EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2026, Vol. 19, Issue 1, Article Number 7205
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Advanced Numerical Methods for Fractional-Order Differential Systems

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Abstract. Fractional-order differential systems have become very popular in the last few years because they can model memory and hereditary effects. However, existing numerical techniques for solving such systems still have certain limitations. We used generalized Caputo-type fractional derivatives for the Abel differential equation in normal form and for the four-dimensional Chen system to avoid such problems. We also used two different methods for their solutions: the adaptive predictor–corrector (P–C) method and the generalized Laplace decomposition method ($\mathcal{T}_\rho$DM). First, the methods are easy to use and give more accurate results than other methods. Second, the methods let you do calculations quickly without having to recalculate fractional sequences for different starting points. Thirdly, the methods are strong enough to accurately identify chaotic attractors and capture the system's dynamics. Lastly, they are flexible and work well with computers, which makes them useful for many different scientific and engineering systems. A lot of numerical tests and comparisons with the Adams–Bashforth–Moulton (ABM) method show that the proposed methods work well, are reliable, and have real-world benefits.

2020 Mathematics Subject Classifications: 26A33, 34A08, 65L07, 37D45

Key Words and Phrases: The adaptive method, fractional Laplace transform, fractional derivatives, chaos.

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DOI: https://doi.org/10.29020/nybg.ejpam.v19i1.7205

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Introduction

Calculus with non-integers as the order is referred to as fractional calculus and now has a general application compared to standard integer forms of derivatives and integrals in several areas such as physics, engineering, applied mathematics, and others. [1–4]. It provides a versatile approach to describe phenomena such as diffusion, viscoelasticity, and control systems. Researchers have developed diverse methods to solve fractional differential equations, aided by advancements in computing and software tools [5–8]. These techniques, rooted in solid mathematical foundations, have applications in physics and engineering and areas like economics, biology, and management [9–16].

Many scientific and engineering sectors use chaotic system modeling articles [17–19], a trend that has grown in recent years. Many of these works address the challenge of incorporating chaotic systems into electrical circuit modeling for chaotic applications. Reliable forecasts are difficult due to the complexity of real-world phenomena and chaotic modeling. We can use phase portraits to analyze the effects of model parameters on system behavior, Lyapunov exponents, and chaotic and hyperchaotic tendencies.

Recent studies have advanced the numerical and analytical treatment of generalized Caputo-type fractional differential equations. Novel predictor–corrector schemes have been developed to enhance stability and accuracy [20, 21], while new approaches address long-term integration and error analysis [22, 23]. Extensions to multidimensional settings and non-uniform meshes further broaden applicability [24]. In parallel, theoretical progress has been made on stochastic and non-Lipschitz cases [25], and alternative methods such as the homotopy analysis method have been proposed [26]. These contributions collectively emphasize the growing importance of generalized Caputo-type derivatives in modeling complex dynamical systems, motivating the present study.

Fractional-order differential systems have attracted increasing attention over the past decades, due to their ability to capture memory and hereditary effects that classical integer-order models neglect [27–32]. Analytical approaches such as the Caputo fractional derivative, Riemann–Liouville fractional derivative, and other generalized definitions provide a solid theoretical foundation for formulating fractional-order dynamics [30–34]. Our goals are to enhance both accuracy and computational efficiency, and to demonstrate the methods' capability to capture complex dynamics including chaotic attractors. The results confirm that the proposed strategies outperform classical approaches and are well-suited for a wide range of fractional-order dynamical systems [35–37].

Which offers advantages when it comes to modeling situations that occur in the actual world. The evidence shown in [38] demonstrates that there has been a growing interest in investigating the impact that memory modeling has on chaotic and hyperchaotic systems, which has revealed a new opportunity for research. It is essential to bring to your attention that even minute modifications to the initial conditions of chaotic systems can potentially reduce the occurrence of chaotic or hyperchaotic behavior. This section aims to provide an overview of existing publications on the topic. In particular, [39] delves into the chaotic behavior of the Chua circuit. In [40], the authors explore and substantiate the Lyapunov fractional exponent approach. Another alternative formulation is discussed in reference [41–46]; their primary focus revolves around Caputo derivatives, bifurcations, and Lyapunov analysis. On the contrary, references [47, 48] encompass a diverse spectrum of chaotic and hyperchaotic systems employing Caputo derivatives. Furthermore, amplitude control is instrumental in capturing dynamically symmetric systems, as elucidated in the reference [49]. The four-dimensional fractional-order Chen system with generalized Caputo-type derivatives and the canonical Abel differential equation are examined in this study. Solutions are achieved utilizing the generalized Laplace decomposition method ($\mathcal{T}_\rho$DM) and adaptive predictor-corrector (P–C) while comparing to the ABM scheme. The offered methods accurately capture chaotic attractors and match ABM results. These results show they can accurately numerically solve complex scientific and engineering models. The $\mathcal{T}_\rho$ decomposition method ($\mathcal{T}_\rho$DM) [50, 51] combines the generalized Laplace transform $\mathcal{T}_\rho$ with the standard Adomian method. It has been displayed by many mathematicians to solve many problems associated with generalized Caputo fractional derivatives. This method is much easier and has fewer mathematical calculations compared with different analytical methods. The results revealed the effectiveness and efficiency of the method. The novelty of this research lies in its innovative application of the adaptive (P-C) method and ($\mathcal{T}_\rho$DM) to solve the fractional Abel equation and the 4-D fractional-order Chen system, as well as its comparative analysis with the ABM. Additionally, it demonstrates the effectiveness of these methods and their utility in identifying chaos. This research can potentially advance numerical and analytical methods in science and engineering, specifically in dealing with complex, non-linear systems.

The paper is structured as follows: Section 2 displays basic definitions. Section 3 provides the adaptive (P-C) method. Section 4 discusses the numerical solution of the first kind of Abel differential equation and the 4-D fractional-order Chen system using the adaptive (P-C) scheme, including numerical results. Section 5 presents a logarithm of the ($\mathcal{T}_\rho$DM).The application of the ($\mathcal{T}_\rho$DM) to our problems is made in Section 6. Section 7 concludes the study.

1. Basic definitions

In this section, we present a short survey of operator derivatives, focusing primarily on the fractional types that play a key role in our analysis.

Definition 1. *[52] For continuous functions f, the left-side generalized fractional integral (FI), denoted by $\mathbf{I}_{a+}^{\alpha,\rho} f(t)$, $\alpha > 0$, and $\rho > 0$, is given by*

$$\mathbf{I}_{a+}^{\alpha,\rho} f(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha-1} f(s)ds, \quad \alpha > 0, t > a. \tag{1}$$

for $m - 1 < \alpha \leq m$ where $m \in \mathbb{N}$.

Definition 2. *For continuous functions f, the generalized fractional derivative (RLFD) denoted by ${}^R\mathbf{D}_{a+}^{\alpha,\rho} f(t)$, of order $\alpha > 0$ is given by*

$${}^R\mathbf{D}_{a+}^{\alpha,\rho} f(t) = \frac{\rho^{\alpha-m+1}}{\Gamma(m-\alpha)} \left(t^{1-\rho}\frac{d}{dt}\right)^m \int_a^t s^{\rho-1}(t^\rho - s^\rho)^{m-\alpha-1} f(s)ds, \quad t > a \geq 0. \tag{2}$$

Definition 3. *[53] For continuous functions f, the generalized fractional derivative of the Caputo type (CFD), denoted by ${}^C\mathbf{D}_{a+}^{\alpha,\rho} f(t)$, of order $\alpha > 0$ is given by*

$${}^C\mathbf{D}_{a+}^{\alpha,\rho} f(t) = {}^R\mathbf{D}_{a+}^{\alpha,\rho} \left(f(x) - \sum_{n=0}^{m-1} \frac{f^{(n)}(a)}{n!}(x-a)^n \right)(t), \quad t > a \geq 0, \tag{3}$$

where $m = \lceil \alpha \rceil$ and $\rho > 0$. In case of $0 < \alpha \leq 1$.

$${}^C\mathbf{D}_{a+}^{\alpha,\rho} f(t) = \frac{\rho^\alpha}{\Gamma(1-\alpha)} \int_a^t (t^\rho - s^\rho)^{-\alpha} s^{1-\rho} f'(s)ds, \quad 0 < \alpha \leq 1, t > a \geq 0. \tag{4}$$

Definition 4. *[49] The new generalized (CFD) operator, $\mathbf{D}_{a+}^{\alpha,\rho}$, $\alpha > 0$ is given by:*

$$\mathbf{D}_{a+}^{\alpha,\rho} f(t) = \frac{\rho^{\alpha-m+1}}{\Gamma(m-\alpha)} \int_a^t s^{\rho-1}(t^\rho - s^\rho)^{m-\alpha-1} \left(s^{1-\rho}\frac{d}{ds}\right)^m f(s)ds, \quad t > a, \tag{5}$$

where $\rho > 0, a \geq 0$, and $m - 1 < \alpha < m$.

Definition 5. *[54] if $f : [0, \infty] \to \mathbb{R}$, then the generalized Laplace transform of f is defined by*

$$\mathcal{T}_\rho\{f(t)\} = \int_0^\infty e^{-\delta t^\rho} f(t) t^{\rho-1} dt. \tag{6}$$

The generalized Laplace transform of the generalized (CFD) $\mathbf{D}_0^{\alpha,\rho}$ is defined by

$$\mathcal{T}_\rho\{\mathbf{D}_0^{\alpha,\rho} f(t)\} = \delta^\alpha \mathcal{T}_\rho\{f(t)\} - \delta^{-1} f(0), \quad 0 < \alpha \leq 1. \tag{7}$$

2. Algorithm of the adaptive predictor-corrector method

This section presents the adaptive (P-C) method, an algorithm designed for the efficient numerical solution of initial value problems containing the generalized (CFD).

$$\mathbf{D}_{a+}^{\alpha,\rho} y(t) = f(t, y(t)), \quad t \in [0, T], \quad y^{(k)}(a) = y_0^k, \quad k = 0, 1, \cdots, \lceil \alpha \rceil, \tag{8}$$

where $\mathbf{D}_{a+}^{\alpha,\rho}$ is CFD, for $m-1<\alpha\leq m, a\geq 0, \rho>0$ and $y\in C^m([a,T])$, the IVP (8) is equivalent, we get:

$$y(t)=u(t)+\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t s^{\rho-1}(t^\rho-s^\rho)^{\alpha-1}f(s,y(s))ds, \tag{9}$$

where

$$u(t)=\sum_{n=0}^{m-1}\frac{1}{\rho^n n!}(t^\rho-a^\rho)^n\left(x^{1-\rho}\frac{d}{dx}\right)^n y(x)\Big|_{x=a}. \tag{10}$$

Assuming that the function f has one solution on $[a,T]$, we partition the interval into N unequal subintervals $[t_k,t_{k+1}], k=0,1,\cdots N-1$ using mesh points.

$$t_0=a, \quad t_{k+1}=(t_k^\rho+h)^{1/\rho}, \quad k=0,1,\cdots,N-1, \tag{11}$$

where $h=\frac{T^\rho-a^\rho}{N}$. [54] $y_k, k=0,1,\cdots,N$, then $y(t_j)$ and $y_j\approx y(t_j)(j=1,2,\cdots,k)$, using the integral equation. $y_{k+1}\approx y(t_{k+1})$.

$$y(t_{k+1})=u(t_{k+1})+\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^{t_{k+1}} s^{\rho-1}(t_{k+1}^\rho-s^\rho)^{\alpha-1}f(s,y(s))ds. \tag{12}$$

Now, substitution

$$z=s^\rho, \tag{13}$$

we get

$$y(t_{k+1})=u(t_{k+1})+\frac{\rho^{-\alpha}}{\Gamma(\alpha)}\int_{a^\rho}^{t_{k+1}^\rho}(t_{k+1}^\rho-z)^{\alpha-1}f(z^{1/\rho},y(z^{1/\rho}))dz. \tag{14}$$

That is

$$y(t_{k+1})=u(t_{k+1})+\frac{\rho^{-\alpha}}{\Gamma(\alpha)}\sum_{j=0}^{k}\int_{t_j^\rho}^{t_{j+1}^\rho}(t_{k+1}^\rho-z)^{\alpha-1}f(z^{1/\rho},y(z^{1/\rho}))dz. \tag{15}$$

Following that, an application of the trapezoidal quadrature method with respect to the weight function $(t_{k+1}^\rho-z)^{\alpha-1}$, we obtain the corrector formula to get a close approximation of the right-hand side of Eq. (15),

$$y(t_{k+1})\approx u(t_{k+1})+\frac{\rho^{-\alpha}h^\alpha}{\Gamma(\alpha+2)}\sum_{j=0}^{k}a_{j,k+1}f(t_j,y(t_j))+\frac{\rho^{-\alpha}h^\alpha}{\Gamma(\alpha+2)}f(t_{k+1},y(t_{k+1})) \tag{16}$$

where

$$a_{j,k+1}=\begin{cases}(k^{\alpha+1}-(k-\alpha)(k+1)^\alpha) & \text{if } j=0\\ (k-j+2)^{\alpha+1}+(k-j)^{\alpha+1}-2(k-j+1)^{\alpha+1} & \text{if } 1\leq j<k\end{cases} \tag{17}$$

Equation (14) is evaluated by substituting $y(t_{k+1})$ with the predictor $y^P(t_{k+1})$ using Eq. (16) where $f(z^{1/\rho}, y(z^{1/\rho}))$ is replaced by $f(t_j, y(t_j))$.

$$\begin{aligned} y^P(t_{k+1}) &\approx u(t_{k+1}) + \frac{\rho^{-\alpha}}{\Gamma(\alpha)} \sum_{j=0}^{k} \int_{t_j^\rho}^{t_{j+1}^\rho} (t_{k+1}^\rho - z)^{\alpha-1} f(t_j, y(t_j)) dz \\ &= u(t_{k+1}) + \frac{\rho^{-\alpha} h^\alpha}{\Gamma(\alpha+1)} \sum_{j=0}^{k} \left((k+1-j)^\alpha - (k-j)^\alpha\right) f(t_j, y(t_j)). \end{aligned} \tag{18}$$

Then,

$$y_{k+1} \approx u(t_{k+1}) + \frac{\rho^{-\alpha} h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{k} a_{j,k+1} f(t_j, y_j) + \frac{\rho^{-\alpha} h^\alpha}{\Gamma(\alpha+2)} f(t_{k+1}, y_{k+1}^p). \tag{19}$$

In [24]. The proposed adaptive (P-C) method uses a non-uniform grid $t_{j+1} = (t_j^\rho + h_\rho)^{1/\rho} : j = 0, 1, \cdots, N-1$, $t_0 = a$ and $h = \frac{T^\rho - a^\rho}{N}$, where N is natural. We cannot use the (P-C) technique to solve IVP defined with the generalized CFD if we employ a uniform grid, as mentioned in [55].

3. Applications

This section explores the applicability of the (P-C) approach for the numerical resolution of initial value issues, utilizing the suggested under-generalized Caputo-type derivatives. Considering this, we turned to numerical simulations to investigate potential solutions to our examples.

3.1. Problem 1

Consider the first kind of Abel differential equation in canonical form, nonhomogeneous equation with cubic nonlinearity [56]

$$\mathbf{D}_0^{\alpha,\rho} y(t) = \sin(t) - y(t)^3 \tag{20}$$

This is associated with the condition $y(0) = 0.5$. By using Eq. (19), the approximations y_{k+1}, and for $N \in \mathbb{N}$ and $T > 0$,

$$y_{k+1} \approx y_0 + \frac{a\rho^{-\alpha} h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{k} a_{j,k+1} (\sin(t_j) - y_j^3) + \frac{a\rho^{-\alpha} h^\alpha}{\Gamma(\alpha+2)} (\sin(t_{k+1}) - (y_{k+1}^k)^3) \tag{21}$$

where $h = \frac{T^\rho}{N}$ and $y_0 = 0.5$, then

$$y_{k+1} \approx 0.5 + \frac{a\rho^{-\alpha} h^\alpha}{\Gamma(\alpha+1)} \sum_{j=0}^{k} \left((k+1-j)^\alpha - (k-j)^\alpha\right) (\sin(t_j) - y_j^3) \tag{22}$$

The tables display solutions of Equation (20) using the adaptive (P-C) method under different settings to display the behavior and precision of the numerical method. Table 1 presents a comparison of solutions using different values of t, utilizing various step sizes of h, in addition to an ABM method solution as a reference. It can be observed from the table that the solutions gradually reduce as the step size is taken forward. Table 2 provides the solutions at $t = 0.1$ for several α, ρ. Table 3 depicts solutions at $t = 0.2$. For the same values of the step size and parameter combinations. These results show that the fractional order parameters influence the solution's stability and its convergence.

Table 1: Solutions of Equation (20), where $\alpha = 1$, $\rho = 1$.

h	$t = 0.1$	$t = 0.2$	$t = 0.5$
1/10	0.49889	0.50233	0.53928
1/20	0.49751	0.49978	0.53459
1/40	0.49686	0.49857	0.53237
1/80	0.49654	0.49798	0.53128
1/160	0.49638	0.49769	0.53075
1/320	0.49630	0.49755	0.53049
ABM	0.49283	0.49575	0.55589

Table 2: Solutions of Equation (20), where $t = 0.1$.

h	$\alpha = 1, \rho = 0.9$	$\alpha = 0.95, \rho = 0.8$	$\alpha = 0.9, \rho = 1.2$
1/10	0.49876	0.49830	0.49856
1/20	0.49724	0.49622	0.49677
1/40	0.49651	0.49509	0.49568
1/80	0.49654	0.49442	0.49493
1/160	0.49598	0.49398	0.49432
1/320	0.49590	0.49363	0.49379
1/640	0.49585	0.49334	0.49327

Table 3: Solutions of Equation (20), where $t = 0.2$.

h	$\alpha = 1, \rho = 0.9$	$\alpha = 0.95, \rho = 0.8$	$\alpha = 0.9, \rho = 1.2$
1/10	0.50259	0.50324	0.50243
1/20	0.49977	0.49962	0.49950
1/40	0.49843	0.49778	0.49788
1/80	0.49778	0.49679	0.49691
1/160	0.49746	0.49621	0.49625
1/320	0.49730	0.49584	0.49574
1/640	0.49722	0.49557	0.49530

Fig. 1 illustrates the curves and parametric plots obtained using the ABM method, and the different formal parameters tend to remain stable and periodic over the solution period. These are presented using parametric plots and curves slightly different from those in Fig. 2, and given the robust nature of the (P-C) method, this shows variations due to the adaptation. The adaptive (P-C) method and the ABM nearly coincide, supporting the adaptive method's accuracy in efficiently capturing the system's dynamics.

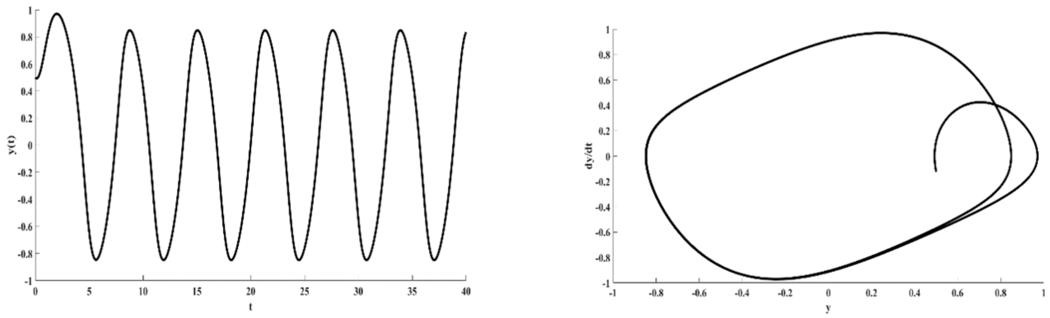

Figure 1: Curves and parametric plots using the ABM method.

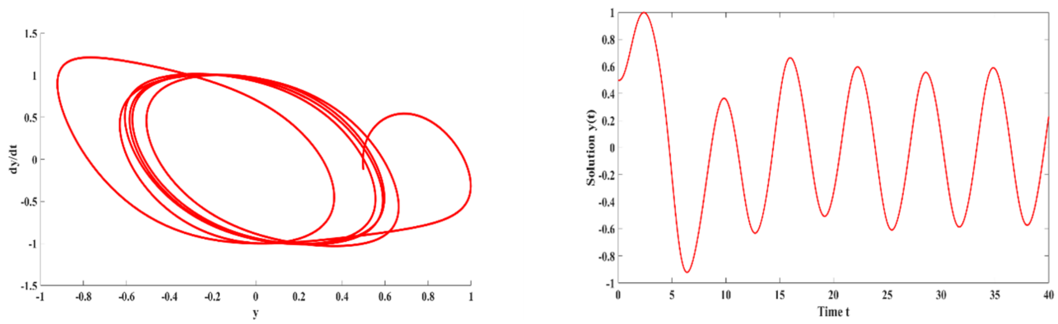

Figure 2: The curves and parametric plots using (P-C) schema when $(\alpha, \rho) = (1, 1)$.

3.2. Problem 2

Consider the following 4-D fractional-order Chen system: [49].

$$\begin{cases} \mathbf{D}_0^{\alpha,\rho} x_1(t) = a(x_2 - x_1) \\ \mathbf{D}_0^{\alpha,\rho} x_2(t) = bx_1 - x_1x_3 + cx_2 - x_4 \\ \mathbf{D}_0^{\alpha,\rho} x_3(t) = x_1x_2 - dx_3 \\ \mathbf{D}_0^{\alpha,\rho} x_4(t) = x_1 + k \end{cases} \tag{23}$$

When $a = 36, b = -16, c = 28, d = 3$, and $k = 0.5$, with $x_1(0) = 0, x_2(0) = 0, x_3(0) = 8$, and $x_4(0) = 6$. By using Eq. (19), the approximations $x_{1_{k+1}}$, $x_{2_{k+1}}$. $x_{3_{k+1}}$ and $x_{4_{k+1}}$, and for $N \in \mathbb{N}$ and $T > 0$,

$$
\begin{aligned}
x_{1_{k+1}} &\approx x_{1_0} + \frac{a\rho^{-\alpha}h^{\alpha}}{\Gamma(\alpha+2)}\sum_{j=0}^{k} a_{j,k+1}a(x_{2_j} - x_{1_j}) + \frac{a\rho^{-\alpha}h^{\alpha}}{\Gamma(\alpha+2)}a(x^P_{2_{k+1}} - x^P_{1_{k+1}}) \\
x_{2_{k+1}} &\approx x_{2_0} + \frac{\rho^{-\alpha}h^{\alpha}}{\Gamma(\alpha+2)}\sum_{j=0}^{k} a_{j,k+1}(bx_{1_j} - x_{1_j}x_{3_j} + cx_{2_j} - x_{4_j}) \\
&\quad + \frac{\rho^{-\alpha}h^{\alpha}}{\Gamma(\alpha+2)}(bx^P_{1_{k+1}} - x^P_{1_{k+1}}x^P_{3_{k+1}} + cx^P_{2_{k+1}} - x^P_{4_{k+1}}) \\
x_{3_{k+1}} &\approx x_{3_0} + \frac{\rho^{-\alpha}h^{\alpha}}{\Gamma(\alpha+2)}\sum_{j=0}^{k} a_{j,k+1}(x_{1_j}x_{2_j} - dx_3) + \frac{\rho^{-\alpha}h^{\alpha}}{\Gamma(\alpha+2)}(x^P_{1_{k+1}}x^P_{2_{k+1}} - bx^P_{3_{k+1}}) \\
x_{4_{k+1}} &\approx x_{3_0} + \frac{\rho^{-\alpha}h^{\alpha}}{\Gamma(\alpha+2)}\sum_{j=0}^{k} a_{j,k+1}(x_{1_j} + k) + \frac{\rho^{-\alpha}h^{\alpha}}{\Gamma(\alpha+2)}(x^P_{1_{k+1}} + k)
\end{aligned}
\tag{24}
$$

where $h = \frac{T^{\rho}}{N}$ and take value of initial condition

$$
\begin{aligned}
x_{1_{k+1}} &\approx \frac{a\rho^{-\alpha}h^{\alpha}}{\Gamma(\alpha+1)}\sum_{j=0}^{k} \left((k+1-j)^{\alpha} - (k-j)^{\alpha}\right) a(x_{2_j} - x_{1_j}) \\
x_{2_{k+1}} &\approx \frac{\rho^{-\alpha}h^{\alpha}}{\Gamma(\alpha+1)}\sum_{j=0}^{k} \left((k+1-j)^{\alpha} - (k-j)^{\alpha}\right)(bx_{1_j} - x_{1_j}x_{3_j} + cx_{2_j} - x_{4_j}) \\
x_{3_{k+1}} &\approx 8 + \frac{\rho^{-\alpha}h^{\alpha}}{\Gamma(\alpha+1)}\sum_{j=0}^{k} \left((k+1-j)^{\alpha} - (k-j)^{\alpha}\right)(x_{1_j}x_{2_j} - dx_3) \\
x_{4_{k+1}} &\approx 6 + \frac{\rho^{-\alpha}h^{\alpha}}{\Gamma(\alpha+1)}\sum_{j=0}^{k} \left((k+1-j)^{\alpha} - (k-j)^{\alpha}\right)(x_{1_j} + k)
\end{aligned}
\tag{25}
$$

In Table 4, we provide a numerical solution using the adaptive (P-C) method to Eq. (20) when $\alpha = 1$, $\rho = 1$ and $t = 0.1$. In Table 5, we provide the numerical solution for the value of $\alpha = 0.95$, $\rho = 1$, and $t = 0.5$.

We analyze phase portraits to study the impact of the generalized Caputo-type fractional derivative on the system dynamics (Eq. (20)). Using fractional orders $\alpha = 0.96, 0.97, 0.98$, and 1, chaotic behavior emerges at $t = 200$. Figures (5 - 8) display the $x_1 - x_3$, $x_1 - x_2$, and $x_2 - x_3$ attractors for these cases.

Table 4: Solutions of Equation (23) where $\alpha = 1$, $\rho = 1$

t	x_1	x_2	x_3	x_4
0.00	0.0	0.0	8.0	6.0
0.01	-0.01022397	-0.06749064	7.76586077	6.00491571
0.02	-0.04060235	-0.15215288	7.53633892	6.00967634
0.03	-0.09006631	-0.25354305	7.31370546	6.01403610
0.04	-0.15798295	-0.37223740	7.09790601	6.01780768
0.05	-0.24415282	-0.50923671	6.88897815	6.02080794
0.06	-0.34880859	-0.66597648	6.68707193	6.02285351
0.07	-0.47261853	-0.84434467	6.49247647	6.02375654
0.08	-0.61669539	-1.04670703	6.30565516	6.02332024
0.09	-0.78261150	-1.27593957	6.12729239	6.02133439
0.10	-0.97242014	-1.53546722	5.95835556.	6.01757063

Table 5: Solutions of Equation (23), where $\alpha = 0.95$, $\rho = 1$.

t	x_1	x_2	x_3	x_4
0.00	0.0	0.0	8.0	6.0
0.01	-0.01460575	-0.07355130	5.92025254	4.62814279
0.02	-0.05216391	-0.15941269	5.50628735	4.47569747
0.03	-0.10921573	-0.26170607	5.20137134	4.39075850
0.04	-0.18483585	-0.38256715	4.94609561	4.33213396
0.05	-0.27936101	-0.52459541	4.72162850	4.28709078
0.06	-0.39408408	-0.69107296	4.51960152	4.24992067
0.07	-0.53114673	-0.88606622	4.33577357	4.21749171
0.08	-0.69351942	-1.11452870	4.16803272	4.18782509
0.09	-0.88503361	-1.38242209	4.01566038	4.15951848
0.10	-1.11045120	-1.69685657	3.87908994	4.13147199

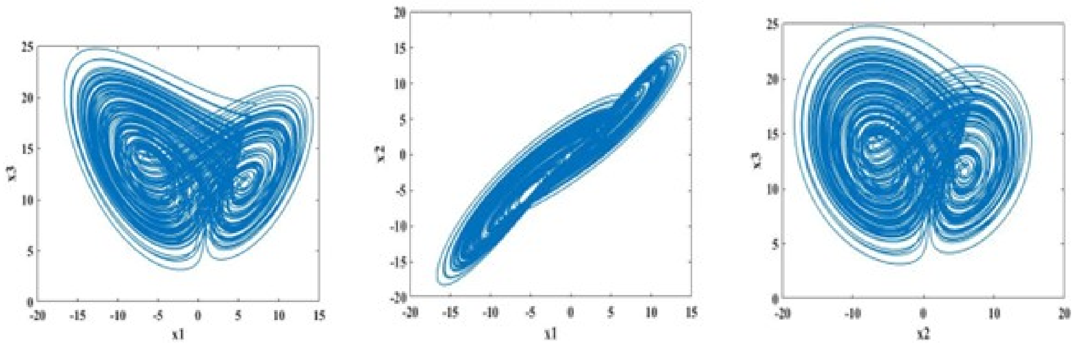

Figure 3: Chaotic dynamics for $(\alpha, \rho) = (0.96, 0.7)$.

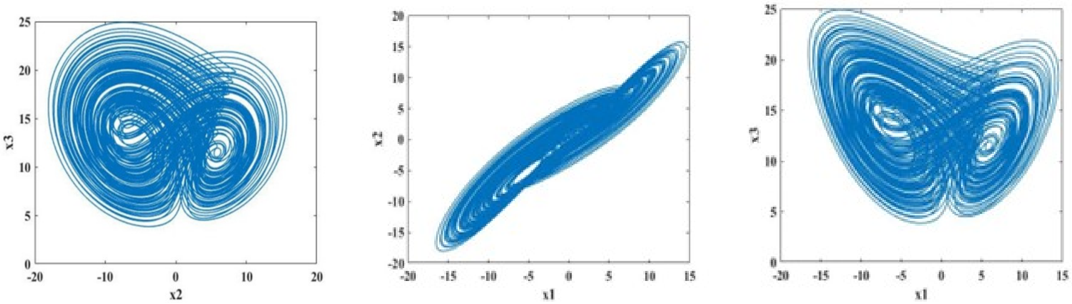

Figure 4: Chaotic dynamics for $(\alpha, \rho) = (0.97, 0.8)$.

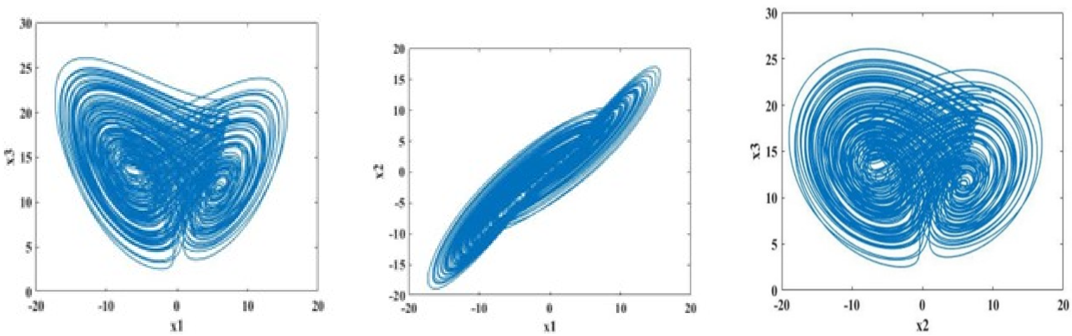

Figure 5: Chaotic dynamics for $(\alpha, \rho) = (0.98, 0.9)$.

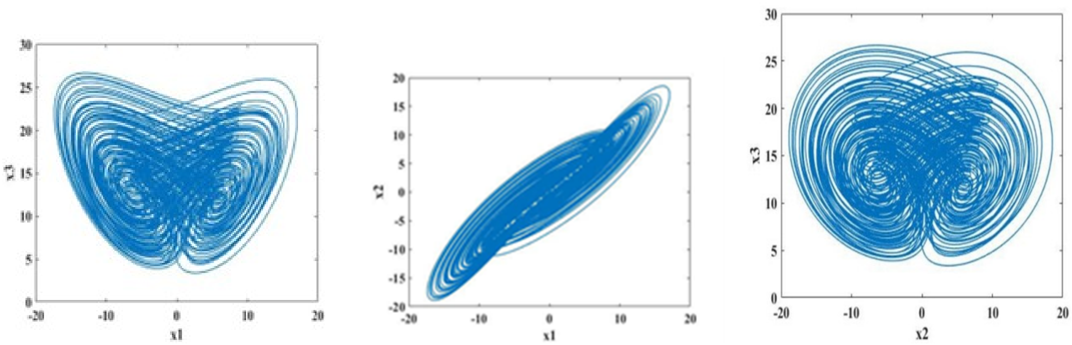

Figure 6: Chaotic dynamics for $(\alpha, \rho) = (1, 1)$.

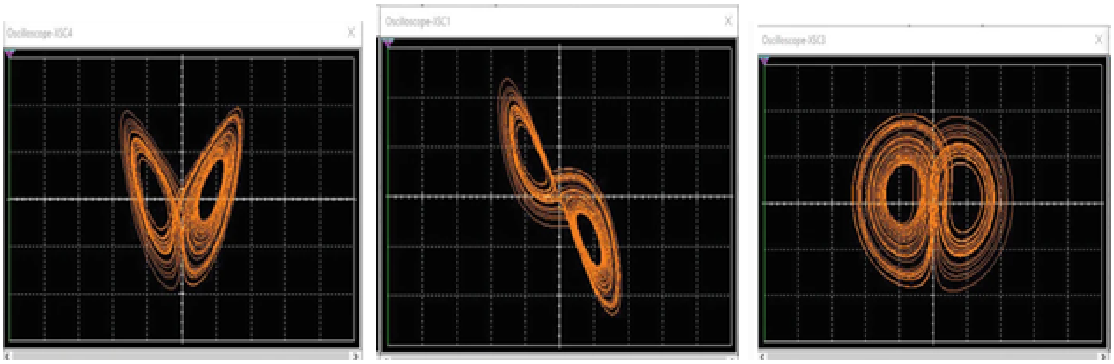

Figure 7: The circuit generated results when $(\alpha, \rho) = (1, 1)$.

In Figures (3–6), we plot numerical solutions to Eq. (20) $(a, b, c, d, k) = (36, -16, 28, 3, 0.5)$, with the initial conditions $x_1(0) = 0, x_2(0) = 0, x_3(0) = 8$, and $x_4(0) = 6$. In Fig. 5, we show the circuital generated results for Eq. (20), when $(\alpha, \rho) = (1, 1)$. We display these figures using the adaptive (P-C) method when $T = 50$ and $N = 1000$ for some different values of the parameters α and ρ in Eq. (20). Furthermore, the importance of changing the α and ρ parameters in generalized Caputo fractional models lies in their capacity to capture a broad spectrum of system behaviors, adapt the model to specific applications, and facilitate model validation and optimization. This flexibility is valuable when dealing with complex and diverse real-world systems. Comparing the circuit simulation in 7 to numerical findings showed that they were in good agreement. Chaos theory has wide-ranging applications, from weather forecasting (where small changes in initial conditions can lead to drastically different weather patterns) to studying turbulent fluid dynamics, the behavior of financial markets, and even biological systems like the human heart's rhythm. It is worth noting that while chaotic systems are deterministic and governed by mathematical equations, they can still appear random and unpredictable over extended periods, making them a fascinating area of study and posing challenges in various fields. Chaos theory has deepened our understanding of the inherent complexity and unpredictability present in many natural and man-made systems. Refer to [55, 57–59] for further insights. Chaos can manifest in a variety of real-world systems, including coronary arteries within blood vessels [60] and even within cancer and tumor cells [61].

4. $\mathcal{T}_\rho$ Decomposition Method ($\mathcal{T}_\rho$DM)

In this section, we present the algorithm of $\mathcal{T}_\rho$ to deal with equations that have the (CFD). Consider the Initial value problem

$$\mathbf{D}_0^{\alpha,\rho} y(t) = f(t, y(t)), \quad \rho > 0, \quad 0 < \alpha \leq 1, \tag{26}$$

Rewire (26) in the following

$$\mathbf{D}_0^{\alpha,\rho} y(t) = f(t) + S(y(t)),$$

Where $f(t)$ is known linear part and $S(y(t))$ is a nonlinear part Taking $\mathcal{T}_\rho$ and its inverse, we obtain

$$y(t) = P(t) + \mathcal{T}_\rho^{-1}\left[\frac{1}{\delta^\alpha}\mathcal{T}_\rho\{S(y(t))\}\right] \tag{27}$$

Where $P(t) = y_0 + \mathcal{T}_\rho^{-1}\left[\frac{1}{\delta^\alpha}\mathcal{T}_\rho\{f(t)\}\right]$ Consider the series expansion

$$y(t) = \sum_{n=0}^{\infty} y_n(t), \quad S(y(t)) = \sum_{n=0}^{\infty} L_n(t) \tag{28}$$

where L_n are the Adomian polynomials corresponding to S. Substituting Eq. (28) into Eq. (24) yields

$$\sum_{n=0}^{\infty} y_n(t) = P(t) + \mathcal{T}_\rho^{-1}\left[\frac{1}{\delta^\alpha}\mathcal{T}_\rho\left\{S(\sum_{n=0}^{\infty} y_n(t))\right\}\right]. \tag{29}$$

As a result, we have the recursive relation

$$y_0(\tau) = P(t), \tag{30}$$

$$y_{n+1}(\tau) = \mathcal{T}_\rho^{-1}\left[\frac{1}{\delta^\alpha}\mathcal{T}_\rho\{L_n(t)\}\right], \quad n \geq 0. \tag{31}$$

After P components, the truncated solution is

$$y^{[P]}(t) = \sum_{n=0}^{P} y_n(t) \tag{32}$$

5. Application ($\mathcal{T}_\rho$DM)

in this section, we display the solutions of problem1 and 2 by using $\mathcal{T}_\rho$DM to demonstrate its capability and efficiency. To substantiate these claims, we benchmark the results with (P-C) as a high-accuracy method.

Table 6: $\mathcal{T}_\rho$DM solutions of Equation (20), where $\alpha = 1, \rho = 1$, $\alpha = 0.9, \rho = 1.2$ and $\alpha = 1, \rho = 0.9$ respectively, together with the ABM method.

t	$y(t)$	$y(t)$	$y(t)$	$y(t)$ (ABM)
0.0	0.50000000	0.5	0.5	0.50000000
0.1	0.49283965	0.495194	0.489829	0.49282671
0.2	0.49581033	0.498667	0.493661	0.49566222
0.3	0.50802144	0.510806	0.507845	0.50772079
0.4	0.52849733	0.531068	0.530572	0.52803985
0.5	0.55618490	0.558590	0.560347	0.56034700

Tables 6, 7, and 8 display the high efficiency results of the $\mathcal{T}_\rho$DM for both the Abel equation and the 4-D fractional Chen system. The results demonstrate strong consistency

Table 7: $\mathcal{T}_\rho$DM Solutions of Equation (23) where $\alpha = 1, \rho = 1$

t	x_1	x_2	x_3	x_4
0.00	0.0	0.0	8.0	6.0
0.01	-0.01053361	-0.06836528	7.76356602	6.00496467
0.02	-0.04121716	-0.15336438	7.53414660	6.00972220
0.03	-0.09100584	-0.25516654	7.31161424	6.01407664
0.04	-0.15929223	-0.37435081	7.09591955	6.01784018
0.05	-0.24590100	-0.51189428	6.88710762	6.02082868
0.06	-0.35108370	-0.66916040	6.68533439	6.02285719
0.07	-0.47551353	-0.84788718	6.49088303	6.02373547
0.08	-0.62028005	-1.05017548	6.30418030	6.02326349
0.09	-0.78688399	-1.27847725	6.12581293	6.02122700
0.1	-0.97723200	-1.53558383	5.95654400	6.01739300

Table 8: $\mathcal{T}_\rho$DM Solutions of Equation (23) where $\alpha = 0.95, \rho = 1$

t	x_1	x_2	x_3	x_4
0.00	0.0	0.0	8.0	6.0
0.01	-0.01811237	-0.09157316	7.69781898	6.00633965
0.02	-0.06585216	-0.20293349	7.42709690	6.01181555
0.03	-0.13942618	-0.33624535	7.17327700	6.01638560
0.04	-0.23733389	-0.49273276	6.93328192	6.01981814
0.05	-0.35932110	-0.67424636	6.70582162	6.02186336
0.06	-0.50607992	-0.88326294	6.49042558	6.02225709
0.07	-0.67909657	-1.12283235	6.28716065	6.02071449
0.08	-0.88055485	-1.39652456	6.09652515	6.01692204
0.09	-1.11326677	-1.70838245	5.91940659	6.01052941
0.1	-1.38061856	-2.06287971	5.75706686	6.00114154

with the adaptive (P-C) solutions obtained in Tables 2–5, confirming the efficiency of $\mathcal{T}_\rho$DM. The comparisons show that $\mathcal{T}_\rho$DM solutions are very close to those obtained by using the (P-C) and ABM scheme in Table 1. Overall, the results validate $\mathcal{T}_\rho$DM as a powerful method for generalized Caputo-type problems.

6. Discussion

The adaptive predictor-corrector method is extremely accurate; its result always lies very close to the result of the Adams–Bashforth–Moulton (ABM) method. Its precision allows safe reconstruction of dynamic behaviors, like the exact determination of chaotic attractors in the fractional-order Chen system. The above characteristic renders it exceptionally suitable to nonlinear dynamics and complex system modeling research, where observation of intricate system behaviors is of utmost concern. We examine in this paper two highly effective numerical approaches to fractional-order systems with generalized (CFD): the (P-C) and $\mathcal{T}_\rho$DM methods. Both approaches were used to solve the canonical Abel differential equation and four-dimensional fractional-order Chen system to compare their inherent strengths and usability. The $\mathcal{T}_\rho$DM method, however, is easier to compute. It is not as accurate as the adaptive (P-C) method, but it's easier to calculate and doesn't require as much in the way of computing power, so it's better for quick simulations and situations where an approximate solution will suffice. Interestingly, the results of $\mathcal{T}_\rho$DM remain as very close to those of both ABM as well as the adaptive (P-C) solutions. This means that $\mathcal{T}_\rho$DM is nearly accurate in a wide range of applications. The comparison shows that both methods give solutions very close to the solutions that are found by standard methods. The flexible (P-C) method's ability to extract order from chaos very nicely complements the usability of $\mathcal{T}_\rho$DM. The implication is that the two methods can supplement one another: one supplies accuracy and chaos detection, and the other supplies ease of implementation of larger or easier simulations. This complementarity makes the flexibility of fractional-order modeling more powerful and enables these methods to be extended to a wider range of nonlinear systems The work shows that the proposed methods are both effective and flexible. Their compatibility with traditional methodologies, coupled with innovative features such as chaos detection and diminished computational expense, underscores their significance for scientific and engineering numerical analyses. Future research may extend their application to additional complex systems, such as higher-dimensional fractional-order models and multi-physics challenges, thereby facilitating the development of sophisticated computational techniques in nonlinear dynamics. In this part, we employed generalized Caputo-type fractional derivatives and implemented two complementary methods: the adaptive predictor–corrector (P-C) scheme and the generalized Laplace decomposition method (T_ρDM). The choice of these methods is supported theoretically by their convergence, consistency, and stability properties, which are essential for accurately solving fractional differential equations with memory and hereditary effects. Numerically, we benchmarked the proposed methods against the classical Adams–Bashforth–Moulton (ABM) approach for the Abel differential equation and the four-dimensional Chen system. The comparison considered solution accuracy, computational efficiency, and the ability to capture complex dynamics, including chaotic attractors. Results demonstrate that the P-C and T_ρDM methods not only reduce computational cost by avoiding repeated evaluations of fractional sequences but also reliably reproduce the system's dynamical features. Therefore, the combination of theoretical justification and systematic numerical evaluation confirms that these methods provide an effective and robust framework for solving nonlinear fractional-order systems.

The study aims to develop robust numerical methods for fractional-order differential systems using generalized Caputo-type derivatives. We implement the adaptive predictor–corrector (P-C) method and the generalized Laplace decomposition method (T_ρDM) to handle nonlinear dynamics and chaotic behavior, demonstrated on the Abel equation and the four-dimensional Chen system. The combination of theoretical justification and numerical testing supports the effectiveness and applicability of the proposed methods.

The study is limited to two example systems with fixed fractional orders. Although the methods accurately capture chaotic attractors, their performance for stiff, high-dimensional, or variable-order systems requires further investigation. Future work will extend the analysis to a broader class of fractional-order differential equations to fully assess the generality and robustness of the proposed approaches.

7. Conclusions

This study aims at solving the Abel differential equation in its standard form and the four-dimensional fractional-order Chen system with the use of the adaptive (P-C) method and the $\mathcal{T}_\rho$DM method. To assess the efficacy of such methods, a comprehensive comparative analysis was done against the existing ABM method. First, the suggested methods are very precise as they always give answers that are highly comparable to those given by the ABM method. Second, a numerical method using MATLAB was created to ensure that comparisons could be made more easily and to ensure compatibility of the methods with established computational techniques. Third, the adaptive (P-C) method shows a remarkable capacity to achieve complex dynamical behavior, such as precise identification of chaotic attractors, through realistic numerical evidence. Lastly, analytical results of the $\mathcal{T}_\rho$DM method are very close to ABM solutions, which is proof that it is robust and reliable. In general, these methods offer scientists and engineers useful and flexible tools for solving enormous types of models numerically and analytically. This enables more accurate and reliable simulations of complex fractional-order systems.

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