**EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS**
2026, Vol. 19, Issue 1, Article Number 7203
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

# Various Properties of Convoluted Special Polynomials Associated with Bell Polynomials and Their Algebraic Characteristics

Sarah Aljohani[1], Shahid Ahmad Wani[2,*], Prakash Jadhav[3]

[1] *Department of Mathematics and Sciences, Prince Sultan University, Riyadh 11586, Saudi Arabia*
[2] *Symbiosis Institute of Technology Pune, Symbiosis International (Deemed) University, Pune, India*
[3] *Department of Mechanical Engineering, SRM University AP, Andhra Pradesh, India*

---

**Abstract.** This paper uses generating functions to present Bell polynomials of two variables associated with Appell sequences. We investigate their diverse properties, including explicit representations, summation formulae, recurrence relations, and addition formulas. Furthermore, we introduce the second kind's Bell-Appell-based Stirling polynomials in two variables and outline their associated findings. This research enhances comprehension regarding the attributes and utility of Bell-Appell and Bell-Appell based Stirling polynomials in mathematical analysis. Further, determinant representation, operational identity, and other characteristics are derived.

**2020 Mathematics Subject Classifications**: 33E20, 33C45, 33B10, 33E30, 11T23
**Key Words and Phrases**: Hybrid special polynomials, monomiality principle, explicit form, summation formulae, Stirling numbers, determinant representation, operational rule

---

## 1. Introduction and preliminaries results

Recent mathematical research has led to significant advances in the study of special polynomials by extending their general forms. These developments have uncovered novel properties, identities, and wide-ranging applications, enriching both pure and applied mathematical disciplines. Special polynomials—distinguished by their intrinsic structures and recurring roles in mathematical contexts—encompass various celebrated families, including Legendre, Chebyshev, Hermite, Bell, Appell, and Touchard polynomials. These families are integral to numerous areas such as mathematical physics, engineering, computer science, and numerical computation. For comprehensive background, refer to sources like [1–5].

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v19i1.7203

*Email addresses:* sjohani@psu.edu.sa (S. Aljohani),
shahidwani177@gmail.com, shahid.wani@sitpune.edu.in (S. A. Wani),
prakash.j@srmap.edu.in (P. Jadhav)

 

The significance of these polynomials lies in their frequent emergence in solving differential equations, exploring orthogonal systems, and performing symbolic computations. Their recurrence relations and algebraic structures make them essential tools in advancing modern algebra, combinatorics, and analysis. These connections underscore their interdisciplinary utility and theoretical depth.

Exponential operators play a central role in the analysis of differential equations, offering elegant techniques for transforming and simplifying complex expressions. Bell's pioneering work [2] highlights how such operators function as generalizations of shift operators, allowing for seamless transformation via variable substitution. Specifically, the shift operator $\exp(\mu\partial_{\lambda_1})$ applied to any differentiable function $f(\lambda_1)$ of degree $r$ yields:

$$\exp(\mu\partial_{\lambda_1})\{f(\lambda_1)\} = \sum_{r=0}^{\infty} \partial_{\lambda_1}^r f(\lambda_1)\frac{\mu^r}{r!} = \sum_{r=0}^{\infty} f^r(\lambda_1)\frac{\mu^r}{r!} = f(\lambda_1+\mu), \tag{1}$$

where $\partial_{\lambda_1}^r = \frac{\partial^r}{\partial_{\lambda_1}^r}$.

Several additional identities can be derived from (8), such as

$$\exp(\mu\ \lambda_1^2\partial_{\lambda_1})\{f(\lambda_1)\} = f\Big(\frac{\lambda_1}{1-\mu\lambda_1}\Big), \tag{2}$$

$$\exp(\mu\partial_{\lambda_1})\{q_1^r\} = \Big(\lambda_1+\mu\Big)^r, \tag{3}$$

$$\exp(\mu\partial_{\lambda_1}^r)\{e^{\lambda_1}\} = e^{\lambda_1+\mu}, \tag{4}$$

$$\exp(\mu\lambda_1\ \partial_{\lambda_1})f\{\lambda_1\} = f(e^{\lambda_1}\mu). \tag{5}$$

Among the classical families, Bell polynomials, denoted by ${}_{B_r}(\lambda_1)$, hold particular prominence in enumerative combinatorics. Introduced by Eric Temple Bell, these polynomials are defined via the exponential generating function:

$$\sum_{r=0}^{\infty} B_r(\lambda_1)\frac{\vartheta^r}{r!} = e^{\lambda_1(e^{\vartheta}-1)}. \tag{6}$$

In the specific case $\lambda_1 = 1$, they yield the classical Bell numbers:

$$\sum_{r=0}^{\infty} B_r\frac{\vartheta^r}{r!} = e^{e^{\vartheta}-1}. \tag{7}$$

The extension to Bell polynomials in two variables provides deeper insights into combinatorial structures such as partitions, algebraic generation, and discrete analysis. These polynomials have far-reaching implications in algorithm analysis, probability theory, and symbolic computation, serving as essential instruments for expressing partial derivatives of exponential generating functions. For further insights, see [6–17]. The Stirling numbers play a fundamental role in combinatorial analysis, particularly in the study of permutations and partitions. The Stirling numbers of the first kind, $S_1(r,\epsilon)$, represent the number of permutations of $r$ distinct objects that decompose into exactly $\epsilon$ cycles. On the other hand, the Stirling numbers of the second kind, $S_2(r,\epsilon)$, quantify the number of distinct ways to divide a set of $r$ elements into $\epsilon$ non-empty subsets, where the subsets are considered indistinguishable.

The generating function for the Stirling polynomials of the second kind, parameterized by $S_2(r,\epsilon;\lambda_1)$ , is given by:

$$\sum_{r=0}^{\infty} \frac{S_2(r,\epsilon;\lambda_1)\ \vartheta^r}{r!} = \frac{(e^{\vartheta\lambda_1}-1)^{\epsilon}}{\epsilon!}, \tag{8}$$

were $\epsilon$ is assumed to be a non-negative integer and for $\lambda_1 = 1$, it simplifies to:

$$\sum_{r=o}^{\infty} \frac{S_2(r,\epsilon)\ \vartheta^r}{r!} = \frac{(e^{\vartheta}-1)^{\epsilon}}{\epsilon!}, \tag{9}$$

Their recurrence relations provide further analytical utility:

$$\lambda_1^r = \sum_{r=0}^{\infty} S_2(r,\epsilon)\ (\lambda_1)_{\epsilon}, \tag{10}$$

or

$$(\lambda_1)_r = \sum_{\epsilon=0}^{r} S_2(r,\epsilon)\ \lambda_1^{\epsilon}, \tag{11}$$

where the falling factorial is given by $(\lambda_1)_\epsilon = \lambda_1(\lambda_1-1)(\lambda_2-2)\cdots(\lambda_1-(\epsilon-1))$.

In addition, the integer power sum $\epsilon \in \mathbb{N}_0$, the expression

$$S_{\epsilon}(r) = \sum_{l=0}^{r} l^{\epsilon} \tag{12}$$

has the exponential generating function:

$$\sum_{\epsilon=0}^{\infty} S_{\epsilon}(r)\ \frac{\vartheta^{\epsilon}}{\epsilon!} = \frac{e^{(r+1)\vartheta-1}}{e^{\vartheta}-1}. \tag{13}$$

Another noteworthy class is that of Appell polynomials, introduced by Paul Appell [18]. Defined as solutions to specific differential equations, Appell polynomials are celebrated for their structured generating functions, recurrence formulas, and explicit expressions. They are foundational in areas such as statistical mechanics, operational calculus, and analytical combinatorics. The generating function for Appell sequences is given by:

$$\mathcal{R}(\vartheta)e^{\lambda_1\vartheta} = \sum_{r=0}^{\infty} \frac{\vartheta^r}{r!}\mathcal{R}_r(\lambda_1), \tag{14}$$

where

$$\mathcal{R}(\vartheta) = \sum_{r=0}^{\infty} \frac{\vartheta^r}{r!}\mathcal{R}_r; \quad \mathcal{R}_0 \neq 0. \tag{15}$$

The structure of article is:

Section 2 introduces the concept of two-variable Bell-based Appell polynomials through generating functions. It elaborates on their closed-form expressions, recurrence and summation relations, and addition formulas. Connections with Stirling polynomials of the second kind are also emphasized.

Section 3 focuses on Bell-Appell based Stirling polynomials of the second kind, outlining their properties and extending their theoretical implications.

Section 4 focuses to derive the determinant form, operational formalism and other properties of these polynomials.

The conclusion summarizes the contributions of the study, highlighting the potential applications and mathematical richness of the explored polynomial families.

## 2. Bell based Appell polynomials

The Bell-based Appell polynomials in two variables provide an effective framework for the analytical treatment of differential equations arising in mathematical physics and engineering. As generalizations of classical polynomial families, they admit compact representations of solutions to partial differential and integral equations, with applications in quantum mechanics, statistical mechanics, and electromagnetic theory. Generating functions play a central role in encoding such polynomial sequences and deriving structural properties. Motivated by this, we introduce the two-variable convoluted Bell–Appell polynomials ($2VBAP$) via the generating function

$$\sum_{r=0}^{\infty} {}_{B}\mathcal{R}_r(\lambda_1,\lambda_2)\frac{\vartheta^r}{n!} = \mathcal{R}(\vartheta)\, e^{\lambda_1(e^{\vartheta}-1)+\lambda_2(e^{\vartheta}-1)^2}. \tag{16}$$

Explicit representations and summation formulas for ${}_{B}\mathcal{R}_r(\lambda_1,\lambda_2)$ are derived, yielding compact expressions that enhance both theoretical insight and computational efficiency.

**Theorem 1.** *The $2VBAP$ ${}_{B}\mathcal{R}_r(\lambda_1,\lambda_2)$ admit the following series representation:*

$${}_{B}\mathcal{R}_r(\lambda_1,\lambda_2) = \sum_{s=0}^{r}\binom{r}{s}\mathcal{R}_{r-s}\; B_s(\lambda_1,\lambda_2). \tag{17}$$

*Proof.* From expression (15), the generating function (16) can be rewritten as

$$\mathcal{R}(\vartheta)e^{\lambda_1(e^{\vartheta}-1)+\lambda_2(e^{\vartheta}-1)^2} = \sum_{r=0}^{\infty}\mathcal{R}_r\frac{\vartheta^r}{r!}\sum_{s=0}^{\infty}B_s(\lambda_1,\lambda_2)\frac{\vartheta^s}{s!}. \tag{18}$$

Substituting the right-hand side of (16) into the left-hand side above and simplifying the resulting product, we obtain

$$\sum_{r=0}^{\infty} {}_{B}\mathcal{R}_r(\lambda_1,\lambda_2)\frac{\vartheta^r}{r!} = \sum_{r=0}^{\infty}\mathcal{R}_r\sum_{s=0}^{\infty}B_s(\lambda_1,\lambda_2)\frac{\vartheta^{r+s}}{r!\; s!}. \tag{19}$$

By reindexing the double series via the substitution $r$ with $r-s$ and rearranging terms, we derive

$$\sum_{r=0}^{\infty} {}_{B}\mathcal{R}_r(\lambda_1,\lambda_2)\frac{\vartheta^r}{r!} = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\binom{r}{s}\mathcal{R}_{r-s}B_s(\lambda_1,\lambda_2)\frac{\vartheta^r}{r!}. \tag{20}$$

Equating the coefficients of $\frac{\vartheta^r}{r!}$ on both sides completes the proof of (17).

**Theorem 2.** *The 2VBAP ${}_{B}\mathcal{R}_r(\lambda_1,\lambda_2)$ admits the explicit form:*

$$ {}_{B}\mathcal{R}_r(\lambda_1,\lambda_2) = \sum_{n=0}^{r}\sum_{m=0}^{n}\binom{r}{n} {}_{B}\mathcal{R}_{r-n}(\lambda_2)\lambda_1^m S_2(n,m). \tag{21}$$

*Proof.* The generating function (16) can be reformulated as

$$\mathcal{R}(\vartheta)e^{\lambda_1(e^{\vartheta}-1)+\lambda_2(e^{\vartheta}-1)^2} = \mathcal{R}(\vartheta)e^{\lambda_1(e^{\vartheta}-1)}\ \exp(\lambda_2(e^{\vartheta}-1)^2). \tag{22}$$

By inserting expressions (6) and (9) into the right-hand side of the preceding equation, we derive

$$\mathcal{R}(\vartheta)e^{\lambda_1(e^{\vartheta}-1)+\lambda_2(e^{\vartheta}-1)^2} = \sum_{r=0}^{\infty} {}_{B}\mathcal{R}_r(\lambda_2)\frac{\vartheta^r}{r!}\sum_{n=0}^{\infty}\sum_{m=0}^{n}{\lambda_1}^m S_2(n,m)\frac{\vartheta^n}{n!}. \tag{23}$$

By reindexing the double series via the substitution $r$ with $r-s$ in the right-hand side of the preceding expression becomes

$$\sum_{r=0}^{\infty} {}_{B}\mathcal{R}_r(\lambda_1,\lambda_2)\frac{\vartheta^r}{r!} = \sum_{r=0}^{\infty}\sum_{n=0}^{r}\sum_{m=0}^{n}\binom{r}{n} {}_{B}\mathcal{R}_{r-n}(\lambda_2)\lambda_1^m S_2(n,m)\frac{\vartheta^r}{r!}. \tag{24}$$

Equating the coefficients of $\frac{\vartheta^r}{r!}$ on both sides completes the proof of (21).

**Theorem 3.** *The 2VBAP ${}_{B}\mathcal{R}_r(\lambda_1,\lambda_2)$ admits series representation form:*

$$ {}_{B}\mathcal{R}_r(\lambda_1+u,\lambda_2+v) = \sum_{n=0}^{r}\binom{r}{r}\ {}_{B}\mathcal{R}_{r-n}(\lambda_1,\lambda_2)B_n(u,v). \tag{25}$$

*Proof.* The generating function (16) can be reformulated as

$$\begin{aligned}
\mathcal{R}(\vartheta)e^{(\lambda_1+u)(e^{\vartheta}-1)+(\lambda_2+v)(e^{\vartheta}-1)^2} &= \mathcal{R}(\vartheta)\ e^{\lambda_1(e^{\vartheta}-1)+\lambda_2(e^{\vartheta}-1)^2}\ e^{u(e^{\vartheta}-1)+v(e^{\vartheta}-1)^2}\\
&= \sum_{r=0}^{\infty}\sum_{n=0}^{\infty} {}_{B}\mathcal{R}_r(\lambda_1,\lambda_2)B_n(u,v)\frac{\vartheta^{r+n}}{r!n!}\\
&= \sum_{r=0}^{\infty}\sum_{n=0}^{r}\binom{r}{r}\ {}_{B}\mathcal{R}_{r-n}(\lambda_1,\lambda_2)B_n(u,v)\frac{\vartheta^r}{r!}.
\end{aligned}$$

Equating the coefficients of $\frac{\vartheta^r}{r!}$ on both sides completes the proof of the assertion (25).

**Theorem 4.** *The $2VBAP$ ${}_B\mathcal{R}_r(\lambda_1, \lambda_2)$ admits the summation formula:*

$$ {}_B\mathcal{R}_r(\lambda_1+\lambda_2, \lambda_3) = \sum_{k=0}^{r} \binom{r}{k} {}_B\mathcal{R}_k(\lambda_1, \lambda_3) B_{r-k}(\lambda_2). \tag{26} $$

*Proof.* By (16), we have

$$ \begin{aligned} \sum_{r=0}^{\infty} {}_B\mathcal{R}_r(\lambda_1+\lambda_2, \lambda_3)\frac{\vartheta^r}{r!} &= \mathcal{R}(\vartheta)e^{(\lambda_1+\lambda_2)(e^{\vartheta}-1)+\lambda_3(e^{\vartheta}-1)^2} \\ &= \sum_{r=0}^{\infty} {}_B\mathcal{R}_r(\lambda_1, \lambda_3)\frac{\vartheta^r}{r!} \sum_{r=0}^{\infty} B_r(\lambda_2)\frac{\vartheta^r}{r!} \\ &= \sum_{r=0}^{\infty} \left[ \sum_{k=0}^{r} \binom{r}{k} {}_B\mathcal{R}_k(\lambda_1, \lambda_3) B_{r-k}(\lambda_2) \right] \frac{\vartheta^r}{r!}. \end{aligned} $$

This gives the required expression as mentioned in 26, upon equating the coefficients of $\frac{\vartheta^r}{r!}$ on both sides.

**Theorem 5.** *Let $r \in \mathbb{N}$ be arbitrary, and assume ${}_B\mathcal{R}_0 = 1$. Then the following identity for the $2VBAP$ ${}_B\mathcal{R}_r(\lambda_1, \lambda_2)$ holds:*

$$ {}_B\mathcal{R}_r(\lambda_1+1, \lambda_2) - {}_B\mathcal{R}_r(\lambda_1, \lambda_2) = \sum_{k=1}^{r} \binom{r}{k} {}_B\mathcal{R}_{r-k}(\lambda_1, \lambda_2)\, B_k. \tag{27} $$

*Proof.* In (16), we have

$$ \begin{aligned} \sum_{r=0}^{\infty} \left[ {}_B\mathcal{R}_r(\lambda_1+1, \lambda_2) - {}_B\mathcal{R}_r(\lambda_1, \lambda_2) \right] \frac{\vartheta^r}{r!} &= \mathcal{R}(\vartheta)\, e^{(\lambda_1+1)(e^{\vartheta}-1)+\lambda_2(e^{\vartheta}-1)^2} - \mathcal{R}(\vartheta)\, e^{\lambda_1(e^{\vartheta}-1)+\lambda_2(e^{\vartheta}-1)^2} \\ &= \mathcal{R}(\vartheta)\, e^{\lambda_1(e^{\vartheta}-1)+\lambda_2(e^{\vartheta}-1)^2} \left( e^{e^{\vartheta}-1} - 1 \right) \\ &= \left( \sum_{m=0}^{\infty} {}_B\mathcal{R}_m(\lambda_1, \lambda_2)\frac{\vartheta^m}{m!} \right) \left( \sum_{k=1}^{\infty} B_k \frac{\vartheta^k}{k!} \right) \\ &= \sum_{r=0}^{\infty} \left[ \sum_{k=1}^{r} \binom{r}{k} {}_B\mathcal{R}_{r-k}(\lambda_1, \lambda_2)\, B_k \right] \frac{\vartheta^r}{r!}. \end{aligned} $$

Equating coefficients of $\vartheta^r/r!$ yields (27).

**Theorem 6.** *For $r > 1$, the $2VBAP$ $B\mathcal{R}r(\lambda_1, \lambda_2)$ satisfy the differential identities:*

$$ \frac{\partial}{\partial \lambda_1} {}_B\mathcal{R}_r(\lambda_1, \lambda_2) = \sum_{k=1}^{r} \binom{r}{k} {}_B\mathcal{R}_k(\lambda_1, \lambda_2). \tag{28} $$

*Proof.* Differentiating both sides of equation (16) with respect to $\lambda_1$, we obtain:

$$\begin{aligned}\frac{\partial}{\partial\lambda_1}\left[\sum_{r=0}^{\infty}{}_B\mathcal{R}_r(\lambda_1,\lambda_2)\frac{\vartheta^r}{r!}\right] &= \frac{\partial}{\partial\lambda_1}\left[\mathcal{R}(\vartheta)e^{\lambda_1(e^\vartheta-1)+\lambda_2(e^\vartheta-1)^2}\right]\\ &= \mathcal{R}(\vartheta)e^{\lambda_1(e^\vartheta-1)+\lambda_2(e^\vartheta-1)^2}(e^\vartheta-1)\\ &= \sum_{k=0}^{\infty}{}_B\mathcal{R}_k(\lambda_1,\lambda_2)\frac{\vartheta^k}{k!}\sum_{r=1}^{\infty}\frac{\vartheta^r}{r!}\\ &= \sum_{r=0}^{\infty}\sum_{k=1}^{r}\binom{r}{k}{}_B\mathcal{R}_k(\lambda_1,\lambda_2)\frac{\vartheta^r}{r!}.\end{aligned}$$

Comparing the coefficient of $\vartheta$, we obtain (28).

**Theorem 7.** *Let $r \geq 0$. Then the 2VBAP $\{{}_B\mathcal{R}_r(\lambda_1,\lambda_2)\}_{r\geq 0}$ satisfy the identity:*

$$\sum_{k=0}^{r}\left(\begin{array}{c} r\\ k\end{array}\right)\left[{}_B\mathcal{R}_k(\lambda_1+\lambda_3,\lambda_2){}_B\mathcal{R}_{r-k}(\lambda_2) - {}_B\mathcal{R}_{r-k}(\lambda_1,\lambda_2){}_B\mathcal{R}_r(\lambda_3,\lambda_2)\right] = 0.$$

*Proof.* From the generating functions:

$$\mathcal{R}(\vartheta)e^{\lambda_1(e^\vartheta-1)+\lambda_2(e^\vartheta-1)^2} = \sum_{r=0}^{\infty}{}_B\mathcal{R}_r(\lambda_1,\lambda_2)\frac{\vartheta^r}{r!} \tag{29}$$

and

$$\mathcal{R}(\vartheta)e^{\lambda_3(e^\vartheta-1)+\lambda_2(e^\vartheta-1)^2} = \sum_{r=0}^{\infty}{}_B\mathcal{R}_r(\lambda_3,\lambda_2)\frac{\vartheta^r}{r!} \tag{30}$$

of (29) and (30), we find

$$\begin{aligned}\mathcal{R}(\vartheta)e^{(\lambda_1+\lambda_3)(e^\vartheta-1)+\lambda_2(e^\vartheta-1)^2}e^{\lambda_2(e^\vartheta-1)^2} &= \left(\sum_{r=0}^{\infty}{}_B\mathcal{R}_r(\lambda_1,\lambda_2)\frac{\vartheta^n}{n!}\right)\left(\sum_{r=0}^{\infty}{}_B\mathcal{R}_r(\lambda_3,\lambda_2)\frac{\vartheta^r}{r!}\right)\\ \left(\sum_{r=0}^{\infty}{}_B\mathcal{R}_r(\lambda_1+\lambda_3,\lambda_2)\frac{\vartheta^r}{r!}\right)\left(\sum_{r=0}^{\infty}{}_B\mathcal{R}_r(0,\lambda_2)\frac{\vartheta^r}{r!}\right) &= \left(\sum_{r=0}^{\infty}{}_B\mathcal{R}_r(\lambda_1,\lambda_2)\frac{\vartheta^r}{r!}\right)\left(\sum_{r=0}^{\infty}{}_B\mathcal{R}_r(\lambda_3,\lambda_2)\frac{\vartheta^r}{r!}\right)\\ \sum_{r=0}^{\infty}\sum_{k=0}^{r}\binom{r}{k}{}_B\mathcal{R}_k(\lambda_1+\lambda_3,\lambda_2){}_B\mathcal{R}_{r-k}(\lambda_2)\frac{\vartheta^r}{r!} &= \sum_{r=0}^{\infty}\sum_{k=0}^{r}\binom{r}{k}{}_B\mathcal{R}_{r-k}(\lambda_1,\lambda_2){}_B\mathcal{R}_r(\lambda_3,\lambda_2)\frac{\vartheta^r}{r!}\\ \sum_{k=0}^{r}\binom{r}{k}{}_B\mathcal{R}_k(\lambda_1+\lambda_3,\lambda_2){}_B\mathcal{R}_{r-k}(\lambda_2) &= \sum_{k=0}^{r}\binom{r}{k}{}_B\mathcal{R}_{r-k}(\lambda_1,\lambda_2){}_B\mathcal{R}_r(\lambda_3,\lambda_2)\end{aligned}$$

Thus,

$$\sum_{k=0}^{r}\left(\begin{array}{c} n\\ k\end{array}\right)\left[{}_B\mathcal{R}_k(\lambda_1+\lambda_3,\lambda_2){}_B\mathcal{R}_{r-k}(\lambda_2) - {}_B\mathcal{R}_{r-k}(\lambda_1,\lambda_2){}_B\mathcal{R}_r(\lambda_3,\lambda_2)\right] = 0.$$

Equating both sides gives the claimed identity.

**Theorem 8.** *Let $r \geq 0$. Then the sequence $\{{}_B\mathcal{R}_r(\lambda_1, \lambda_2, \lambda_3, \lambda_4)\}_{r\geq 0}$ of $2VBAP$ satisfies:*

$$ {}_B\mathcal{R}_r(\lambda_1+\lambda_2, \lambda_3+\lambda_4) = \sum_{k=0}^{r} \binom{r}{k} {}_B\mathcal{R}_{r-k}(\lambda_1, \lambda_3) B_k(\lambda_2, \lambda_4). \tag{31} $$

*Proof.* Start with the identity:

$$ \begin{aligned} \mathcal{R}(\vartheta)e^{(\lambda_1+\lambda_2)(e^\vartheta-1)+(\lambda_3+\lambda_4)(e^\vartheta-1)^2} &= \mathcal{R}(\vartheta)e^{\lambda_1(e^\vartheta-1)+\lambda_2(e^\vartheta-1)+\lambda_3(e^\vartheta-1)^2+\lambda_4(e^\vartheta-1)^2} \\ &= \mathcal{R}(\vartheta)e^{\lambda_1(e^\vartheta-1+\lambda_3(e^\vartheta-1)^2}e^{\lambda_2(e^\vartheta-1+\lambda_4(e^\vartheta-1)^2} \end{aligned} $$

Applying this to the generating function in (16), we have

$$ \begin{aligned} \sum_{r=0}^{\infty} {}_B\mathcal{R}_r(\lambda_1+\lambda_2, \lambda_3+\lambda_4)\frac{\vartheta^r}{r!} &= \mathcal{R}(\vartheta)e^{(\lambda_1+\lambda_2)(e^\vartheta-1)+(\lambda_3+\lambda_4)(e^\vartheta-1)^2} \\ &= \mathcal{R}(\vartheta)e^{\lambda_1(e^\vartheta-1+\lambda_3(e^\vartheta-1)^2}e^{\lambda_2(e^\vartheta-1+\lambda_4(e^\vartheta-1)^2} \\ &= \sum_{r=0}^{\infty} {}_B\mathcal{R}_r(\lambda_1, \lambda_3)\frac{\vartheta^r}{r!} \sum_{k=0}^{\infty} B_k(\lambda_2, \lambda_4)\frac{\vartheta^k}{k!}. \end{aligned} $$

By rearranging the resulting series, we find

$$ \sum_{r=0}^{\infty} {}_B\mathcal{R}_r(\lambda_1+\lambda_2, \lambda_3+\lambda_4)\frac{\vartheta^r}{r!} = \sum_{r=0}^{\infty}\sum_{k=0}^{r} \binom{r}{k} {}_B\mathcal{R}_{r-k}(\lambda_1, \lambda_3) B_k(\lambda_2, \lambda_4)\frac{\vartheta^r}{r!}. $$

Hence, the identity (31) is established on equating the same coefficients on both sides.

## 3. Two-variable Bell-Appell based Stirling polynomials of the second kind

This section introduces the two-variable Bell-Appell-based Stirling polynomials of the second kind ($2VBASP$), detailing their core properties and structural relations. These polynomials form a distinct class with both theoretical and applied significance. Their definition is presented below.

**Definition 1.**

$$ \sum_{r=0}^{\infty} {}_B\mathcal{S}_2(r, \epsilon; \lambda_1, \lambda_2)\frac{\vartheta^r}{r!} = \frac{(e^\vartheta-1)^\epsilon}{\epsilon!}\ \mathcal{R}(\vartheta)e^{\lambda_1(e^\vartheta-1)+\lambda_2(e^\vartheta-1)^2}. \tag{32} $$

This definition serves as a foundational element, offering a theoretical platform for advancing the study of these polynomials and their significant roles across various branches of mathematical research and applications.

By setting $\lambda_2 = 0$ in equation (32), we derive a specialized subclass of polynomials known as the Bell-Appell-Stirling polynomials of the second kind. These are formally given by

$$ \sum_{r=0}^{\infty} {}_B\mathcal{S}_2(r, \epsilon; \lambda_1)\frac{\vartheta^r}{r!} = \frac{(e^\vartheta-1)^\epsilon}{\epsilon!}\ \mathcal{R}(\vartheta)e^{\lambda_1(e^\vartheta-1)}. \tag{33} $$

**Remark 1.** *Furthermore, upon additional substitution of* $\lambda_1 = \lambda_2 = 0$ *into the expression provided by* (32)*, we derive a collection of polynomials recognized as the Stirling numbers of the second kind, as delineated in* (9)*.*

**Theorem 9.** *For any non-negative integer* $r$*, the* $2VBASP$ *of the second kind* ${}_B\mathcal{S}_2(r,\epsilon;\lambda_1,\lambda_2)$ *admits the following identity:*

$$\sum_{l=0}^{r}\binom{r}{l}\mathcal{S}_2(l,\epsilon)B_{r-l}(\lambda_1,\lambda_2) = {}_B\mathcal{S}_2(r,\epsilon;\lambda_1,\lambda_2). \tag{34}$$

*Proof.* In consideration of expression (32), we find

$$\sum_{r=0}^{\infty}{}_B\mathcal{S}_2(r,\epsilon;\lambda_1,\lambda_2)\frac{\vartheta^r}{r!} = \frac{(e^{\vartheta}-1)^{\epsilon}}{\epsilon!}\ \mathcal{R}(\vartheta)e^{\lambda_1(e^{\vartheta}-1)+\lambda_2(e^{\vartheta}-1)^2}$$
$$= \sum_{r=\epsilon}^{\infty}\mathcal{S}_2(r,\epsilon)\frac{\vartheta^r}{r!}\ \sum_{r=0}^{\infty}{}_B\mathcal{R}_r(\lambda_1,\lambda_2)\frac{\vartheta^r}{r!} \tag{35}$$

and can be reformulated as follows:

$$\sum_{r=0}^{\infty}{}_B\mathcal{S}_2(r,\epsilon;\lambda_1,\lambda_2)\frac{\vartheta^r}{r!} = \sum_{r=0}^{\infty}\sum_{l=0}^{r}\binom{r}{l}\mathcal{S}_2(l,\epsilon)\quad {}_B\mathcal{R}_{r-l}(\lambda_1,\lambda_2)\frac{\vartheta^r}{r!}. \tag{36}$$

By aligning the exponents of corresponding powers of $\vartheta$, the desired result is derived.

**Remark 2.** *Upon substituting* $\lambda_2 = 0$ *into the expression provided by* (32)*, the correlation satisfied by the* $2VBASP$ *of the second kind is as follows:*

$$\sum_{l=0}^{r}\binom{r}{l}\mathcal{S}_2(l,\epsilon)B_{r-l}(\lambda_1) = {}_B\mathcal{S}_2(r,\epsilon;\lambda_1), \tag{37}$$

*for any non-negative integer* $n$*.*

**Theorem 10.** *For the* $2VBASP$ ${}_B\mathcal{S}_2(r,\epsilon;\lambda_1,\lambda_2)$*, the succeeding summation formula hold:*

$${}_B\mathcal{S}_2(r,\epsilon;\lambda_1+\lambda_3,\lambda_2) = \sum_{k=0}^{r}\binom{r}{k}{}_B\mathcal{S}_2(r-k,\epsilon;\lambda_1,\lambda_2)B_k(\lambda_3). \tag{38}$$

*Proof.* In consideration of expressions (32) and (6), we find

$$\sum_{r=0}^{\infty}{}_B\mathcal{S}_2(r,\epsilon;\lambda_1+\lambda_3,\lambda_2) = \frac{(e^{\vartheta}-1)^{\epsilon}}{\epsilon!}\ \mathcal{R}(\vartheta)e^{(\lambda_1+\lambda_3)(e^{\vartheta}-1)+\lambda_2(e^{\vartheta}-1)^2}$$
$$= \sum_{r=0}^{\infty}{}_B\mathcal{S}_2(r,\epsilon;\lambda_1,\lambda_2)\frac{\vartheta^r}{r!}\sum_{k=0}^{\infty}B_k(\lambda_3)\frac{\vartheta^k}{k!}$$

$$= \sum_{r=0}^{\infty}\left[\sum_{k=0}^{r}\binom{r}{k}{}_{B}\mathcal{S}_2(r-k,\epsilon;\lambda_1,\lambda_2)B_k(\lambda_3)\right]\frac{\vartheta^r}{r!}.$$

Finally, through the comparison of coefficients of $\frac{\vartheta^r}{r!}$ on both sides, we establish Theorem 10.

**Theorem 11.** *For any arbitrary $r \in \mathbb{N}$, the following correlation for the 2VBASP ${}_{B}\mathcal{S}_2(r,\epsilon;\lambda_1,\lambda_2)$ holds:*

$$ {}_{B}\mathcal{S}_2(r,\epsilon;\lambda_1+1,\lambda_2) - {}_{B}\mathcal{S}_2(r,\epsilon;\lambda_1,\lambda_2) = \sum_{k=0}^{r}\binom{r}{k}{}_{B}\mathcal{S}_2(r-k,\epsilon;\lambda_1,\lambda_2)B_k - {}_{B}\mathcal{S}_2(r,\epsilon;\lambda_1,\lambda_2). \tag{39}$$

*Proof.* In consideration of expression (16), we find

$$\sum_{r=0}^{\infty}\left[{}_{B}\mathcal{S}_2(r,\epsilon;\lambda_1+1,\lambda_2) - {}_{B}\mathcal{S}_2(r,\epsilon;\lambda_1,\lambda_2)\right]\frac{\vartheta^r}{r!} = \frac{(e^{\vartheta}-1)^{\epsilon}}{\epsilon!}\mathcal{R}(\vartheta)e^{(\lambda_1+1)(e^{\vartheta}-1)+\lambda_2(e^{\vartheta}-1)^2} -$$

$$\frac{(e^{\vartheta}-1)^{\epsilon}}{\epsilon!}\mathcal{R}(\vartheta)e^{\lambda_1(e^{\vartheta}-1)+\lambda_2(e^{\vartheta}-1)^2} = \frac{(e^{\vartheta}-1)^{\epsilon}}{\epsilon!}\mathcal{R}(\vartheta)e^{\lambda_1(e^{\vartheta}-1)+\lambda_2(e^{\vartheta}-1)^2}\left[e^{e^{z}-1}-1\right]$$

$$= \sum_{r=0}^{\infty}\left[\sum_{k=0}^{r}\binom{r}{k}{}_{B}\mathcal{S}_2(r-k,\epsilon;\lambda_1,\lambda_2)B_k - {}_{B}\mathcal{S}_2(r,\epsilon;\lambda_1,\lambda_2)\right]\frac{\vartheta^r}{r!}.$$

Finally, through the comparison of coefficients of $\frac{\vartheta^r}{r!}$ on both sides, we establish (39).

## Determinant Form and Operational Identities

**Theorem 12.** *For each $r \geq 0$, the 2VBAP ${}_{B}\mathcal{R}_r(\lambda_1,\lambda_2)$ admit the determinantal representation*

$$ {}_{B}\mathcal{R}_r(\lambda_1,\lambda_2) = \frac{(-1)^r}{\big(m_0(\lambda_2)\big)^{r+1}}\begin{vmatrix} m_0(\lambda_2) & B_1(\lambda_1,\lambda_2) & B_2(\lambda_1,\lambda_2) & \cdots & B_r(\lambda_1,\lambda_2) \\ m_1(\lambda_2) & m_0(\lambda_2) & 0 & \cdots & 0 \\ m_2(\lambda_2) & m_1(\lambda_2) & m_0(\lambda_2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m_r(\lambda_2) & m_{r-1}(\lambda_2) & m_{r-2}(\lambda_2) & \cdots & m_0(\lambda_2) \end{vmatrix}, \tag{40}$$

*where the moments $m_j(\lambda_2)$ are defined via the expansion*

$$G(\vartheta;\lambda_2) := \mathcal{R}(\vartheta)\,\exp\big(\lambda_2(e^{\vartheta}-1)^2\big) = \sum_{j=0}^{\infty} m_j(\lambda_2)\,\frac{\vartheta^j}{j!}. \tag{41}$$

*Proof.* By definition

$$\sum_{r=0}^{\infty} {}_B\mathcal{R}_r(\lambda_1,\lambda_2)\frac{\vartheta^r}{r!} = \mathcal{R}(\vartheta)\,\exp\bigl(\lambda_1(e^{\vartheta}-1)+\lambda_2(e^{\vartheta}-1)^2\bigr). \tag{42}$$

Introduce

$$G(\vartheta;\lambda_2) = \mathcal{R}(\vartheta)\,\exp\bigl(\lambda_2(e^{\vartheta}-1)^2\bigr) = \sum_{j\geq 0} m_j(\lambda_2)\frac{\vartheta^j}{j!}, \tag{43}$$

and

$$\sum_{s\geq 0} B_s(\lambda_1,\lambda_2)\frac{\vartheta^s}{s!} = \exp\bigl(\lambda_1(e^{\vartheta}-1)+\lambda_2(e^{\vartheta}-1)^2\bigr). \tag{44}$$

Equation (42) can be rewritten in two equivalent ways:

$$\sum_{r\geq 0} {}_B\mathcal{R}_r\frac{\vartheta^r}{r!} = G(\vartheta;\lambda_2)\,\exp\bigl(\lambda_1(e^{\vartheta}-1)\bigr) = \mathcal{R}(\vartheta)\sum_{s\geq 0} B_s(\lambda_1,\lambda_2)\frac{\vartheta^s}{s!}. \tag{45}$$

Utilizing the second factorization to set up a lower–triangular linear system. By the Cauchy product rule, we find

$${}_B\mathcal{R}_r(\lambda_1,\lambda_2) = \sum_{s=0}^{r}\binom{r}{s}\mathcal{R}_{r-s}\,B_s(\lambda_1,\lambda_2), \qquad r\geq 0, \tag{46}$$

with $\mathcal{R}_j = j![\vartheta^j]\mathcal{R}(\vartheta)$.

Likewise, from $G(\vartheta;\lambda_2) = \mathcal{R}(\vartheta)\exp(\lambda_2(e^{\vartheta}-1)^2)$ we obtain

$$m_r(\lambda_2) = \sum_{s=0}^{r}\binom{r}{s}\mathcal{R}_{r-s}\,c_s(\lambda_2), \qquad \sum_{s\geq 0} c_s(\lambda_2)\frac{\vartheta^s}{s!} = \exp\bigl(\lambda_2(e^{\vartheta}-1)^2\bigr). \tag{47}$$

Eliminating $\mathcal{R}_r$ between (46) and (47) (substitute the unique solution of the lower–triangular Toeplitz system (47) into (46)) yields another lower–triangular system of the form

$${}_B\mathcal{R}_r(\lambda_1,\lambda_2) = \sum_{s=0}^{r}\binom{r}{s} m_{r-s}(\lambda_2)\,B_s(\lambda_1,\lambda_2) \;-\; \sum_{s=0}^{r-1}\binom{r}{s}\eta_{r-s}(\lambda_2)\,{}_B\mathcal{R}_s(\lambda_1,\lambda_2), \tag{48}$$

for certain coefficients $\eta_j(\lambda_2)$ determined uniquely by (47). In matrix form, for each fixed $r$,

$$\underbrace{\begin{bmatrix} m_0(\lambda_2) & 0 & \cdots & 0 \\ m_1(\lambda_2) & m_0(\lambda_2) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ m_r(\lambda_2) & \cdots & m_1(\lambda_2) & m_0(\lambda_2) \end{bmatrix}}_{=:T_r(\lambda_2)} \begin{bmatrix} B_0(\lambda_1,\lambda_2) \\ B_1(\lambda_1,\lambda_2) \\ \vdots \\ B_r(\lambda_1,\lambda_2) \end{bmatrix} = \begin{bmatrix} {}_B\mathcal{R}_0(\lambda_1,\lambda_2) \\ {}_B\mathcal{R}_1(\lambda_1,\lambda_2) \\ \vdots \\ {}_B\mathcal{R}_r(\lambda_1,\lambda_2) \end{bmatrix}, \tag{49}$$

where $T_r(\lambda_2)$ is a lower–triangular Toeplitz matrix with diagonal entries $m_0(\lambda_2) = \mathcal{R}(0) \neq 0$; hence it is invertible. Solving this system by Cramer's rule for the last component (indexed by $r$) gives exactly the determinant in (40):

$$
{}_B\mathcal{R}_r(\lambda_1,\lambda_2) = \frac{\det\big(\widehat{T}_r(\lambda_2)\big)}{\det\big(T_r(\lambda_2)\big)} = \frac{(-1)^r}{\big(m_0(\lambda_2)\big)^{r+1}} \begin{vmatrix} m_0(\lambda_2) & B_1(\lambda_1,\lambda_2) & \cdots & B_r(\lambda_1,\lambda_2) \\ m_1(\lambda_2) & m_0(\lambda_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ m_r(\lambda_2) & m_{r-1}(\lambda_2) & \cdots & m_0(\lambda_2) \end{vmatrix}, \quad (50)
$$

where $\widehat{T}_r$ is obtained from $T_r$ by replacing its first column by the vector $\big(B_0, B_1, \ldots, B_r\big)^\top$ and expanding along the first row. This is the classical Sheffer–Toeplitz determinantal form adapted to the present data, and it proves (40).

**Theorem 13.** *Let $\Delta := e^{D_{\lambda_1}} - 1$, with $D_{\lambda_1} = \frac{\partial}{\partial\lambda_1}$. Then, for all $r \geq 0$, the 2VBAP ${}_B\mathcal{R}_r(\lambda_1,\lambda_2)$ admits the operational identity*

$$
{}_B\mathcal{R}_r(\lambda_1,\lambda_2) = \Big[\ \mathcal{R}(D_{\lambda_1})\ \exp\big(\lambda_2\,\Delta^2\big)\ \Big]\ B_r(\lambda_1), \quad (51)
$$

*where $B_r(\lambda_1)$ denotes the classical Bell polynomials. Moreover, the 2VBASP ${}_B\mathcal{S}_2(r,\epsilon;\lambda_1,\lambda_2)$ satisfy*

$$
{}_B\mathcal{S}_2(r,\epsilon;\lambda_1,\lambda_2) = \frac{1}{\epsilon!}\,\Delta^\epsilon\,\big({}_B\mathcal{R}_r(\lambda_1,\lambda_2)\big), \qquad r,\epsilon \in \mathbb{N}_0. \quad (52)
$$

*Proof.* We use the standard umbral interpretation of shift operators. For any sufficiently regular $F$, $e^{aD_{\lambda_1}}F(\lambda_1) = F(\lambda_1 + a)$; hence

$$
\Delta F(\lambda_1) = (e^{D_{\lambda_1}} - 1)F(\lambda_1) = F(\lambda_1 + 1) - F(\lambda_1), \quad (53)
$$

and

$$
\exp\big(\lambda_2\Delta^2\big) = \sum_{m\geq 0} \frac{\lambda_2^m}{m!}\Delta^{2m}. \quad (54)
$$

Recall the exponential generating function of the classical Bell polynomials:

$$
\sum_{r\geq 0} B_r(\lambda_1)\frac{\vartheta^r}{r!} = \exp\big(\lambda_1(e^\vartheta - 1)\big). \quad (55)
$$

Apply the operator $\mathcal{R}(D_{\lambda_1})\exp(\lambda_2\Delta^2)$ term wise to this exponential generating function. Since $D_{\lambda_1}$ acts only on $\lambda_1$ and commutes with $\vartheta$,

$$
\begin{aligned}
\sum_{r=0}^{\infty} \mathcal{R}(D_{\lambda_1})\exp(\lambda_2\Delta^2)\mathscr{B}_r(\lambda_1)\frac{\vartheta^r}{r!} &= \mathcal{R}(D_{\lambda_1})\exp(\lambda_2\Delta^2)\exp\Big(\lambda_1(e^\vartheta - 1)\Big) \\
&= \mathcal{R}(D_{\lambda_1})\exp\big(\lambda_2(e^{D_{\lambda_1}} - 1)^2\big)\exp(\lambda_1(e^\vartheta - 1)). \quad (56)
\end{aligned}
$$

Use the shift rule $e^{aD_{\lambda_1}}F(\lambda_1) = F(\lambda_1 + a)$ with $a = (e^{\vartheta} - 1)$ to evaluate $(e^{D_{\lambda_1}} - 1)$ on the exponential:

$$(e^{D_{\lambda_1}}-1)\ \exp\big(\lambda_1(e^{\vartheta}-1)\big) = \exp\big((\lambda_1+1)(e^{\vartheta}-1)\big)-\exp\big(\lambda_1(e^{\vartheta}-1)\big) = \big(e^{\vartheta}-1\big)\ \exp\big(\lambda_1(e^{\vartheta}-1)\big), \tag{57}$$

so that $(e^{D_{\lambda_1}} - 1)^2$ multiplies the same exponential by $(e^{\vartheta} - 1)^2$. Therefore

$$\exp\Big(\lambda_2\,(e^{D_{\lambda_1}} - 1)^2\Big)\ \exp\big(\lambda_1(e^{\vartheta} - 1)\big) = \exp\big(\lambda_2(e^{\vartheta} - 1)^2\big)\ \exp\big(\lambda_1(e^{\vartheta} - 1)\big). \tag{58}$$

Next, $\mathcal{R}(D_{\lambda_1})$ replaces $\vartheta$ by $D_{\lambda_1}$ inside $\mathcal{R}$ and then acts on the $\lambda_1$–exponential; but $D_{\lambda_1}$ does not affect $\vartheta$, hence it simply multiplies the exponential generating function by $\mathcal{R}(\vartheta)$:

$$\mathcal{R}(D_{\lambda_1})\ \exp\big(\lambda_1(e^{\vartheta} - 1)\big) = \mathcal{R}(\vartheta)\ \exp\big(\lambda_1(e^{\vartheta} - 1)\big). \tag{59}$$

Putting these steps together,

$$\sum_{r\geq 0}\Big(\mathcal{R}(D_{\lambda_1})\exp(\lambda_2\Delta^2)\,B_r(\lambda_1)\Big)\frac{\vartheta^r}{r!} = \mathcal{R}(\vartheta)\ \exp\big(\lambda_2(e^{\vartheta} - 1)^2\big)\ \exp\big(\lambda_1(e^{\vartheta} - 1)\big), \tag{60}$$

which is exactly the exponential generating function of ${}_B\mathcal{R}_r(\lambda_1, \lambda_2)$ by (42). Equating coefficients proves (51).

For (52), use the defining exponential generating function of 2VBASP:

$$\sum_{r\geq 0} {}_B\mathcal{S}_2(r, \epsilon; \lambda_1, \lambda_2)\frac{\vartheta^r}{r!} = \frac{(e^{\vartheta} - 1)^{\epsilon}}{\epsilon!}\,\mathcal{R}(\vartheta)\ \exp\big(\lambda_1(e^{\vartheta} - 1) + \lambda_2(e^{\vartheta} - 1)^2\big). \tag{61}$$

But $(e^{\vartheta} - 1)^{\epsilon}$ is the exponential generating function of $\Delta^{\epsilon}$ acting on functions of $\lambda_1$; precisely,

$$\Delta^{\epsilon}\ \exp\big(\lambda_1(e^{\vartheta} - 1)\big) = (e^{\vartheta} - 1)^{\epsilon}\ \exp\big(\lambda_1(e^{\vartheta} - 1)\big). \tag{62}$$

Hence

$$\sum_{r\geq 0} {}_B\mathcal{S}_2(r, \epsilon; \lambda_1, \lambda_2)\frac{\vartheta^r}{r!} = \frac{1}{\epsilon!}\,\Delta^{\epsilon}\big[\mathcal{R}(\vartheta)\ \exp\big(\lambda_2(e^{\vartheta} - 1)^2\big)\ \exp\big(\lambda_1(e^{\vartheta} - 1)\big)\big] = \frac{1}{\epsilon!}\,\Delta^{\epsilon}\left[\sum_{r\geq 0} {}_B\mathcal{R}_r(\lambda_1, \lambda_2)\frac{\vartheta^r}{r!}\right]. \tag{63}$$

Comparing coefficients yields (52).

**Theorem 14.** *For any $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{C}$ and $r \geq 0$, one has*

$${}_B\mathcal{R}_r(\lambda_1 + \mu_1, \lambda_2 + \mu_2) = \sum_{k=0}^{r}\binom{r}{k}\ {}_B\mathcal{R}_{r-k}(\lambda_1, \lambda_2)\ B_k(\mu_1, \mu_2), \tag{64}$$

*where $B_k(\mu_1, \mu_2)$ are the two-variable Bell polynomials. Similarly, the* 2VBASP *satisfy*

$${}_B\mathcal{S}_2(r, \epsilon; \lambda_1 + \mu_1, \lambda_2 + \mu_2) = \sum_{k=0}^{r}\binom{r}{k}\ {}_B\mathcal{S}_2(r - k, \epsilon; \lambda_1, \lambda_2)\ B_k(\mu_1, \mu_2). \tag{65}$$

*Proof.* Start from the defining generating function

$$\sum_{r\geq 0} {}_{B}\mathcal{R}_r(\lambda_1,\lambda_2)\frac{\vartheta^r}{r!} = \mathcal{R}(\vartheta)\ \exp\bigl(\lambda_1(e^{\vartheta}-1)+\lambda_2(e^{\vartheta}-1)^2\bigr). \tag{66}$$

Evaluate the same expression with $(\mu_1,\mu_2)$ in place of $(\lambda_1,\lambda_2)$ and multiply:

$$\begin{aligned}\left(\sum_{r\geq 0} {}_{\mathscr{B}}\mathcal{R}_r(\lambda_1,\lambda_2)\frac{\vartheta^r}{r!}\right)\left(\sum_{k\geq 0} \mathscr{B}_k(\mu_1,\mu_2)\frac{\vartheta^k}{k!}\right) &= \mathcal{R}(\vartheta)\ \exp\bigl(\lambda_1(e^{\vartheta}-1)+\lambda_2(e^{\vartheta}-1)^2\bigr)\\ \exp\left(\mu_1(e^{\vartheta}-1)+\mu_2(e^{\vartheta}-1)^2\right) &\\ &= \mathcal{R}(\vartheta)\ \exp\bigl((\lambda_1+\mu_1)(e^{\vartheta}-1)+(\lambda_2+\mu_2)(e^{\vartheta}-1)^2\bigr)\\ &= \sum_{r\geq 0} {}_{\mathscr{B}}\mathcal{R}_r(\lambda_1+\mu_1,\lambda_2+\mu_2)\frac{\vartheta^r}{r!}.\end{aligned}$$

Comparing coefficients of $\vartheta^r/r!$ on both sides gives (64).

For (65), use the exponential generating function definition

$$\sum_{r\geq 0} {}_{B}\mathcal{S}_2(r,\epsilon;\lambda_1,\lambda_2)\frac{\vartheta^r}{r!} = \frac{(e^{\vartheta}-1)^{\epsilon}}{\epsilon!}\,\mathcal{R}(\vartheta)\ \exp\bigl(\lambda_1(e^{\vartheta}-1)+\lambda_2(e^{\vartheta}-1)^2\bigr). \tag{67}$$

Multiply the generating functions corresponding to $(\lambda_1,\lambda_2)$ and $(\mu_1,\mu_2)$ exactly as above, now with the prefactor $(e^{\vartheta}-1)^{\epsilon}/\epsilon!$ common to both sides. The same Cauchy product argument yields

$$\sum_{r\geq 0} {}_{B}\mathcal{S}_2(r,\epsilon;\lambda_1+\mu_1,\lambda_2+\mu_2)\frac{\vartheta^r}{r!} = \left(\sum_{r\geq 0} {}_{B}\mathcal{S}_2(r,\epsilon;\lambda_1,\lambda_2)\frac{\vartheta^r}{r!}\right)\left(\sum_{k\geq 0} B_k(\mu_1,\mu_2)\frac{\vartheta^k}{k!}\right), \tag{68}$$

and coefficient extraction gives (65).

**Theorem 15.** *For each $r,\epsilon\geq 0$, the* 2VBASP *admit the determinantal representation*

$$ {}_{B}\mathcal{S}_2(r,\epsilon;\lambda_1,\lambda_2) = \frac{(-1)^r}{\bigl(m_0(\lambda_2)\bigr)^{r+1}}\begin{vmatrix} m_0(\lambda_2) & S_2(1,\epsilon;\lambda_1,\lambda_2) & \cdots & S_2(r,\epsilon;\lambda_1,\lambda_2)\\ m_1(\lambda_2) & m_0(\lambda_2) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ m_r(\lambda_2) & m_{r-1}(\lambda_2) & \cdots & m_0(\lambda_2)\end{vmatrix}, \tag{69}$$

*where $m_j(\lambda_2)$ are the same moments defined by*

$$G(\vartheta;\lambda_2) = \mathcal{R}(\vartheta)\exp\bigl(\lambda_2(e^{\vartheta}-1)^2\bigr) = \sum_{j=0}^{\infty} m_j(\lambda_2)\,\frac{\vartheta^j}{j!}. \tag{70}$$

*Proof.* From the defining generating function (32), one sees that

$$ {}_B\mathcal{S}_2(r,\epsilon;\lambda_1,\lambda_2)=\frac{1}{\epsilon!}\Delta^\epsilon\big({}_B\mathcal{R}_r(\lambda_1,\lambda_2)\big), \tag{71}$$

with ${}_B\mathcal{R}_r$ the two-variable Bell–Appell polynomials. Since ${}_B\mathcal{R}_r$ itself admits the determinantal representation (40), applying $\Delta^\epsilon$ preserves the Sheffer–Toeplitz structure and yields the determinant (69), with the replacement of the Bell–Appell column by the Stirling-type column $\{S_2(s,\epsilon;\lambda_1,\lambda_2)\}$. This follows from Cramer's rule exactly as in the Bell–Appell case.

**Theorem 16.** *Let $\Delta=e^{D_{\lambda_1}}-1$. Then for all $r,\epsilon\geq 0$, the 2VBASP ${}_B\mathcal{S}_2(r,\epsilon;\lambda_1,\lambda_2)$ admits the operational identity:*

$$ {}_B\mathcal{S}_2(r,\epsilon;\lambda_1,\lambda_2)=\frac{1}{\epsilon!}\Big[\,\mathcal{R}(D_{\lambda_1})\,\exp(\lambda_2\Delta^2)\,\Big]\,\Delta^\epsilon\,B_r(\lambda_1). \tag{72}$$

*Proof.* Starting with the expression (32), we observe

$$ \frac{(e^\vartheta-1)^\epsilon}{\epsilon!}\exp(\lambda_1(e^\vartheta-1))=\sum_{r\geq 0}\frac{1}{\epsilon!}\Delta^\epsilon\big(B_r(\lambda_1)\big)\frac{\vartheta^r}{r!}. \tag{73}$$

Multiplying by $\mathcal{R}(\vartheta)\exp(\lambda_2(e^\vartheta-1)^2)$ and arguing exactly as in the expression (51), we conclude that

$$ \sum_{r\geq 0}{}_B\mathcal{S}_2(r,\epsilon;\lambda_1,\lambda_2)\frac{\vartheta^r}{r!}=\sum_{r\geq 0}\left(\frac{1}{\epsilon!}\,\mathcal{R}(D_{\lambda_1})\,\exp(\lambda_2\Delta^2)\,\Delta^\epsilon B_r(\lambda_1)\right)\frac{\vartheta^r}{r!}. \tag{74}$$

Comparing coefficients yields (72).

**Theorem 17.** *For all $\lambda_1,\lambda_2,\mu_1,\mu_2\in\mathbb{C}$ and $r,\epsilon\geq 0$, the 2VBASP ${}_B\mathcal{S}_2(r,\epsilon;\lambda_1,\lambda_2)$ admits the identity:*

$$ {}_B\mathcal{S}_2(r,\epsilon;\lambda_1+\mu_1,\lambda_2+\mu_2)=\sum_{k=0}^{r}\binom{r}{k}\,{}_B\mathcal{S}_2(r-k,\epsilon;\lambda_1,\lambda_2)\,B_k(\mu_1,\mu_2). \tag{75}$$

*Proof.* Multiply the generating functions corresponding to parameters $(\lambda_1,\lambda_2)$ and $(\mu_1,\mu_2)$:

$$ \begin{aligned} &\left(\sum_{r\geq 0}{}_B\mathcal{S}_2(r,\epsilon;\lambda_1,\lambda_2)\frac{\vartheta^r}{r!}\right)\left(\sum_{k\geq 0}B_k(\mu_1,\mu_2)\frac{\vartheta^k}{k!}\right)\\ &=\frac{(e^\vartheta-1)^\epsilon}{\epsilon!}\,\mathcal{R}(\vartheta)\,\exp\big(\lambda_1(e^\vartheta-1)+\lambda_2(e^\vartheta-1)^2\big)\,\exp\big(\mu_1(e^\vartheta-1)+\mu_2(e^\vartheta-1)^2\big)\\ &=\frac{(e^\vartheta-1)^\epsilon}{\epsilon!}\,\mathcal{R}(\vartheta)\,\exp\big((\lambda_1+\mu_1)(e^\vartheta-1)+(\lambda_2+\mu_2)(e^\vartheta-1)^2\big). \end{aligned}$$

This is exactly the generating function of ${}_B\mathcal{S}_2(r,\epsilon;\lambda_1+\mu_1,\lambda_2+\mu_2)$. Coefficient comparison gives (75).

## 4. Conclusion

This work introduced and analyzed the two-variable Bell-Appell polynomials, highlighting their core properties through generating functions. We derived explicit forms, summation formulas, recurrence relations, and addition identities, providing a robust analytical foundation. Matrix formulations and product expressions were also established to enhance their computational tractability. Furthermore, we defined and examined the Bell-Appell-based Stirling polynomials of the second kind, extending the theoretical landscape.

These results contribute significantly to the structural understanding and functional utility of Bell-Appell polynomials, laying the groundwork for further theoretical and applied developments.

Promising extensions include the study of Bell-Appell polynomials in higher dimensions, where their behavior in multivariate settings could yield novel mathematical insights. Development of optimized algorithms for their evaluation can enhance applicability in computational contexts such as symbolic computation, numerical simulation, and machine learning.

Moreover, establishing deeper connections with areas like combinatorics, algebraic geometry, and representation theory may reveal new structural correspondences and broaden their interdisciplinary relevance. These avenues hold potential for both theoretical enrichment and practical impact across mathematical and applied sciences.

## Availability of data and materials

Not applicable.

## Competing interests

The authors declare no competing interests.

## Acknowledgements

The author S. Aljohani would like to thank Prince Sultan University for paying the APC and for the support through the TAS research lab.

## References

[1] L. Carlitz. Some remarks on the bell numbers. *Fibonacci Quarterly*, 18(1):66–73, 1980.

[2] G. Dattoli, P. L. Ottaviani, A. Torre, and L. Vázquez. Evolution operator equations: Integration with algebraic and finite difference methods. *Rivista del Nuovo Cimento*, 20:3–133, 1997.

[3] H. M. Srivastava and H. L. Manocha. *A Treatise on Generating Functions*. Ellis Horwood, New York, 1984.

[4] W. Wang and T. Wang. Identities on bell polynomials and sheffer sequences. *Discrete Mathematics*, 309:1637–1648, 2009.

[5] E. T. T. Bell. Exponential polynomials. *Annals of Mathematics*, 35:258–277, 1934.

[6] R. Alyusof and S. A. Wani. Certain properties and applications of $\delta_h$ hybrid special polynomials associated with appell sequences. *Fractal and Fractional*, 7:233, 2023.

[7] L. Comtet. *Advanced Combinatorics*. D. Reidel Publishing Company, Dordrecht, 1974.

[8] S. R. Mondal, K. S. Nisar, and T. Abdeljawad. Some subordination involving polynomials induced by lower triangular matrices. *Advances in Difference Equations*, page 538, 2020.

[9] K. Shah, F. Jarad, and T. Abdeljawad. Stable numerical results to a class of time-space fractional partial differential equations via spectral method. *Journal of Advanced Research*, 19:39–48, 2020.

[10] T. Kim. Identities involving frobenius–euler polynomials arising from non-linear differential equations. *Journal of Number Theory*, 132(12):2854–2865, 2012.

[11] T. Kim and B. J. Lee. Some identities of the frobenius–euler polynomials. *Abstract and Applied Analysis*, pages 1–7, 2009.

[12] T. Kim and J. J. Seo. Some identities involving frobenius–euler polynomials and numbers. *Proceedings of the Jangjeon Mathematical Society*, 19:39–46, 2016.

[13] A. Bayad and T. Kim. Identities for apostol-type frobenius–euler polynomials resulting from the study of a nonlinear operator. *Russian Journal of Mathematical Physics*, 23:164–171, 2016.

[14] T. Kim. An identity of the symmetry for the frobenius–euler polynomials associated with the fermionic $p$-adic invariant $q$-integrals on $\mathbb{Z}_p$. *Rocky Mountain Journal of Mathematics*, 41:239–247, 2011.

[15] D. S. Kim and T. Kim. Some new identities of frobenius–euler numbers and polynomials. *Journal of Inequalities and Applications*, page 307, 2012.

[16] S. A. Wani, T. R. Shah, W. Ramírez, and C. Clemente. Exploring the properties of multivariable hermite polynomials in relation to apostol-type frobenius–genocchi polynomials. *Georgian Mathematical Journal*, 32(3):515–528, 2025.

[17] S. Khan and S. A. Wani. Fractional calculus and generalized forms of special polynomials associated with appell sequences. *Georgian Mathematical Journal*, 28(2):261–270, 2021.

[18] P. Appell. Sur une classe de polynômes. *Annales Scientifiques de l'École Normale Supérieure*, 9:119–144, 1880.