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Total Neighborhood Number in the Join and Corona of Graphs

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Abstract. A set S of vertices in a graph G is called a *neighborhood set* of G if G is the union of the subgraphs induced by the closed neighborhoods of the vertices in S. A subset $S_T \subseteq V(G)$ is a *total neighborhood set* of G if S_T is a neighborhood set and every vertex $u \in V(G)$ is adjacent to at least one vertex $v \in S_T$. The *neighborhood number* $n_0(G)$ (respectively, the *total neighborhood number* $n_t(G)$) of G is defined as the minimum cardinality of a neighborhood set (respectively, total neighborhood set) of G.
In this paper, the author characterizes the class of graphs that attain the lower bound of the neighborhood number, which is two. Furthermore, the paper presents the characterization of neighborhood sets in the join of graphs and of total neighborhood sets in both the join and corona of graphs. Exact values for the neighborhood number of the join of graphs and for the total neighborhood number of the join and corona are also established.

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1. Introduction

Let $G = (V(G), E(G))$ be a finite, simple, connected graph. For a vertex $v \in V(G)$, the *open neighborhood* of v is the set $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$, and the *closed neighborhood* is $N_G[v] = N_G(v) \cup \{v\}$.

A subset $S \subseteq V(G)$ is called a *neighborhood set* of G if

$$G = \bigcup_{v \in S} \langle N_G[v] \rangle,$$

that is, the union of the subgraphs induced by the closed neighborhoods of vertices in S covers G. The minimum cardinality of a neighborhood set is called the *neighborhood number* of G and is denoted by $n_0(G)$.

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The concept of neighborhood number was first introduced by Sampathkumar and Neeralagi [1] in 1985. They investigated fundamental properties, bounds, and extremal cases for graphs attaining small neighborhood numbers. Subsequent studies, such as those by Tahmasbzadehbaee, Soner, and Mojdeh [2], examined the behavior of $n_0(G)$ for various graph classes, extending the scope of the original work. In [3], Pescueso and Benacer provided a characterization of the neighborhood set of the corona of graphs and established the exact value of its neighborhood number.

A related invariant, the *total neighborhood number*, was introduced by Kulli and Patwari [4]. A subset $S \subseteq V(G)$ is a *total neighborhood set* of G if (i) S is a neighborhood set of G, and (ii) every vertex $u \in V(G)$ has a neighbor in S, that is, $N_G(u) \cap S \neq \varnothing$. The minimum cardinality of such a set is called the *total neighborhood number* of G and is denoted by $n_t(G)$. Kulli and Patwari established basic properties and bounds for $n_t(G)$ and explored its relationship with other domination-related parameters.

In this paper, we build upon these foundational studies to provide new characterizations and exact results for the neighborhood and total neighborhood numbers of graphs under some binary operations. Specifically, we:

(i) Characterize graphs that attain the lower bound $n_t(G) = 2$;

(ii) Present necessary and sufficient conditions for a subset S to be a (total) neighborhood set in the join $G + H$ and corona $G \circ H$ of graphs;

(iii) Derive exact values for $n_0(G + H)$, $n_t(G + H)$, and $n_t(G \circ H)$.

These results unify and extend the earlier works of Sampathkumar and Neeralagi [1], Kulli and Patwari [4], and Pescueso and Benacer [3], offering a deeper understanding of how neighborhood-based parameters behave under graph operations.

2. Results

Unless otherwise stated, the terminology and notation adopted in this paper are consistent with those of Harary [5].

Definition 1. [4] Let G be a simple graph with no isolated vertices. A set $S \subseteq V(G)$ is a *total neighborhood set* of G if:

(i) S is a neighborhood set of G, and

(ii) for every vertex u in G, there exists a vertex v in S such that u is adjacent to v.

The neighborhood number(respectively, *total neighborhood number*) of G, denoted by $n_0(G)$(respectively, $n_t(G)$), is the minimum cardinality of a neighborhood set (resp. total neighborhood set) of G.

Definition 2. The *degree* of a vertex v in graph G, denoted by $deg(v)$, is the number of edges incident with v, that is, $deg(v) = |N(v)|$. The minimum degree among the vertices of G is denoted $\delta(G)$ while $\Delta(G)$ is the largest such number.

Definition 3. The *distance* $d(u, v)$ between two vertices u and v in G is the length of a shortest path joining them if any; otherwise $d(u, v) = \infty$. The *eccentricity* $e(v)$ of a vertex v in a connected graph G is $\max d(u, v)$ for all u in G. The *radius* $r(G)$ is the minimum eccentricity of the vertices.

Definition 4. A *cut vertex* of a graph G is one whose removal increases the number of components.

Theorem 1. [4] *For any connected graph G with $p \geq 3$ vertices,*

$$2 \leq n_t(G) \leq p - 1.$$

The next result gives a characterization of graphs G whose total neighborhood number equals two.

Proposition 1. *Let G be a connected graph of order $n \geq 3$. Then $n_t(G) = 2$ if and only if one of the following conditions holds:*

(i) $\Delta(G) = n - 1$, *or*

(ii) there exists two adjacent cut vertices in G and $r(G) = 3$.

Proof. Suppose that $n_t(G) = 2$. Let $S = \{u, v\} \subseteq V(G)$ be a total neighborhood set of G. Then $e = uv \in E(G)$ and $G = \langle N[u]\rangle \cup \langle N[v]\rangle$. Consider the following cases:
Case 1: If $\langle N[u]\rangle$ is a subgraph of $\langle N[v]\rangle$, then every vertex $w \in V(G)\backslash\{v\}$, $wv \in E(G)$. This implies that $deg(v) = n - 1$. Similarly, if $\langle N[v]\rangle$ is a subgraph of $\langle N[u]\rangle$, then $deg(u) = n - 1$. In both cases, $\Delta(G) = n - 1$.
Case 2: If $\langle N[u]\rangle$ is not a subgraph of $\langle N[v]\rangle$ and $\langle N[v]\rangle$ is not a subgraph of $\langle N[u]\rangle$, let $T = N[v]\backslash N[u]$ and $U = N[u]\backslash N[v]$. Then $\langle T\rangle$ is a component in $G\backslash\{v\}$. Hence, v is a cut vertex in G. Also, $\langle U\rangle$ is a component in $G\backslash\{u\}$. Thus, u is a cut vertex in G. Let $a \in T$ and $b \in U$. Then $d(a, b) = 3$, $d(a, v) = d(b, u) = 1$, and $d(a, u) = d(b, v) = 2$. Thus, $e(a) = e(b) = 3$. This shows that $r(G) = 3$.

Conversely, suppose that condition (i) holds. Let $z \in V(G)$ and let $S = \{z, w\}$, where $deg(z) = n - 1$. Then clearly, S is a total neighborhood set in G. Hence $n_t(G) = 2$. Moreover, if condition (ii) holds, let $S = \{x, y\} \subseteq V(G)$ such that x and y are adjacent cut vertices in G. Since $r(G) = 3$, $d(c, x) = 1$ for every vertex c in $V(G)\backslash N[y]$. Similarly, $d(r, y) = 1$ for every vertex r in $V(G)\backslash N[x]$. Hence, $G = \langle N[x]\rangle \cup \langle N[y]\rangle$. This shows that S is a total neighborhood set in G. Thus, $n_t(G) = 2$. □

Let G_1 and G_2 be any graphs. The *join* of graphs G_1 and G_2, denoted by $G_1 + G_2$, is the graph with vertex set $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup C$, where $C = \{xy \mid x \in V(G_1), y \in V(G_2)\}$.

Theorem 1. *Let G and H be nontrivial connected graphs. Then $S \subseteq V(G + H)$ is a neighborhood set of $G + H$ if and only if one of the following conditions holds:*

(i) $S \cap V(G)$ is a neighborhood set of G, or

(ii) $S \cap V(H)$ *is a neighborhood set of* H.

Proof. Let $S \subseteq V(G+H)$ be a neighborhood set of $G+H$. Suppose that $S_1 = S \cap V(G)$ is not a neighborhood set of G. Then there exists $e = uv \in E(G)$ such that $e \notin E(\bigcup_{w \in S_1} \langle N_G[w] \rangle)$. This implies that either $u \notin N_G[w]$ or $v \notin N_G[w]$ for each $w \in S_1$. Without loss of generality, suppose that $u \notin N_G[w]$ for each $w \in S_1$. Since S is a neighborhood set of $G+H$, $uy \in E(\bigcup_{z \in S_2} \langle N_{G+H}[z] \rangle)$, for all $y \in V(H)$, where $S_2 = S \cap V(H)$. This implies that for every $y \in V(H)$, there exists $z \in S_2$ such that $y \in N_H[z]$. Thus, $H = \bigcup_{z \in S_2} \langle N[z] \rangle$. This shows that S_2 is a neighborhood of H.

The proof of the converse is straightforward. □

Corollary 1. *Let* G *and* H *be nontrivial connected graphs. Then*

$$n_0(G+H) = \min\{n_0(G), n_0(H)\}.$$

Proof. This result follows directly from Theorem 1. □

Theorem 2. *Let* G *and* H *be nontrivial connected graphs. Then* $S \subseteq V(G+H)$ *is a total neighborhood set of* $G+H$ *if and only if one of the following holds:*

(i) $S \cap V(G)$ *is a total neighborhood set of* G*, or*

(ii) $S \cap V(H)$ *is a total neighborhood set of* H*, or*

(iii) $S \cap V(G)$ *is a neighborhood set of* G *and* $S \cap V(H) \neq \varnothing$*, or*

(iv) $S \cap V(H)$ *is a neighborhood set of* H *and* $S \cap V(G) \neq \varnothing$.

Proof. Let G and H be nontrivial connected graphs and write $J = G+H$. For $S \subseteq V(J)$ put $S_G = S \cap V(G)$ and $S_H = S \cap V(H)$. Assume S is a total neighborhood set of J. In particular S is a neighborhood set of J. By Theorem 1, we know that one of the following holds:

(i) S_G is a neighborhood set of G, or (ii) S_H is a neighborhood set of H.

If $S_H = \varnothing$ then $S = S_G \subseteq V(G)$. Since S is a total neighborhood set of J, the restriction S_G covers G by closed neighborhoods and S_G totally dominates every vertex of G. Hence S_G is a total neighborhood set of G, which is case (i). Symmetrically, if $S_G = \varnothing$ then S_H is a total neighborhood set of H, which is case (ii).

So we may assume $S_G \neq \varnothing$ and $S_H \neq \varnothing$. By Theorem 1, at least one of S_G or S_H is a neighborhood set in its factor. If S_G is a neighborhood set of G, then we are in case (iii) (and note $S_H \neq \varnothing$); if S_H is a neighborhood set of H, then we are in case (iv).

Conversely, suppose one of the four listed conditions (i)–(iv) holds; we show S is a total neighborhood set of J.

Case (i). Assume S_G is a total neighborhood set of G (and $S_H = \varnothing$). Then S_G covers G by closed neighborhoods and totally dominates G. For each $g \in S_G$ we have

$N_J[g] = N_G[g] \cup V(H)$, so the closed neighborhoods of members of S_G also cover every vertex of H; hence they cover all of J. Total domination in J holds because S_G totally dominates G, and every vertex of H is adjacent to S_G. Thus, S is a total neighborhood set of J.

Case (ii) is symmetric.

Case (iii). Assume S_G is a neighborhood set of G and $S_H \neq \varnothing$. By Theorem 1, the fact that S_G is a neighborhood set of G and $S_H \neq \varnothing$ implies S is a neighborhood set of J. Note that every vertex of G is adjacent to some $h \in S_H$, and every vertex of H is adjacent to every vertex of S_G. Moreover, elements of S_G and S_H are mutually adjacent. Hence, every vertex of J has a neighbor in S, so S is a total dominating set of J. Therefore, S is a total neighborhood set of J.

Case (iv) is symmetric.

This completes the proof of Theorem 2. □

Corollary 2. *Let G and H be nontrivial connected graphs. Then for the join $G + H$ we have*

$$n_t(G + H) = \min\{\, n_t(G),\ n_t(H),\ n_0(G) + 1,\ n_0(H) + 1 \,\}.$$

Proof. Let $J = G + H$ and, for any $S \subseteq V(J)$, denote $S_G = S \cap V(G)$ and $S_H = S \cap V(H)$. By Theorem 2, every total neighborhood set S of J falls into exactly one of the four mutually exclusive types (i)–(iv) of that theorem.

Let S be an arbitrary total neighborhood set of J.

- If S is of type (i) in Theorem 2, i.e. S_G is a total neighborhood set of G and $S_H = \varnothing$, then $|S| = |S_G| \geq n_t(G)$ by definition of $n_t(G)$.
- If S is of type (ii), symmetrically $|S| \geq n_t(H)$.
- If S is of type (iii), i.e. S_G is a neighborhood set of G and $S_H \neq \varnothing$, then the restriction S_G must have size at least $n_0(G)$ (by definition of $n_0(G)$). Moreover, S_H contains at least one vertex, so $|S| \geq n_0(G) + 1$.
- If S is of type (iv), symmetrically $|S| \geq n_0(H) + 1$.

Thus, every total neighborhood set S of J has size at least

$$\min\{\, n_t(G),\ n_t(H),\ n_0(G) + 1,\ n_0(H) + 1 \,\},$$

which proves the lower bound

$$n_t(J) \geq \min\{\, n_t(G),\ n_t(H),\ n_0(G) + 1,\ n_0(H) + 1 \,\}.$$

We now show that each of the four candidate values is realized by some total neighborhood set of J, so the minimum of them is attainable.

(i) Take T_G a minimum total neighborhood set of G with $|T_G| = n_t(G)$. Regard T_G as a subset of $V(J)$ (that is, take the same vertices in the G-part and none from H). By Theorem 2 (case (i)), this set is a total neighborhood set of J, so $n_t(J) \leq n_t(G)$.

(ii) Similarly, take $T_H \subseteq V(H)$ with $|T_H| = n_t(H)$ to obtain $n_t(J) \leq n_t(H)$ (case (ii)).

(iii) Let $U_G \subseteq V(G)$ be a minimum neighborhood set of G with $|U_G| = n_0(G)$. Pick any single vertex $h \in V(H)$ and set $S = U_G \cup \{h\}$. Then $S_G = U_G$ is a neighborhood set of G and $S_H \neq \varnothing$, so by Theorem 2 (case (iii)) S is a total neighborhood set of J. Hence $n_t(J) \leq n_0(G) + 1$.

(iv) Symmetrically, taking a minimum neighborhood set $U_H \subseteq V(H)$ and any vertex $g \in V(G)$ produces a total neighborhood set of size $n_0(H)+1$, so $n_t(J) \leq n_0(H)+1$.

Combining these four constructions yields the upper bound

$$n_t(J) \leq \min\{\, n_t(G),\ n_t(H),\ n_0(G) + 1,\ n_0(H) + 1\,\}.$$

Therefore

$$n_t(G + H) = \min\{\, n_t(G),\ n_t(H),\ n_0(G) + 1,\ n_0(H) + 1\,\},$$

as claimed. □

The corona of two graphs G and H, denoted by $G \circ H$, is the graph obtained by taking one copy of G of order n and n copies of H, and then joining the i-th vertex of G to every vertex in the i-th copy of H. For every $v \in V(G)$, we denote by $B_v = v + H^v$ the subgraph of $G \circ H$ corresponding to the join $\langle\{v\}\rangle + H^v$. The next result characterizes a total neighborhood set for a corona of two graphs.

Theorem 3. *Let G be a connected graph with $|V(G)| \geq 2$, let H be any graph, and let $J = G \circ H$ be the corona of G by H. A set $S \subseteq V(J)$ is a total neighborhood set of J if and only if the following three conditions hold:*

(i) For every $v \in V(G)$ the set $S_v = S \cap V(B_v)$ is a neighborhood set of B_v.

(ii) For every $v \in V(G)$ and every $x \in V(H^v)$ we have $N_{B_v}(x) \cap S_v \neq \varnothing$.

(iii) $S_G = S \cap V(G)$ is a vertex cover of G, and for every $v \in V(G)$ either $S_v \cap V(H^v) \neq \varnothing$ or $N_G(v) \cap S_G \neq \varnothing$.

Proof. Assume S is a total neighborhood set of J. Thus $J = \bigcup_{s \in S} \langle N_J[s] \rangle$ and for every $y \in V(J)$, there exists an $s \in S$ such that $s \in N_J(y)$. Let $v \in V(G)$. Let $x \in V(H^v)$ be arbitrary. Since S is a total dominating set of J, there exists some $s \in S$ with $s \in N_J(x)$. However, any neighbor of x in J lies in B_v. Therefore s lies in B_v, i.e. $s \in S_v$. This proves that every $x \in V(H^v)$ has a neighbor in S_v, which is condition (ii).

Next, let $y \in V(B_v)$ be arbitrary. Because S covers J by closed neighborhoods, there exists some $s \in S$ such that $y \in N_J[s]$. If $y \in H^v$ then by the same adjacency restriction, s must belong to B_v, hence $y \in N_{B_v}[s]$ with $s \in S_v$. If $y = v$, then $s \in V(B_v)$ or $s \in V(B_{v'})$

for some $v' \in V(G)$ with $v \neq v'$. In the former case we get $y \in N_{B_v}[s]$ with $s \in S_v$. Thus, every vertex of B_v is contained in the union of closed neighborhoods of elements of S_v, so S_v is a neighborhood set of B_v. This proves (i).

Let $uv \in E(G)$ be any edge of G. The edge uv is an edge of J, so it must be contained in the union $\bigcup_{s\in S}\langle N_J[s]\rangle$. Hence there exists some $s \in S$ with $\{u, v\} \subseteq N_J[s]$. No vertex of any copy H^w can be adjacent to two distinct vertices in G, so $s \in V(G)$. Consequently, at least one of u or v belongs to $S_G = S \cap V(G)$. Since uv was arbitrary, S_G is a vertex cover of G.

Finally, let $v \in V(G)$, if $S_v \cap V(H^v) \neq \varnothing$ then we are done. So suppose that $S_v \cap V(H^v) = \varnothing$. Then $v \in S$, that is, $v \in S_G$; otherwise, every vertex in H^v is not dominated by any element of S, a contradiction. It implies that S_v is not a total dominating set of B_v, and $S_v = \{v\}$. Since S is a total dominating set of J, it follows that $N_G(v) \cap S_G \neq \varnothing$. Thus (iii) is satisfied.

Conversely, assume (i)–(iii) hold, and prove S is a total neighborhood set of J. Let $z \in V(J)$. Then $z \in V(B_v)$ for some $v \in V(G)$. By (i), S_v is a neighborhood set of B_v, so there exists $s \in S_v \subseteq S$ with $z \in N_{B_v}[s]$. Since $N_{B_v}[s] \subseteq N_J[s]$, we have $z \in N_J[s]$. As z was arbitrary, $J = \bigcup_{s\in S}\langle N_J[s]\rangle$, i.e. S is a neighborhood set of J.

Moreover, for any edge $uv \in E(G)$ condition (iii) guarantees at least one of u or v lies in $S_G \subseteq S$; if $u \in S$ then $N_J[u]$ contains both u and v, hence the edge uv is contained in $\langle N_J[u]\rangle$. Thus, every edge of J is contained in the union of the $\langle N_J[s]\rangle$.

Let $z \in V(J)$ be arbitrary. If $z \in V(H^v)$ for some $v \in V(G)$, then by (ii) there exists $s \in S_v \subseteq S$ with $s \in N_{B_v}(z) \subseteq N_J(z)$; hence z has a neighbor in S. If $z = v$, then by (iii) either S_v contains a vertex of H^v (which is adjacent to v) or v is adjacent to some vertex in S_G; in either case v has a neighbor in S. Therefore, every vertex of J has a neighbor in S, i.e. S is a total dominating set of J. Hence, S is a total neighborhood set of J. □

Corollary 3. *Let G be a connected graph of order n and let H be any graph of order m. Then the total neighborhood number of the corona $G \circ H$ is given by*

$$n_t(G \circ H) = \begin{cases} 2, & \text{if } n = 1, \\ n, & \text{otherwise.} \end{cases}$$

Proof. Let G be a connected graph of order n and let H be any graph of order m. If $n = 1$, then $\Delta_{G+H}(v) = m = |V(G + H)| - 1$, where $v \in V(G)$. By Proposition 1, $n_t(G + H) = 2$. Let $n \geq 2$ and let $S = V(G)$. By Theorem 3, S is a total neighborhood of $G \circ H$. Thus, $n_t(G \circ H) \leq |S| = n$.

Next, let S^* be a minimum total neighborhood set of $G \circ H$. By Theorem 3, $|S^* \cap V(B_v)| \geq 1$ for every $v \in V(G)$. It follows that $n_t(G \circ H) = |S^*| \geq |V(G)| = n$. Hence, $n_t(G \circ H) = n$. □

3. Conclusion

This paper establishes the neighborhood and total neighborhood numbers of graphs under the join and corona operations. We first characterized all connected graphs with total neighborhood number $n_t(G) = 2$, showing that these are exactly the graphs with a universal vertex or those containing adjacent cut vertices with radius three.

For the join $G + H$, we provided necessary and sufficient conditions for a set to be a (total) neighborhood set and derived exact formulas for $n_0(G + H)$ and $n_t(G + H)$. In particular, we proved that $n_0(G + H) = \min\{n_0(G), n_0(H)\}$.

For the corona $G \circ H$, we completely described all total neighborhood sets and determined the exact value of $n_t(G \circ H)$.

These results extend existing work on neighborhood-based parameters and clarify their behavior under two fundamental graph operations.

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