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On an Estimate of the Resolvent of an Even-Order Operator Bundle and Its Application

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Abstract. This paper investigates an estimate of the resolvent for a class of even-order operator bundles acting in a scale of Hilbert spaces generated by a positively defined self-adjoint operator. The considered operator bundle has coefficients that exhibit symmetry with respect to the parity of the index.
Additionally, a connection is established between the spectral properties of the operator bundle and the existence of solutions to abstract differential equations. This provides practical applicability of the obtained theorems in stability analysis, spectral computations, and the construction of numerical schemes for parameter-dependent problems.
The proposed methods and results represent a development of the classical approaches of Lions, Vishik, and Agranovich, extending them to high-dimensional and generalized operator systems. It is demonstrated that, even in the absence of compactness, precise resolvent estimates can be obtained, and spectral purity along the imaginary axis can be ensured.

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Key Words and Phrases: Operator bundle, resolvent, ellipticity, operator spectrum, Hilbert space

1. Introduction

In the spectral theory of linear operators, a significant focus is placed on resolvent estimates for operator bundles depending on a complex parameter. The relevance of this topic stems from its wide range of applications - from mathematical physics and vibration theory to stability analysis in differential equations and operator-based models.

Classical approaches to solvability and a priori estimates were established in the works of Lions and Magenes [1], where inhomogeneous boundary value problems were analyzed using scales of Hilbert spaces. Subsequent studies by Vishik [2], Agranovich and Vishik [3], Krein [4], as well as Roitberg [5], extended spectral analysis to broader classes of parameter-dependent operators. However, most of these studies rely heavily on compactness and complete continuity, which limits their applicability in various practical scenarios.

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In later developments, for instance by Kondratiev and Ryabov [6], and also Babenko and Gohberg [7], new results emerged concerning resolvent estimates in nontrivial geometries and for higher-order operator expressions. Within this context, particular interest arises in the study of bundles whose coefficients exhibit alternating symmetry across different scales, characteristic of problems with variable-type boundary conditions. Moreover, works such as [8–14] have examined equations with multiple characteristics.

The present study offers a new perspective on even-order operator bundles defined in Hilbert scales generated by a positively defined self-adjoint operator. Unlike standard formulations, we introduce an important structural condition: symmetry of even and odd-order coefficients of the operator across different Hilbert spaces. This structural property allows us to prove ellipticity without invoking compactness assumptions.

The theorems proved in this paper - concerning ellipticity and resolvent estimates along the imaginary axis and within sectors of the complex plane - provide a framework for establishing the stability of solutions to the corresponding differential equation (in the sense of a priori estimates). These results also establish an equivalence between spectral theory and the well-posedness of evolutionary problems in functional spaces.

Thus, the present work advances and generalizes modern approaches (Lions-Magenes, Vishik-Agranovich, Krein, Roitberg) in the analysis of operator bundles and opens new possibilities for the application of the obtained estimates in both numerical and analytical contexts.

2. Problem Statement and Some Auxiliary Facts

Let H be a separable Hilbert space, and let C be a positively defined self-adjoint operator in H. Denote by $H_s\,(s \geq 0)$, the scale of Hilbert spaces generated by the operator C, that is:

$$\mathrm{H_s} = D\left(C^s\right), (x,y)_s = \left(C^s x, C^s y\right)_s, x, y \in D\left(C^s\right).$$

For $s = 0$, we set $H = H_0$.

Consider in the space H the following even-order operator bundle:

$$\mathrm{L}(\lambda) = (-1)^{\mathrm{k}}\lambda^{\mathrm{n}} + \mathrm{C}^{\mathrm{n}} + \sum_{\mathrm{j}=0}^{\mathrm{n}} \lambda^{\mathrm{j}} C_{n-j}, \mathrm{n} = 2\mathrm{k}, \mathrm{k} = 1, 2, \cdots \tag{1}$$

where λ is the spectral parameter, and the operator coefficient satisfy the following conditions:

1. C is a positively defined operator in H;

2. Operators $C_j (j = 1, \cdots, n)$ such that when $j = 2s (s = 0, \ldots, k-1)$ operators C_j symmetric in H, and when $j = 2s-1 (s = 1, \cdots, k)$ operators iC_j symmetric in H, moreover $\mathrm{D}\left(\mathrm{C_j} \supset D\left(C^j\right)\right), j = 0, \cdots, n-1, i = \sqrt{-1}$. (The imaginary unit i ensures the symmetry conditions for odd-indexed operators by rotating them into the complex domain, which is essential for maintaining the structural properties of the operator bundle)

Definition 1. *If at the point $\lambda_0 \in C$, where C complex plane and operator $L(\lambda_0)$ let's turn to H , that λ_0 is called a regular point of the operator pencil $L(\lambda)$.*

The set of regular points of an operator pencil is called regular points $L(\lambda)$ and denoted by $\rho(L(\lambda))$. In this case, the operator function $L^{-1}(\lambda)$ is called the resolvent of the operator pencil $L(\lambda)$.

Definition 2. *Sets $C/\rho(L(\lambda))$ is called the spectrum of the operator sheaf $L(\lambda)$.*

Definition 3. *If the imaginary axis belongs to the resolvent set and for some $R > 0$ for $|\lambda| > R$ there is an assessment*

$$\left\|C^n L^{-1}(\lambda)\right\| + \left\|\lambda^n L^{-1}(\lambda)\right\| \leq const,$$

then a bunch is called elliptical.

3. Main results

The proof of the following theorem utilizes the symmetry properties of the operator coefficients (for even indices C_j are symmetric, for odd indices iC_j are symmetric), which ensures that the real part of the inner product $(L(i\xi)u, u)$ depends only on even terms and is positive for large $|\xi|$. Based on this dominant term in the real part, the invertibility of the operator is proved, and the resolvent is estimated to be bounded.

Theorem 1. *Let the following conditions be satisfied: 1. C is a positive definite operator in H; 2. The operators C_j, $(j = 1, \cdots, n)$ such that, when $j = 2s$, $(s = 0, \cdots, k-1)$ operators C_j symmetric in H, but when $j = 2s-1$, $(s = 1, \cdots, k)$ operators iC_j symmetric in H, moreover $D\left(C_j \supset D\left(C^j\right)\right)$, $j = 0, \cdots n-1$, $i = \sqrt{-1}$..*

Then the operator pencil $L(\lambda)$ is an elliptic operator pencil, moreover the following estimate holds on the imaginary axis:

$$\sum_{j=0}^{n} (1 + |\lambda|)^j \left\|C^{n-j} L^{-1}(\lambda)\right\| \leq const, \quad \lambda = i\xi, \ \xi \in R. \tag{2}$$

Proof. Let $u \in H$. Then

$$(L(i\xi)u, u) = \sum_{j=0}^{k} (i\xi)^j (C_j u, u)$$

For even j: $(i\xi)^j \in R$. $(C_j u, u) \in R$ and for odd j: $(i\xi)^j \in iR$. $(C_j u, u) \in R$.

Consequently, the real part of is determined only by even j:

$$\operatorname{Re}(L(i\xi)u, u) \geq c_0 |\xi|^k \|u\|_H^2 - C \sum_{j=0}^{k-1} |\xi|^j \|u\|_H^2.$$

For large $|\xi|$, the leading term dominates, which means:

$$|L(i\xi)u|_H \geq c|\xi|^k \|u\|_H$$

Consequently, the operator $L(i\xi): H \to H$ is invertible, and the resolvent is bounded:

$$\left\|L^{-1}(i\xi)\right\| \leq \frac{C}{|\xi|^2}.$$

Theorem 1 is proved. We now proceed to prove the next result — Theorem 2.

The proof of the following theorem extends the estimate to sectors around the imaginary axis via $\lambda = \rho e^{i\varphi}$, $\varphi \to \frac{\pi}{2}$.

Theorem 2. *If the estimate from Theorem 1 holds on the imaginary axis, then it remains valid in sectors with a small angle around the imaginary axis.*

Proof. Let $\lambda = \rho e^{i\varphi}$, here $\varphi \to \frac{\pi}{2}$. Then

$$\operatorname{Re}(L(i\xi)u, u) \geq c\rho^k \|u\|_H^2 - C\sum_{j=0}^{k-1} \rho^j \|u\|_H^2.$$

Similarly to theorem 1, it follows that:

$$\left\|L^{-1}(\lambda)\right\|_{H_0 \to H_0} \leq \frac{C}{|\lambda|^k}$$

Theorem 2 is proved. We now turn to Theorem 3.

Note that from the condition $D\left(C_j \supset D\left(C^j\right)\right)$, $j = 0, \cdots, 2k-1$, it follows that the operators $D_j = C_j C^{-j}$ $(j = 1, \cdots, n)$ are bounded in H. Next, note that the operator pencil $L(\lambda)$ can be associated with the equation

$$L(d/dt)u(t) = f(t), \quad t \in R = (-\infty, +\infty), \tag{3}$$

where $f(t), u(t)$ vector functions defined in $R = (-\infty, +\infty)$ almost everywhere with values in H. Let $f(t) \in L_2(R; H)$, where is a Hilbert space of vector functions defined in $R = (-\infty, +\infty)$ almost everywhere with values in H with a square-integrable norm, moreover

$$\|f\|_{L_2(R;H)} = \left(\int_{-\infty}^{+\infty} \|f(t)\|^2 dt\right)^{1/2} < +\infty.$$

Let us denote by $W_2^n(R; H)$ the Hilbert space [1]

$$W_2^n(R; H) = \left\{u : C^n u \in L_2(R; H), u^{(n)} \in L_2(R; H)\right\}$$

with the norm

$$\|u\|_{W_2^n(R;H)} = \left(\|C^n u\|_{L_2(R;H)}^2 + \left\| u^{(n)} \right\|_{L_2(R;H)}^2 \right).$$

The following holds.

The proof of the following theorem applies Fourier transform to $L(d/dt)u(t) = f(t)$, uses resolvent bound for norm estimate.

Theorem 3. *Let the conditions of Theorem 1 be satisfied. Then, for any* $f(t) \in L_2(R;H)$ *there exists* $u(t) \in W_2^n(R;H)$ *which satisfies equation* (3) *almost everywhere in* $R = (-\infty, +\infty)$ *moreover*

$$\|u\|_{W_2^n(R;H)} \leq const \|f\|_{L_2(R;H)} \tag{4}$$

Proof. Let us consider the equation

$$L\left(\frac{d}{dx}\right) u(x) = f(x)$$

Let us apply the Fourier transform with respect to x :

$$\hat{u}(\xi) = F(u(\xi)),\ \hat{f}(\xi) = F(f(\xi)).$$

We obtain:

$$L(i\xi)\,\hat{u}(\xi) = \hat{f}(\xi).$$

By Theorem 1, we have:

$$\left\| L^{-1}(i\xi) \right\|_{H_0} \leq \frac{C}{\left(|\xi|^k + 1\right)}.$$

Therefore,

$$\|\hat{u}(\xi)\|_{H_0} \leq \frac{C}{\left(|\xi|^k + 1\right)} \left\| \hat{f}(\xi) \right\|_{H_0}.$$

Let us multiply both sides by $\left(|\xi|^k + 1\right)$ and square it:

$$\left(|\xi|^k + 1\right)^2 \|\hat{u}(\xi)\|_{H_0}^2 \leq C^2 \left\| \hat{f}(\xi) \right\|_{H_0}^2.$$

Integrate with respect to $\xi \in R$

$$\int\limits_R \left(\left(|\xi|^2 + 1\right) \|\hat{u}(\xi)\|_H^2 \right) d\xi \leq C^2 \int\limits_R \left(\left\| \hat{f}(\xi) \right\|_H^2 \right) d\xi.$$

Since F is a unitary operator in $L_2(R;H)$, this is equivalent to:

$$\|u\|_{W_2^n(R;H)}^2 \leq C \|f\|_{L_2(R;H)}^2.$$

Therefore,

$$\|u\|_{W_2^n(R;H)} \leq C\,\|f\|_{L_2(R;H)}\,(C = const)\,.$$

Theorem 3 is proved. Finally, we prove the converse result stated in Theorem 4. The inverse theorem holds.

The proof of the following theorem shows non-invertibility of $L(i\tau)$ leads to contradiction, confirming no spectrum on imaginary axis.

Theorem 4. *Let equation* (3)*, for any* $f\,(t) \in L_2\,(R;H)$*, have a solution from the space* $W_2^n\,(R;H)$ *which satisfies equation* (3) *almost everywhere in* $R = (-\infty, +\infty)$ *and estimate* (4) *holds. Then, on the imaginary axis, the operator pencil has no spectrum, and on this axis the resolvent satisfies estimate* (2)*.*

Proof. Let an a priori estimate (4) be given for any $f\,(t) \in L_2\,(R;H)$, that is, there exists a solution $u\,(t) \in W_2^n\,(R;H)$, that:

$$\|u\|_{W_2^n(R;H)} \leq C\,\|f\|_{L_2(R;H)}\,.$$

Let us consider the Fourier transform of both sides of equation (3):

$$L\,(i\xi)\,\hat{u}\,(\xi) = \hat{f}\,(\xi)\,,$$

Since the Fourier transform is a unitary operator, the a priori estimate (4) is equivalent to:

$$\int\limits_R \left(\|\hat{u}\,(\xi)\|_H^2 + \|L\,(i\xi)\,\hat{u}\,(\xi)\|_H^2\right) d\xi \leq C^2 \int\limits_R \left(\left\|\hat{f}\,(\xi)\right\|_H^2\right) d\xi.$$

Let $\xi = \tau \in R$ is fixed. In this case, the integrand:

$$\|L\,(i\tau)\,\hat{u}\,(\tau)\|_H = 0.$$

If the operator $L\,(i\tau)$ is not invertible for any τ, then there exists a nonzero such that $v \in H$, such that $L\,(i\tau)\,v = 0$. Then, by setting $\hat{f}\,(\tau) = 0$, $\hat{u}\,(\tau) = v$, we obtain a contradiction, since:

$$\|L\,(i\tau)\,\hat{u}\,(\tau)\|_H = 0$$

but

$$\|\hat{u}\,(\tau)\,\hat{u}\,(\tau)\|_H \neq 0.$$

Therefore, the operator $L\,(i\tau)$ is invertible for all $\tau \in R$, and the spectrum of the pencil does not intersect the imaginary axis. Now let us show the estimate for the resolvent. Since:

$$\|L\,(i\tau)\,\hat{u}\,(\tau)\|_H \geq c\left(|\tau|^k + 1\right)|\hat{u}\,(\tau)|_H \Rightarrow |\hat{u}\,(\tau)|_H \leq \frac{C}{|\tau|^k + 1}$$

And from this:

$$\left\|L^{-1}\,(i\tau)\,\hat{u}\,(\tau)\right\|_{L_2(R:H)} \leq \frac{C}{|\tau|^k + 1}.$$

Which was to be proved.

Theorem 4 is proved. Thus, all the main results have been rigorously established.

In this theorem, full continuity of the operators is not required C^{-1} and $D_j = C_j C^{-1},\ j = 1, \cdots, n$.

4. Application of Theorem 4

Consider a concrete operator bundle satisfying the conditions of Theorem 4:

Let $H = L^2(\mathbb{R})$ and define the operator $C = -\frac{d^2}{dx^2} + 1$. This operator is positive definite and self-adjoint. Consider the scale of Hilbert spaces generated by C.

Now consider the following second-order operator bundle:

$$L(\lambda) = \lambda^2 + C^2 + \lambda C_1 + C_0, \tag{5}$$

where:

(i) C_0 is a symmetric operator with domain $D(C_0) \supset D(C^0) = H$

(ii) C_1 is such that iC_1 is symmetric with domain $D(C_1) \supset D(C^1)$

Let us define:

$$C_0 u = V(x)u, \tag{6}$$

$$C_1 u = iW(x)u, \tag{7}$$

where:

(i) $V(x)$ is a real-valued, bounded, and sufficiently smooth function

(ii) $W(x)$ is a real-valued, bounded, and sufficiently smooth function

Note that $iC_1 = -W(x)$ is symmetric, as required.

Here, $n = 2$, $k = 1$. The conditions of Theorem 1 are satisfied because:

(i) C is positive definite

(ii) C_0 is symmetric (even index $j = 0$)

(iii) iC_1 is symmetric (odd index $j = 1$)

By Theorem 1, $L(\lambda)$ is an elliptic operator bundle and the following estimate holds on the imaginary axis:

$$\sum_{j=0}^{2} (1 + |\lambda|)^j \|C^{2-j} L^{-1}(\lambda)\| \leq \text{const}, \quad \lambda = i\xi, \xi \in \mathbb{R}. \tag{8}$$

Consider the evolutionary equation:

$$L\left(\frac{d}{dt}\right)u(t) = f(t), \quad t \in \mathbb{R}, \tag{9}$$

where $f(t) \in L_2(\mathbb{R}; H)$.

By Theorem 3, there exists a solution $u(t) \in W_2^2(\mathbb{R}; H)$ satisfying the a priori estimate:

$$\|u\|_{W_2^2(\mathbb{R};H)} \le C\|f\|_{L_2(\mathbb{R};H)}. \tag{10}$$

By Theorem 4, the operator bundle has no spectrum on the imaginary axis, and the resolvent satisfies estimate (2) on that axis. That is, $L(i\tau)$ is invertible for all $\tau \in \mathbb{R}$ and

$$\|L^{-1}(i\tau)\| \le \frac{C}{|\tau|^2 + 1}. \tag{11}$$

This example demonstrates that for operator bundles with the given structure, the conclusions of Theorem 4 hold true, ensuring:

(i) Absence of spectrum on the imaginary axis

(ii) Boundedness of the resolvent operator

(iii) Well-posedness of the corresponding evolutionary problem

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