomitant $Y_{[r,n,w,l]}$, the rth w-GOS's extropy is thus provided by*

$$\begin{aligned}\varepsilon(Y_{[r,n,w,l]}) = \varepsilon(Y) - \delta R^*_{[r,n,w,l]}\mathbb{E}((1-u)^{\alpha-1}u^{\lambda-1}(\lambda-(\alpha+\lambda)u)g(G_Y^{-1}(u))) \\ - \frac{(\delta R^*_{[r,n,w,l]})^2}{2}\mathbb{E}((1-u)^{2(\alpha-1)}u^{2(\lambda-1)}(\lambda-(\alpha+\lambda)u)^2 g(G_Y^{-1}(u))),\end{aligned} \tag{2.1}$$

where $\varepsilon(Y)$ is extropy for r.v. Y and U is an uniformly distributed on $U(0,1)$.

Proof. From (1.1) and (1.11), we have

$$\begin{aligned}
\varepsilon(Y_{[r,n,w,l]}) &= \frac{-1}{2}\int_0^\infty g^2_{(r,n,w,l)}(y)dy \\
&= \frac{-1}{2}\int_0^\infty g_Y^2(y)\left[1+\delta R^*_{[r,n,w,l]}\bar{G}_Y(y)^{\alpha-1}G_Y^{\lambda-1}(y)\left(\lambda-(\alpha+\lambda)G_Y(y)\right)\right]^2 dy \\
&= \varepsilon(Y)-\delta R^*_{[r,n,w,l]}\int_0^\infty g_Y^2(y)\bar{G}_Y(y)^{\alpha-1}G_Y^{\lambda-1}(y)\left(\lambda-(\alpha+\lambda)G_Y(y)\right)dy \\
&\quad -\frac{(\delta R^*_{[r,n,w,l]})^2}{2}\int_0^\infty g_Y^2(y)\bar{G}_Y(y)^{2(\alpha-1)}G_Y^{2(\lambda-1)}(y)\left(\lambda-(\alpha+\lambda)G_Y(y)\right)^2 dy \\
&= \varepsilon(Y)-\delta R^*_{[r,n,w,l]}\int_0^1 (1-u)^{\alpha-1}u^{\lambda-1}\left(\lambda-(\alpha+\lambda)u\right)g(G_Y^{-1}(u))du \\
&\quad -\frac{(\delta R^*_{[r,n,w,l]})^2}{2}\int_0^1 (1-u)^{2(\alpha-1)}u^{2(\lambda-1)}\left(\lambda-(\alpha+\lambda)u\right)^2 g(G_Y^{-1}(u))du \\
&= \varepsilon(Y)-\delta R^*_{[r,n,w,l]}\mathbb{E}((1-u)^{\alpha-1}u^{\lambda-1}\left(\lambda-(\alpha+\lambda)u\right)g(G_Y^{-1}(u))) \\
&\quad -\frac{(\delta R^*_{[r,n,w,l]})^2}{2}\mathbb{E}((1-u)^{2(\alpha-1)}u^{2(\lambda-1)}\left(\lambda-(\alpha+\lambda)u\right)^2 g(G_Y^{-1}(u))),
\end{aligned}$$

Theorem 2.1.2. *Suppose that $w=0$ and $l=1$, the w-Gos reduces to OS. Therefore, the PDF of concomitants of OS from Lai and Xie extensions is given by*

$$g_{[r:n]}(y)=g_Y(y)\left[1+\delta T^*_{[r:n]}\bar{G}_Y(y)^{\alpha-1}G_Y^{\lambda-1}(y)\left(\lambda-(\alpha+\lambda)G_Y(y)\right)\right], \tag{2.2}$$

where

$$T^*_{[r:n]}=\frac{n!}{(r-1)!(n-r)!}\beta(\alpha+n-r,\lambda+r-1)\frac{\lambda(n-r)+\alpha(1-r)}{\alpha+n+\lambda-1}. \tag{2.3}$$

Proof: Since the PDF of the r-th OS is given by

$$g_{[r:n]}(x)=\frac{n!}{(r-1)!(n-r)!}\left[1-G_X(x)\right]^{n-r}G_X^{r-1}(x)g_X(x). \tag{2.4}$$

From (1.9) and (2.4), thus the OS's concomitant PDF is provided by

$$\begin{aligned}
g_{[r:n]}(y) &= \int_{-\infty}^\infty g_{Y|X}(y\mid x)g_{[r:n]}(x)dx \\
&= g_Y(y)+\delta\frac{n!}{(r-1)!(n-r)!}\bar{G}_Y(y)^{\alpha-1}G_Y^{\lambda-1}(y)\left(\lambda-(\alpha+\lambda)G_Y(y)\right)g_Y(y) \\
&\quad\times\int_{-\infty}^\infty\left[\lambda G_X(x)^{\lambda+r-2}-(\lambda+\alpha)G_X(x)^{\lambda+r-1}\right]\overline{G}_X(x)^{n-r+\alpha-1}g_X(x)dx.
\end{aligned}$$

Let $t=\overline{G}_X(x)$, then we have

$$\begin{aligned}
g_{[r:n]}(y) &= g_Y(y)+\delta\frac{n!}{(r-1)!(n-r)!}\bar{G}_Y(y)^{\alpha-1}G_Y^{\lambda-1}(y)\left(\lambda-(\alpha+\lambda)G_Y(y)\right)g_Y(y) \\
&\quad\times\int_0^1\left[\lambda t^{n-r+\alpha-1}(1-t)^{\lambda+r-2}-(\lambda+\alpha)t^{n-r+\alpha-1}(1-t)^{\lambda+r-1}\right]dt.
\end{aligned}$$

which proves the theorem.

Corollary 2.1.1. *According to Theorem (2.1.1), under the rth concomitants OS's extropy,* $\varepsilon(Y_{[r:n]})$*, is given by*

$$\begin{aligned}\varepsilon(Y_{[r:n]}) = \varepsilon(Y) - \delta T^*_{[r:n]}\mathbb{E}((1-u)^{\alpha-1}u^{\lambda-1}(\lambda-(\alpha+\lambda)u)g(G_Y^{-1}(u)))\\ - \frac{(\delta T^*_{[r:n]})^2}{2}\mathbb{E}((1-u)^{2(\alpha-1)}u^{2(\lambda-1)}(\lambda-(\alpha+\lambda)u)^2g(G_Y^{-1}(u))).\end{aligned} \tag{2.5}$$

Therefore, we have

$$\begin{aligned}\varepsilon(Y_{[n:n]}) = \varepsilon(Y) - \delta\frac{\alpha n(1-n)}{\alpha+\lambda+n-1}\beta(\alpha,\lambda+n-1)\mathbb{E}((1-u)^{\alpha-1}u^{\lambda-1}(\lambda-(\alpha+\lambda)u)g(G_Y^{-1}(u))))\\ - \frac{1}{2}\{\frac{\delta\alpha n(1-n)}{\alpha+\lambda+n-1}\beta(\alpha,\lambda+n-1)\}^2\mathbb{E}((1-u)^{2(\alpha-1)}u^{2(\lambda-1)}(\lambda-(\alpha+\lambda)u)^2g(G_Y^{-1}(u)))),\end{aligned}$$

$$\begin{aligned}\varepsilon(Y_{[1:n]}) = \varepsilon(Y) - \delta\frac{\lambda n(n-1)}{\alpha+\lambda+n-1}\beta(\alpha+n-1,\lambda)\mathbb{E}((1-u)^{\alpha-1}u^{\lambda-1}(\lambda-(\alpha+\lambda)u)g(G_Y^{-1}(u)))\\ - \frac{1}{2}\{\frac{\delta\lambda n(n-1)}{\alpha+\lambda+n-1}\beta(\alpha+n-1,\lambda)\}^2\mathbb{E}((1-u)^{2(\alpha-1)}u^{2(\lambda-1)}(\lambda-(\alpha+\lambda)u)^2g(G_Y^{-1}(u))).\end{aligned}$$

Remark 2.1. *We get a recurrence relation between extropy for concomitants of OS from Lai and Xie extensions* $Y_{[r+1:n]}$ *and* $Y_{[r:n]}$*, then we have the result from corollary (2.1.1) as*

$$\begin{aligned}\varepsilon(Y_{[r+1:n]}) - \varepsilon(Y_{[r:n]}) = \delta\Omega_{r,n}\mathbb{E}((1-u)^{\alpha-1}u^{\lambda-1}(\lambda-(\alpha+\lambda)u)g(G_Y^{-1}(u)))\\ - \delta^2\frac{(\Omega_{r,n})^2 - 2T^*_{[r:n]}\Omega_{r,n}}{2}\mathbb{E}((1-u)^{2(\alpha-1)}u^{2(\lambda-1)}(\lambda-(\alpha+\lambda)u)^2g(G_Y^{-1}(u))),\end{aligned}$$

where

$$\begin{aligned}\Omega_{r,n} = T^*_{[r:n]} - T^*_{[r+1:n]} = c_{(r-1:n)}\{\frac{\lambda(n-r)+\alpha(1-r)}{(n-r)(\alpha+n+\lambda-1)}\beta(\alpha+n-r,\lambda+r-1)\\ - \frac{(\lambda(n-r-1)-\alpha r)}{r(\alpha+n-\lambda-1)}\beta(\alpha+n-r-1,\lambda+r)\},\end{aligned}$$

and

$$c_{(r-1:n)} = \frac{n!}{(r-1)!(n-r-1)!}.$$

Remark 2.2. *We get a recurrence relation between extropy for concomitants of OS from Lai and Xie extensions* $Y_{[r:n-1]}$ *and* $Y_{[r:n]}$*, then we have the result from corollary (2.1.1)* *as*

$$\varepsilon(Y_{[r:n-1]}) - \varepsilon(Y_{[r:n]}) = \delta\Omega^*_{r,n}\mathbb{E}((1-u)^{\alpha-1}u^{\lambda-1}(\lambda-(\alpha+\lambda)u)g(G_Y^{-1}(u))) - \delta^2\frac{(\Omega^*_{r,n})^2 - 2T^*_{[r:n-1]}\Omega^*_{r,n}}{2}\mathbb{E}((1-u)^{2(\alpha-1)}u^{2(\lambda-1)}(\lambda-(\alpha+\lambda)u)^2 g(G_Y^{-1}(u))), \tag{2.6}$$

where

$$\Omega^*_{r,n} = T^*_{[r:n]} - T^*_{[r:n-1]} = \frac{(n-1)!}{(r-1)!(n-r-1)!}\{\frac{n(\lambda(n-r)+\alpha(r-1))(\alpha+n+r-1)}{(n-r)(\alpha+n+\lambda)(\alpha+n+\lambda-1)} - \frac{\lambda(n-r-1)+\alpha(r-1)}{\alpha+n+\lambda-2}\}\beta(\alpha+n-r-1,\lambda+r-1).$$

Theorem 2.1.3. *The record value is a special case of the w-GOS with $w=-1$ and $l=1$. Therefore, the PDF of concomitants of record value from Lai and Xie extensions is given by*

$$g_{L(n)}(y) = g_Y(y)\left[1+\delta\mu^*_{L(n)}\bar{G}_Y(y)^{\alpha-1}G_Y^{\lambda-1}(y)\left(\lambda-(\alpha+\lambda)G_Y(y)\right)\right], \tag{2.7}$$

where

$$\mu^*_{L(n)} = \sum_{\epsilon=0}^{\alpha-1}\binom{\alpha-1}{\epsilon}(-1)^\epsilon[\frac{\alpha+\lambda}{(-(\lambda+\epsilon)+1)^n} - \frac{\lambda}{(-(\lambda+\epsilon))^n}]. \tag{2.8}$$

Proof: Since the PDF of the nth lower record is given by

$$g_{L(n)}(x) = \frac{1}{(n-1)!}\left[-\ln G_X(x)\right]^{n-1}g_X(x). \tag{2.9}$$

From (1.9) and (2.9), then the PDF of the concomitant of the nth lower record is given by

$$\begin{aligned} g_{L(n)}(y) &= \int_{-\infty}^{\infty} g_{Y|X}(y \mid x)g_{L(n)}(x)dx \\ &= g_Y(y) + \frac{\delta}{(n-1)!}\bar{G}_Y(y)^{\alpha-1}G_Y^{\lambda-1}(y)\left(\lambda-(\alpha+\lambda)G_Y(y)\right)g_Y(y) \\ &\quad\times\int_{-\infty}^{\infty}\bar{G}_X(x)^{\alpha-1}G_X^{\lambda-1}(x)\left(\lambda-(\alpha+\lambda)G_X(x)\right)\left[-\ln G_X(x)\right]^{n-1}g_X(x)dx, \end{aligned}$$

let $t = [-\ln G_X(x)]$, then we have

$$g_{L(n)}(y) = g_Y(y) + \frac{\delta}{(n-1)!}\bar{G}_Y(y)^{\alpha-1}G_Y^{\lambda-1}(y)\left(\lambda - (\alpha+\lambda)G_Y(y)\right)g_Y(y) \times \int_0^1 \left[(\lambda+\alpha)e^{(-\lambda+1)t} - \lambda e^{-\lambda t}\right](1-e^{-t})^{\alpha-1}t^{n-1}dt.$$

$$= g_Y(y) + \frac{\delta}{(n-1)!}\bar{G}_Y(y)^{\alpha-1}G_Y^{\lambda-1}(y)\left(\lambda - (\alpha+\lambda)G_Y(y)\right)g_Y(y) \times \sum_{\epsilon=0}^{\alpha-1}\binom{\alpha-1}{\epsilon}(-1)^{\epsilon}\int_0^1 \left[(\lambda+\alpha)e^{(-(\lambda+\epsilon)+1)t} - \lambda e^{-(\lambda+\epsilon)t}\right]t^{n-1}dt.$$

which proves the theorem.

Corollary 2.1.2. *According to Theorem (2.1.1), the concomitant extropy of* $Y_{L(n)}$ *is given by*

$$\varepsilon(Y_{L(n)}) = \varepsilon(Y) - \delta\mu^*_{L(n)}\mathbb{E}((1-u)^{\alpha-1}u^{\lambda-1}(\lambda-(\alpha+\lambda)u)g(G_Y^{-1}(u))) - \frac{1}{2}(\delta\mu^*_{L(n)})^2\mathbb{E}((1-u)^{2(\alpha-1)}u^{2(\lambda-1)}(\lambda-(\alpha+\lambda)u)^2g(G_Y^{-1}(u))). \tag{2.10}$$

Now, we will give an application of Theorem (2.1.1) by the following examples.

Example 2.1.1. *Assume that the exponential distribution* $(Exp(\theta))$ *with CDF produces the non-negative continuous r.v.* Y *as follows*

$$G(y) = 1 - e^{-\theta y}, \theta > 0, y > 0. \tag{2.11}$$

According to Theorem (2.1.1), the concomitant extropy $\varepsilon(Y_{[r,n,w,l]})$ *is given by*

$$\varepsilon(Y_{[r,n,w,l]}) = -0.25\theta - \frac{3\delta R^*_{[r,n,w,l]}\theta\Gamma(3+\alpha)\Gamma(1+\lambda)}{\Gamma(4+\alpha+\lambda)} - (\delta R^*_{[r,n,w,l]})^2\theta \times \frac{\lambda(10\lambda+\alpha(4+\alpha+\lambda))\Gamma(4+2\alpha)\Gamma(-1+2\lambda)}{\Gamma(5+2\alpha+2\lambda)}. \tag{2.12}$$

Based on OS, Figure 1 shows some plots of $\varepsilon(Y_{[r,n,0,1]})$ for $Exp(\theta)$.

Example 2.1.2. *Assume that the uniform distribution with CDF produces the non-negative continuous r.v.* Y *as follows*

$$G(y) = \left(\frac{y}{\sigma}\right), 0 < y < \sigma. \tag{2.13}$$

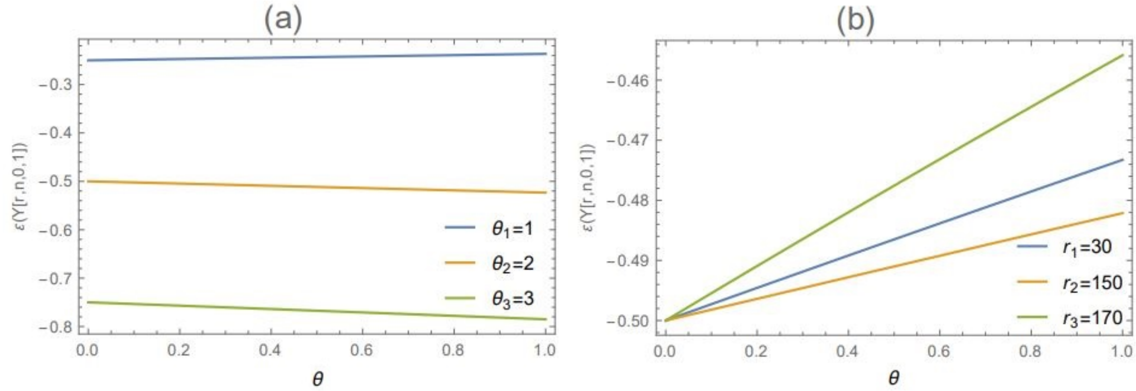

Figure 1: plots of $\varepsilon(Y_{[r,n,0,1]})$ form $Exp(\theta)$ for a sample of size $n = 200, a = 1, b = 2$ with various selections: (a) $r = 30, \theta_1 = 1, \theta_2 = 2, \theta_3 = 3$, (b) $\theta = 2, r_1 = 30, r_2 = 150, r_3 = 170$.

According to Theorem (2.1.1), the concomitant extropy $\varepsilon(Y_{[r,n,w,l]})$ *is given by*

$$\varepsilon(Y_{[r,n,w,l]}) = \frac{-0.5}{\sigma} - \frac{2\delta R^*_{[r,n,w,l]}\Gamma(2+\alpha)\Gamma(1+\lambda)}{\sigma\Gamma(3+\alpha+\lambda)} - \frac{(\delta R^*_{[r,n,w,l]})^2}{\sigma} \times \frac{\lambda(6\lambda+\alpha(3+\alpha+\lambda))\Gamma(3+2\alpha)\Gamma(-1+2\lambda)}{\Gamma(2(2+\alpha+\lambda))}. \tag{2.14}$$

Based on OS, Figure 2 shows some plots of $\varepsilon(Y_{[r,n,0,1]})$ for a uniform distribution

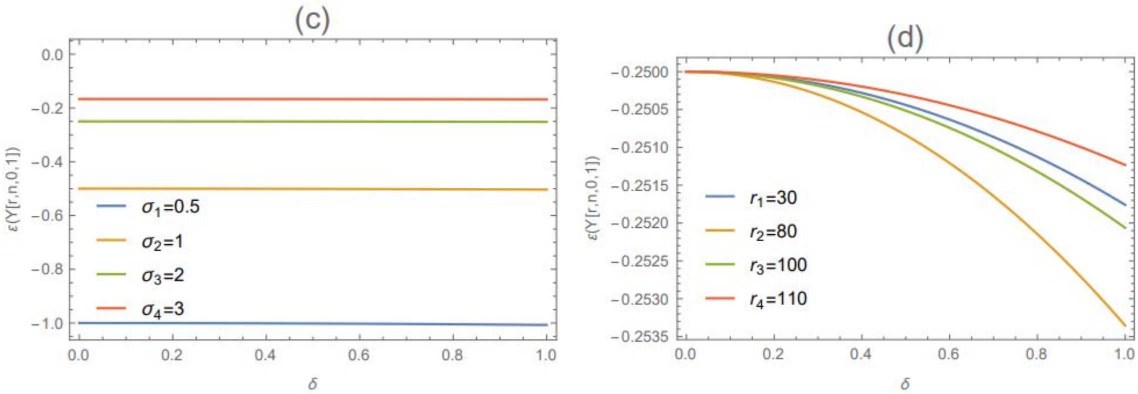

Figure 2: plots of $\varepsilon(Y_{[r,n,0,1]})$ form a uniform distribution for a sample of size $n = 200, \alpha = 1, \lambda = 2$ with various selections: (c) $r = 30, \sigma_1 = 0.5, \sigma_2 = 1, \sigma_3 = 2, \sigma_4 = 3$, (d) $\sigma = 2, r_1 = 30, r_2 = 80, r_3 = 100, r_4 = 110$.

2.2. Residual and past extropies of concomitants for w-generalized order statistics

In this subsection, we will derive the measures of the residual and past extropies of w-GOS, $Y_{[r,n,w,l]}$, from Lai and Xie extensions by following theorems.

Theorem 2.2.1. *Assume that the r-th w-GOS concomitant $Y_{[r,n,w,l]}$ is a continuous and non-negative r.v. from the Lai and Xie extensions. Then, the residual extropy of the r-th w-GOS, based on the concomitant $Y_{[r,n,w,l]}$, is given by*

$$\begin{aligned}\varepsilon^t(Y_{[r,n,w,l]}) =& \varepsilon^R(Y_{[r,n,w,l]};t) = [\frac{1}{1+\delta\eta^*_{[r,n,w,l]}\bar{G}_Y(t)^{\alpha-1}G_Y^{\lambda}(t)}]^2[\varepsilon^R(Y) \\ &- \frac{\delta R^*_{[r,n,w,l]}}{\bar{G}_Y^2(t)}\mathbb{E}((1-u)^{\alpha-1}\,u^{\lambda-1}g(G_Y^{-1}(u))\,(\lambda-(\alpha+\lambda)u)) \\ &- \frac{(\delta R^*_{[r,n,w,l]})^2}{2\bar{G}_Y^2(t)}\mathbb{E}((1-u)^{2(\alpha-1)}\,u^{2(\lambda-1)}\,(\lambda-(\alpha+\lambda)u)^2\,g(G_Y^{-1}(u)))],\end{aligned}$$

where $\varepsilon^R(Y)$ is residual extropy for r.v. of Y and U_R is r.v. from $U(1, G_Y(t))$.

Proof. From (1.2), (1.11) and (1.12), we have

$$\begin{aligned}\varepsilon^R(Y_{[r,n,w,l]};t) &= \frac{-1}{2}\int_t^{\infty}\frac{g_Y^2(y)\left[1+\delta R^*_{[r,n,w,l]}\bar{G}_Y(y)^{\alpha-1}G_Y^{\lambda-1}(y)\,(\lambda-(\alpha+\lambda)G_Y(y))\right]^2dy}{\bar{G}_Y^2(t)\left[1+\delta\eta^*_{[r,n,w,l]}\bar{G}_Y(t)^{\alpha-1}G_Y^{\lambda}(t)\right]^2} \\ &= [\frac{1}{1+\delta\eta^*_{[r,n,w,l]}\bar{G}_Y(t)^{\alpha-1}G_Y^{\lambda}(t)}]^2[\varepsilon^R(Y) - \frac{\delta R^*_{[r,n,w,l]}}{\bar{G}_Y^2(t)}\int_t^{\infty}g_Y^2(y) \\ &\quad\times \bar{G}_Y(y)^{\alpha-1}G_Y^{\lambda-1}(y)\,(\lambda-(\alpha+\lambda)G_Y(y))\,dy - \frac{(\delta R^*_{[r,n,w,l]})^2}{2\bar{G}_Y^2(t)}\int_t^{\infty}g_Y^2(y) \\ &\quad\times \bar{G}_Y(y)^{2(\alpha-1)}G_Y^{2(\lambda-1)}(y)\,(\lambda-(\alpha+\lambda)G_Y(y))^2\,dy]. \\ &= [\frac{1}{1+\delta\eta^*_{[r,n,w,l]}\bar{G}_Y(t)^{\alpha-1}G_Y^{\lambda}(t)}]^2[\varepsilon^R(Y) - \frac{\delta R^*_{[r,n,w,l]}}{\bar{G}_Y^2(t)} \\ &\quad\times\int_{G_Y(t)}^1(1-u)^{\alpha-1}\,u^{\lambda-1}\,(\lambda-(\alpha+\lambda)u)\,g(G_Y^{-1}(u))du - \frac{(\delta R^*_{[r,n,w,l]})^2}{2\bar{G}_Y^2(t)} \\ &\quad\times\int_{G_Y(t)}^1(1-u)^{2(\alpha-1)}\,u^{2(\lambda-1)}\,(\lambda-(\alpha+\lambda)u)^2\,g(G_Y^{-1}(u))du]\end{aligned}$$

$$= [\frac{1}{1+\delta\eta^*_{[r,n,w,l]}\bar{G}_Y(t)^{\alpha-1}G_Y^{\lambda}(t)}]^2[\varepsilon^R(Y) - \frac{\delta R^*_{[r,n,w,l]}}{\bar{G}_Y^2(t)}$$

$$\times \mathbb{E}((1-u)^{\alpha-1}\, u^{\lambda-1}\,(\lambda-(\alpha+\lambda)u)\, g(G_Y^{-1}(u))) - \frac{(\delta R^*_{[r,n,w,l]})^2}{2\bar{G}_Y^2(t)}$$

$$\times \mathbb{E}((1-u)^{2(\alpha-1)}\, u^{2(\lambda-1)}\,(\lambda-(\alpha+\lambda)u)^2\, g(G_Y^{-1}(u)))].$$

Theorem 2.2.2. *Assume that the r-th w-GOS concomitant $Y_{[r,n,w,l]}$ is a continuous and non-negative r.v. from the Lai and Xie extensions. From (1.3), (1.10), and (1.11), the residual extropy of the r-th w-GOS, based on the concomitant $Y_{[r,n,w,l]}$, is given by*

$$\varepsilon_t(Y_{[r,n,w,l]}) = \varepsilon_P(Y_{[r,n,w,l]};t) = [\frac{1}{1+\delta\eta^*_{[r,n,w,l]}\bar{G}_Y(t)^{\alpha}G_Y^{\lambda-1}(t)}]^2[\varepsilon_P(Y)$$

$$- \frac{\delta R^*_{[r,n,w,l]}}{G_Y^2(t)}\mathbb{E}((1-u)^{\alpha-1}\, u^{\lambda-1}\,(\lambda-(\alpha+\lambda)u)\, g(G_Y^{-1}(u)))$$

$$- \frac{(\delta R^*_{[r,n,w,l]})^2}{2G_Y^2(t)}\mathbb{E}((1-u)^{2(\alpha-1)}\, u^{2(\lambda-1)}\,(\lambda-(\alpha+\lambda)u)^2\, g(G_Y^{-1}(u)))],$$

where $\varepsilon_P(Y)$ is past extropy for r.v. of Y and U_P is r.v. from $U(0, G_Y(t))$.

Proof. An analogous set of steps to those used in the proof of Theorem 2.2.1 could be applied here.

2.3. Cumulative residual extropy of concomitants for w-generalized order statistics

In this subsection, we will derive the measure of CREX of concomitants for w-GOS, $Y_{[r,n,w,l]}$, from Lai and Xie extensions by following theorem.

Theorem 2.3.1. *Assume that the r-th w-GOS concomitant $Y_{[r,n,w,l]}$ is a continuous and non-negative r.v. from the Lai and Xie extensions. Then, the CREX extropy of the r-th w-GOS, based on the concomitant $Y_{[r,n,w,l]}$, is given by*

$$\xi_{[r,n,w,l]}(Y) = \xi(Y) - \delta\eta^*_{[r,n,w,l]}\mathbb{E}((1-u)^{\alpha+1}u^{\lambda}\frac{1}{g(G_Y^{-1}(u))}) - \frac{(\delta\eta^*_{[r,n,w,l]})^2}{2}$$

$$\times \mathbb{E}((1-u)^{2\alpha}u^{2\lambda}\frac{1}{g(G_Y^{-1}(u))}).$$

Proof. *From (1.4) and (1.12), we have*

$$\begin{aligned}
\xi_{[r,n,w,l]}(Y) &= \frac{-1}{2}\int_0^\infty (\bar{G}_{[r,n,w,l]}(y))^2 dy \\
&= \frac{-1}{2}\int_0^\infty \left[\bar{G}_Y^2(y) + 2\delta\eta^*_{[r,n,w,l]}\bar{G}_Y(y)^{\alpha+1}G_Y^{\lambda}(y) + (\delta\eta^*_{[r,n,w,l]})^2\bar{G}_Y(y)^{2\alpha}G_Y^{2\lambda}(y)\right] dy. \\
&= \xi(Y) - \delta\eta^*_{[r,n,w,l]}\int_0^\infty (1-u)^{\alpha+1}u^{\lambda}\frac{1}{g(G_Y^{-1}(u))}du - \frac{(\delta\eta^*_{[r,n,w,l]})^2}{2} \\
&\quad \times \int_0^\infty (1-u)^{2\alpha}u^{2\lambda}\frac{1}{g(G_Y^{-1}(u))}du,
\end{aligned}$$

with noting that $\xi(Y)$ is the CREX of r.v. Y, and U is r.v. from $U(0,1)$.

2.4. The negative cumulative extropy of concomitants for w-generalized order statistics

In this subsection, we will derive the measure of NCEX of concomitants for w-GOS, $Y_{[r,n,w,l]}$, from Lai and Xie extensions by following theorem.

Theorem 2.4.1. *Assume that the r-th w-GOS concomitant $Y_{[r,n,w,l]}$ is a continuous and non-negative r.v. from the Lai and Xie extensions. Then, the NCEX extropy of the r-th w-GOS, based on the concomitant $Y_{[r,n,w,l]}$, is given by*

$$\begin{aligned}
\xi^*(Y) = \xi^*(Y) &+ \delta\eta^*_{[r,n,w,l]}\mathbb{E}((1-u)^{\alpha}u^{\lambda+1}\frac{1}{g(G_Y^{-1}(u))}) + \frac{(\delta\eta^*_{[r,n,w,l]})^2}{2} \\
&\times \mathbb{E}((1-u)^{2\alpha}u^{2\lambda}\frac{1}{g(G_Y^{-1}(u))}).
\end{aligned}$$

Proof. *From (1.5) and (1.10), we have*

$$\begin{aligned}
\xi^*(Y) &= \frac{1}{2}\int_0^\infty (1 - G^2_{[r,n,w,l]}(y))dy \\
&= \frac{1}{2}\int_0^\infty \left[(1 - G_Y^2(y)) + 2\delta\eta^*_{[r,n,w,l]}\bar{G}_Y(y)^{\alpha}G_Y^{\lambda+1}(y) + (\delta\eta^*_{[r,n,w,l]})^2\bar{G}_Y(y)^{2\alpha}G_Y^{2\lambda}(y)\right] dy \\
&= \xi^*(Y) + \delta\eta^*_{[r,n,w,l]}\int_0^\infty (1-u)^{\alpha}u^{\lambda+1}\frac{1}{g(G_Y^{-1}(u))}du + \frac{(\delta\eta^*_{[r,n,w,l]})^2}{2} \\
&\quad \times \int_0^\infty (1-u)^{2\alpha}u^{2\lambda}\frac{1}{g(G_Y^{-1}(u))}du,
\end{aligned}$$

with noting that $\xi^*(Y)$ is the NCEX of r.v. Y, and U is r.v. on $U(0,1)$.

3. Non-parametric estimation

Using the empirical data, we derive a non-parametric estimate of the CREX of w-GOS concomitants under the Lai and Xie extensions in this section. Consider the random sample $Y_1, ..., Y_n$ from a population with CDF and its empirical estimator G_n. According to (1.4) and (1.12), the empirical CREX of w-GOS concomitants is provided by

$$\begin{aligned}\xi_{[r,n,w,l]}(G_n) &= \frac{-1}{2}\int_0^\infty \bar{G}_n^2(y)(1+\delta\eta^*_{[r,n,w,l]}G_n^\lambda(y)(1-G_n(y))^{\alpha-1}))^2 dy \\ &= \frac{-1}{2}\sum_{j=1}^{n-1}\int_{Y_{(j)}}^{Y_{(j+1)}} \bar{G}_n^2(y)(1+\delta\eta^*_{[r,n,w,l]}G_n^\lambda(y)(1-G_n(y))^{\alpha-1}))^2 dy,\end{aligned} \tag{3.1}$$

where $\bar{G}_n(y) = 1 - G_n(y)$, and the corresponding OS of the random sample is $Y_{(1)} \leq Y_{(2)} \leq ... \leq Y_{(n)}$, whereas $G_n(y)$ is the empirical CDF. .

We take into consideration the first empirical estimator $\xi_{1[r,n,w,l]}(G_n)$ in order to estimate $\xi_{[r,n,w,l]}$ as follows

$$\xi_{1[r,n,w,l]}(G_n) = \frac{-1}{2}\sum_{j=1}^{n-1} W_{j+1}\left(1-\frac{j}{n}\right)^2\left(1+\delta\eta^*_{[r,n,w,l]}(\frac{j}{n})^\lambda(1-\frac{j}{n})^{\alpha-1}\right)^2, \tag{3.2}$$

where $G_n(y) = \frac{j}{n}$, $j = 1, 2, ..., n-1$, $W_{j+1} = Y_{(j+1)} - Y_{(j)}$, $W_1 = Y_{(1)}$. Additionally, the kernel-smoothed estimator, or second empirical estimator, $\xi_{2[r,n,w,l]}(G_n)$, is provided by

$$\xi_{2[r,n,w,l]}(G_n) = \frac{-1}{2}\sum_{j=1}^{n-1} W_{j+1}\left(1-G_n(y_j)\right)^2\left(1+\delta\eta^*_{[r,n,w,l]}(G_n(y_j))^\lambda(1-G_n(y_j))^{\alpha-1}\right)^2, \tag{3.3}$$

where

$$G_n(y_j) = \frac{1}{n}\sum_{i=1}^{n} B\left(\frac{y-Y_i}{h}\right),$$

$B(y) = \int_{-\infty}^{y} S(t)dt$ and h are bandwidth parameters; refer to Nadaraya [24].

By using 100 samples of size n=25, are generated with the uniform distribution $U(0, 1)$. These data satisfy the asymptomatic normality of the empirical estimator's assumption in (3.2) and (3.3). The histogram shown in Figure 3 is presented.

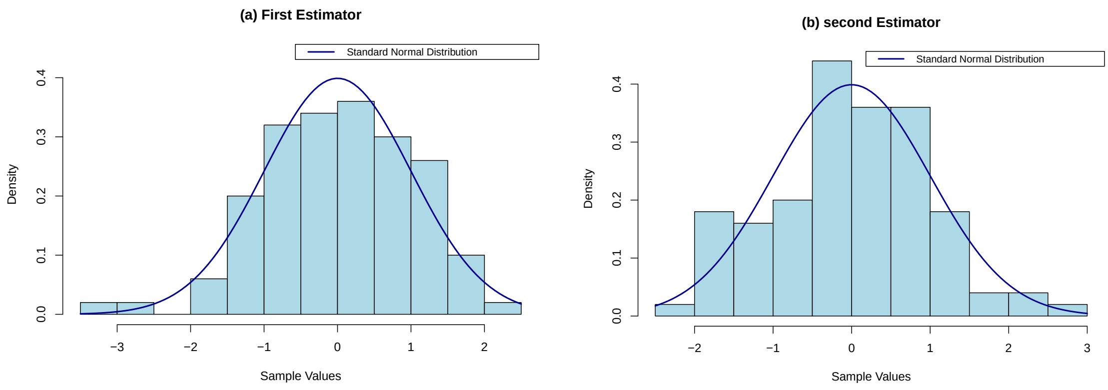

Figure 3: Histogram of empirical estimators for sample values under Lai and Xie extensions at $\alpha = 3$, $\lambda = 4$, $\delta = 0.6$ with standard normal distribution.

In the following, we use the suggested techniques in the examples to describe how the empirical and kernel estimators work.

Example 3.0.1. *Consider the random sample $X_1, ..., X_n$ with $U(0,1)$. Pyke states in [25] that the sample spacing W_{j+1} is in accordance with the beta distribution $Beta(1,n)$. Thus, derived from (3.2) and (3.3), we obtain*

$$\mathbb{E}(\xi_{1[r,n,w,l]}(G_n)) = \frac{-1}{2(1+n)} \sum_{j=1}^{n-1} \left(1 - \frac{j}{n}\right)^2 \left(1 + \delta\eta^*_{[r,n,w,l]}(\frac{j}{n})^\lambda(1 - \frac{j}{n})^{\alpha-1}\right)^2,$$

$$Var(\xi_{1[r,n,w,l]}(G_n)) = \frac{n}{4(2+n)(1+n)^2} \sum_{j=1}^{n-1} \left(1 - \frac{j}{n}\right)^4 \left(1 + \delta\eta^*_{[r,n,w,l]}(\frac{j}{n})^\lambda(1 - \frac{j}{n})^{\alpha-1}\right)^4,$$

$$\mathbb{E}(\xi_{2[r,n,w,l]}(G_n)) = \frac{-1}{2(1+n)} \sum_{j=1}^{n-1} (1 - G_n(y_j))^2 \left(1 + \delta\eta^*_{[r,n,w,l]}(G_n(y_j))^\lambda(1 - G_n(y_j))^{\alpha-1}\right)^2,$$

$$Var(\xi_{2[r,n,w,l]}(G_n)) = \frac{n}{4(2+n)(1+n)^2} \sum_{j=1}^{n-1} (1 - G_n(y_j))^4 \left(1 + \delta\eta^*_{[r,n,w,l]}(G_n(y_j))^\lambda(1 - G_n(y_j))^{\alpha-1}\right)^4.$$

Example 3.0.2. *Consider the random sample $X_1, ..., X_n$ with $Exp(\theta)$. Pyke states in [25] that the sample spacing W_{j+1} is in accordance with $Exp(\theta(n-j))$. Thus, derived from (3.2) and (3.3), we get*

$$\mathbb{E}(\xi_{1[r,n,w,l]}(G_n)) = \frac{-1}{2n\theta} \sum_{j=1}^{n-1} \left(1 - \frac{j}{n}\right) \left(1 + \delta\eta^*_{[r,n,w,l]}(\frac{j}{n})^\lambda(1 - \frac{j}{n})^{\alpha-1}\right)^2,$$

$$Var(\xi_{1[r,n,w,l]}(G_n)) = \frac{1}{4n^2\theta^2}\sum_{j=1}^{n-1}\left(1-\frac{j}{n}\right)^2\left(1+\delta\eta^*_{[r,n,w,l]}(\frac{j}{n})^\lambda(1-\frac{j}{n})^{\alpha-1}\right)^4,$$

$$\mathbb{E}(\xi_{2[r,n,w,l]}(G_n)) = \frac{-1}{2\theta}\sum_{j=1}^{n-1}\frac{(1-G_n(y_j))^2}{n-j}\left(1+\delta\eta^*_{[r,n,w,l]}(G_n(y_j))^\lambda(1-G_n(y_j))^{\alpha-1}\right)^2,$$

$$Var(\xi_{2[r,n,w,l]}(G_n)) = \frac{1}{4\theta^2}\sum_{j=1}^{n-1}\frac{(1-G_n(y_j))^4}{(n-j)^2}\left(1+\delta\eta^*_{[r,n,w,l]}(G_n(y_j))^\lambda(1-G_n(y_j))^{\alpha-1}\right)^4.$$

Based on OS, Table (1) presents the mean and variance of $\xi_{1[r;n,0,1]}$ and $\xi_{2[r;n,0,1]}$ from $EXP(\theta)$, For varying sample size values $(n = 10, 30, 60, 100)$ using $(\theta = 0.5, 1, 2)$.

Table 1: The empirical estimators' mean and variance for CREX of concomitants of OS under Lai and Xie extensions at $\alpha = 5$, $\lambda = 7$, $r = 3$, and $\delta = 0.8$.

n	θ	$EXP(\theta)$ distribution			
		The first estimator		The second estimator	
		$\mathbb{E}(\xi_{1[r;n,0,1]})$	$Var(\xi_{1[r;n,0,1]})$	$\mathbb{E}(\xi_{2[r;n,0,1]})$	$Var(\xi_{2[r;n,0,1]})$
10	0.5	-0.4500	0.0285	-0.5543	0.0387
	1	-0.2250	0.0071	-0.2837	0.0095
	2	-0.1125	0.0017	-0.1538	0.0027
30	0.5	-0.4833	0.0105	-0.5178	0.0115
	1	-0.2416	0.0026	-0.2600	0.0028
	2	-0.1208	0.0006	-0.1325	0.0006
60	0.5	-0.4916	0.0054	-0.5088	0.0056
	1	-0.2458	0.0013	-0.2550	0.0013
	2	-0.1229	0.0003	-0.1290	0.0004
100	0.5	-0.4950	0.0032	-0.5054	0.0033
	1	-0.2475	0.0008	-0.2531	0.0008
	2	-0.1237	0.0002	-0.1278	0.0001

Table 1 provides the characteristics listed below:

(i) When n is a fixed value, the values of the mean increase as the values of θ increase, whereas the values of variance decrease with θ.

(ii) For a fixed θ, both the values of mean and variance are declining as n increases in value.

(iii) When n approaches infinity, the value of the variance tends to zero.

Simulated data are shown in Figure 4. Thus, we can infer that the empirical estimators go closer to the theoretical value as r and θ increase, and vice versa.

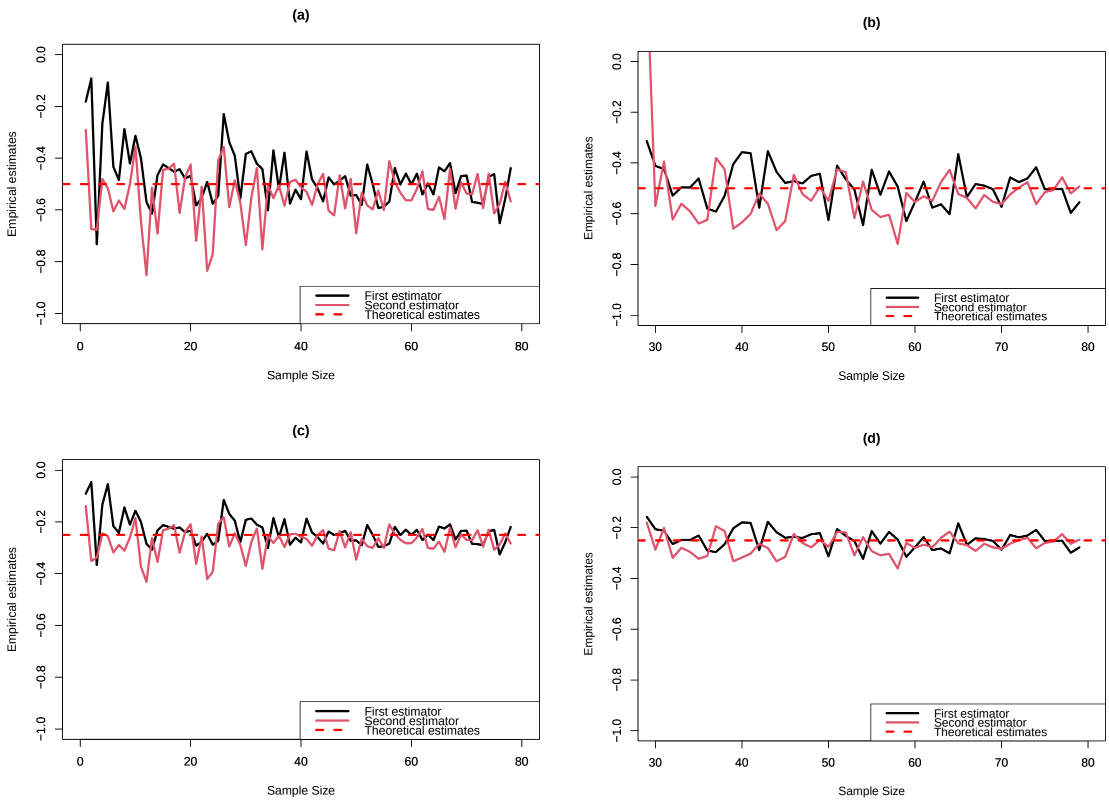

Figure 4: Empirical estimators of simulated data for CREX of concomitants of OS under Lai and Xie extensions. (a)$\alpha = 5$, $\lambda = 7$, $n = 80$, $r = 3$, $\delta = 0.8$, $\theta = 0.5$. (b)$\alpha = 5$, $\lambda = 7$, $n = 80$, $r = 30$, $\delta = 0.8$, $\theta = 0.5$. (c)$\alpha = 5$, $\lambda = 7$, $n = 80$, $r = 3$, $\delta = 0.8$, $\theta = 1$. (d)$\alpha = 5$, $\lambda = 7$, $n = 80$, $r = 30$, $\delta = 0.8$, $\theta = 1$.

3.1. Saudi Arabia Real industrial data analysis

The Industrial Production Index (IPI) is a key economic indicator that measures the monthly output of factories, mines, and utilities within an economy. It reflects the real volume of industrial production and provides valuable insights into the performance of the industrial sector, which is a major driver of overall economic growth. A higher IPI value typically indicates an expansion in industrial activity, suggesting that factories are producing more goods to meet rising demand, which in turn contributes to employment and economic development. Conversely, a lower IPI value points to reduced industrial activity, signaling a potential slowdown in the economy due to weaker demand or production challenges.

In Saudi Arabia, the IPI serves as an important measure for assessing the progress of the Kingdom's industrial diversification efforts under Vision 2030, particularly in sectors such as manufacturing, mining, and energy. The data presented below, obtained from the General Authority for Statistics `https://www.stats.gov.sa/en/statistics-tabs?tab=436312&category=123454` , shows the monthly index for the manufacture of basic metals from January 2023 to August 2025:
116.6, 106.3, 119.6, 111.3, 113.8, 120.4, 122.1, 120.8, 114.3, 121.3, 117.0, 117.1, 120.7, 123.7, 123.4, 131.4, 127.7, 125.9, 125.1, 124.8, 122.4, 123.0, 121.8, 118.8, 115.9, 111.1, 112.4, 113.6, 109.5, 107.7, 108.6, 108.7

The $Weibull(\beta_1, \beta_2)$ distribution was able to properly fit this data set with parameter $\beta_1 = 20.6709$ and $\beta_2 = 120.993$, which a p-value equal to 0.971468. Furthermore, see Figure 5, which shows the estimated PDF and CDF.

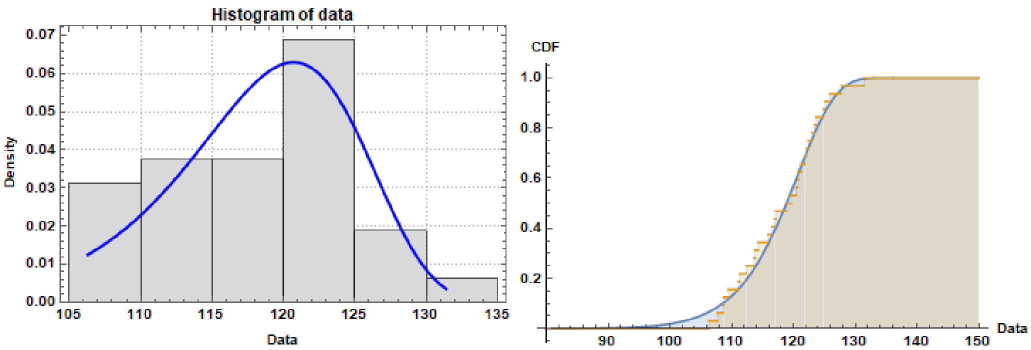

Figure 5: Estimated PDF and CDF of Weibull distribution for data

Based on the above real data, the empirical estimators for CREX of concomitants of OS under Lai and Xie extensions at $\alpha = 4$, $\lambda = 6$, and $\delta = 0.8$ will be $\xi_{1[20;32,0,1]} = -4.06006$ and $\xi_{2[20;32,0,1]} = -4.26736$. Meanwhile, the theoretical value is $\xi_{[20;32,0,1]} = -56.9977$. As we can see from the previous data, the second estimator is better in representation to the theoretical than the first one is.

4. Conclusions

In this work, we have investigated several measures of extropy, including residual and past extropy, CREX, and NCEX, in the context of the concomitant of w-GOS based on the extensions of Lai and Xie. A number of new properties of extropy related to record values and order statistics were derived and discussed. In addition, empirical estimation techniques were employed to estimate CREX, and the performance of the proposed estimators was assessed through both numerical illustrations for the exponential distribution $EXP(\theta)$ and simulation studies.

The simulation results demonstrate that the empirical estimators converge toward the theoretical values as both r and θ increase. Moreover, the application to real-life data, originally considered in the Lai and Xie framework, provided further evidence of the practical relevance of the proposed methodology. In particular, the non-parametric estimation of CREX for w-GOS concomitants showed that the kernel-smoothed estimator, corresponding to the second empirical estimator, consistently outperformed alternatives across all values of n and r. Furthermore, the results revealed that the mean squared error decreases as the sample size n and the order r increase, confirming the efficiency of the estimators.

Overall, the findings of this study enrich the theoretical understanding of extropy-related measures within generalized order statistics and highlight their potential applicability in practical data analysis. Future work may focus on extending these concepts to broader distributional families, exploring their connections with other information-theoretic measures, and developing further applications in reliability analysis, survival studies, and statistical modeling.

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Author Contributions

Each author contributed equally to the study's inception and design.

Competing Interests

The writers are not required to disclose any relevant financial or non-financial interests.

Ethics approval

The authors approve of all the ethical issues.

Data Availability

The article contains the datasets created and/or analyzed during the present investigation.

Declarative use of AI techniques

The authors confirm that AI tools were not utilized in crafting this article.

Conflict of interest

The authors confirm the lack of any conflict of interest.

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