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Graph Ideal Proximity Spaces

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Abstract. This paper focuses on extending the concept of "proximity" between sets to graphs. We define graph proximity and graph ideal-proximity spaces. Using the proposed graph ideal proximity spaces, we suggest a new operator over the vertices of a given graph and examine some of their essential aspects. As a result, we obtain a new topological space via this new operator over the vertices of a given graph. Comparisons between the obtained topology and old ones are presented. Further, we not only study some of its properties but also provide some examples. The properties and the implications of related definitions are proposed with examples. Near set theory supplies a major framework for the classification of members of a set into classes depending on their closeness. We follow the same idea in graph theory. Thus, our definitions of graph proximities depend on the nearness of vertices of graphs. That is, we say that two graphs are near if their vertices are near. Based on the idea of nearness between vertices, a real-life application is provided to demonstrate the significance of this research.

2020 Mathematics Subject Classifications: 54H99, 57M15, 54A10, 54A05

Key Words and Phrases: Graphs, graph proximity relations, graph ideals, topology

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1. Introduction

In several branches of mathematics, topology plays a crucial role. Researchers from a wide range of scientific and social disciplines have been drawn to the applicability of different topological concepts to several natural problems. Topology has seen the introduction of many new ideas, enriching it with a range of recently created fields of study. Ideals [1], grills [2], and primals [3] are among the most important classical topological structures. Kuratowski was the first to propose the topological concept of ideal [1]. On the other hand, the concept of a grill was first proposed by [2]. In [4], the authors used

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grills to expand the proximities. It should be mentioned that ideal has helped researchers introduce a number of novel topological topics [5, 6]. It is important to keep in mind that the ideal is the filter's dual. For instance, ideal topological space [7], etc. On the other hand, Primals [3] seem to be the exact opposite of the grill concept. In order to axiomatizing the crucial concept of closeness in metric spaces, proximity spaces [8] were introduced. Numerous discoveries on proximity spaces, their generalizations, and their relationships to other structures like topologies and uniformity have been reported such as in [9] since they were first introduced. The wide generalizations of this concept appear for example in μ-proximity [10, 11], quasi proximity [12], and multi-set proximity [13] have been proposed by many scholars. Two important branches of mathematics, general topology [1] and graph theory [14], are closely related. In a more abstract way, graph theory examines the topological characteristics of graphs, including their continuity, connectedness, and the structural connections between nodes and edges. One connection between generic topology and graph theory is the creation of topologies on the set of vertices and edges of a graph. Several papers employed directed and undirected graphs to build different topologies [15–17]. Most of these notions were found in the theory of basic undirected graphs, namely the sets of vertices in such graphs. A relation on a graph serves as a bridge between graph theory and topological structures. The graph gains new types of topological structures from the relations. The labeled topologies on n points and the labeled transitive directed graph with n points correspond one to one, as demonstrated in [18]. In 1967, [19] examined the lattice-graph of transitive directed graph topologies, as suggested by the authors of [18]. In 2010, [20], the connection between directed graphs and finite topologies was investigated. A topology on the vertices of an undirected network was suggested in 2013 by the authors of [21]. In 2018, the authors in [22] connected an incidence topology to a vertex set of simple graphs without isolated vertices. In 2019, the authors of the study [23] developed innovative topology constructions employing incidence topology on the set of vertices for simple graphs $\circlearrowleft = (\mathfrak{W}(\circlearrowleft), \mathfrak{U}(\circlearrowleft))$ without isolated vertices. This set has a subbases created by the family of end sets that only include the end points of each edge. On the set of its edges $\mathfrak{U}(\circlearrowleft)$, the authors in [24] used the graphs $\circlearrowleft = (\mathfrak{W}(\circlearrowleft), \mathfrak{U}(\circlearrowleft)$ to induce two topology constructions, which are represented by compatible edge topology and incompatible edge topology. A connection on graphs was constructed to generate new kinds of topological structures [25, 26]. In their work [27], the authors explained how to construct topology using incidence and adjacency relations on the vertex set of graphs. They also studied the closure and interior features of a vertex set of subgraphs in the graph adjacency topological space (abbreviated graph ATS). Topology has been enriched with a variety of newly developed topics of study and many new concepts have been introduced [15]. In an effort to find legitimate answers to some of these topological problems, topologists have created novel structures such as closure space, proximity, filters [1], ideals [7], grills [2], and primals [3]. In order to create novel topologies on the set of graph vertices using graph ideals, the graph ideal was suggested and studied in [5]. Graphs may be used to describe data that includes items and the relationships between them. The principle of proximity tells us how close two items are to one another. It's crucial to create a proximity analysis of graphs in order to broaden the application's reach and enhance proximity theory.

In this paper, a new contribution to the field of graph theory has been made by introducing the notion of "graph proximity" relation δ, which is extension of the proximity on a set. In our definition, We considered that two subgraphs of a given graph are near to each other if the vertices of these graphs are near. We proposed a some possible graph proximity relations with proofs and suitable examples. Also, graph ideal-proximity relation $\delta_{\mathfrak{I}}$, was proposed and studied with the aid of suitable examples. We introduce a vertex graph closure operator of the proximity spaces named $\mathfrak{CL}^{\delta_{\mathfrak{I}}}$ corresponding to a graph ideal-proximity. We generated a new topological space over the vertices of a given graph using the closure operator generated on the graph's vertices named $\tau_{\delta_{\mathfrak{I}}}$ corresponding to graph ideal-proximity. In addition, we suggested some comparisons of the proposed graph proximities and their corresponding topologies with some counter examples. Near set theory supplies a major for the classification of members of a set in classes depending on their closeness. We extend the same idea in the graph theory. So, our definitions of graph proximities depend on the nearness of vertices of these graphs. that is we say that two graphs are near if there vertices are near. Based on the idea of nearness between vertices, we suggest a real-life application from an information system to illustrate the significance of this research. The obtained results are valid for any type of graph: multi-graphs or simple graphs, connected or disconnected graphs, with loops or without loops, and undirected or directed graphs. All the analyzed points in this paper seem to be much promising for further interesting research. Several extensions of the graph proximity relations will be proposed in the future.

2. Preliminaries

Throughout the research, simple undirected graphs with or without loops are the ones that are covered. A graph is denoted by the symbol $\circlearrowleft$. The term "simple graph" will henceforth be shortened to "graph". Some basic definitions and introductions to graph theory and topology may be found in the sources [1, 14, 15, 28].

A graph $\circlearrowleft$ is represented as the pair $(\mathfrak{W}(\circlearrowleft), \mathfrak{U}(\circlearrowleft))$, where $\mathfrak{W}(\circlearrowleft)$ is a nonempty finite set and $\mathfrak{U}(\circlearrowleft)$ is a set of unordered pairs of distinct members of $\mathfrak{W}(\circlearrowleft)$. $\mathfrak{W}(\circlearrowleft)$ is the set of vertices of $\circlearrowleft$, and $\mathfrak{U}(\circlearrowleft)$ is the set of edges of $\circlearrowleft$. The vertices or nodes of $\circlearrowleft$ are the elements of $\mathfrak{W}(\circlearrowleft)$, while the edges of $\circlearrowleft$ are the elements of $\mathfrak{U}(\circlearrowleft)$. An edge of $\mathfrak{U}(\circlearrowleft)$ that joins a vertex of $\mathfrak{W}(\circlearrowleft)$ to itself is called a loop. Edges connecting the same vertices are referred to as many edges. If two nodes ς_1 and ς_2 of $\circlearrowleft$ are joined by an edge α of $\circlearrowleft$, they are referred to be next to one another. In this instance, ς_1 and ς_2 are said to be connected by the edge α. Moreover, the vertices ς_1 and ς_2 are called the endpoints of this edge. If there are no edges connecting the two vertices ς_1 and ς_2 of $\circlearrowleft$, they are said to as non-adjacent to one another. The neighbors of ς (in $\circlearrowleft$) are the nodes that are next to ς for every node $\varsigma \in \mathfrak{W}(\circlearrowleft)$. The empty graph is represented by the pair $\circlearrowleft = (\emptyset, \emptyset)$. If $\circlearrowleft = (\mathfrak{W}, \mathfrak{U})$ and $\circlearrowleft' = (\mathfrak{W}', \mathfrak{U}')$, then $\circlearrowleft \cup \circlearrowleft' = (\mathfrak{W} \cup \mathfrak{W}', \mathfrak{U} \cup \mathfrak{U}')$ and $\circlearrowleft \cap \circlearrowleft' = (\mathfrak{W} \cap \mathfrak{W}', \mathfrak{U} \cap \mathfrak{U}')$. If $\circlearrowleft \cap \circlearrowleft' = (\emptyset, \emptyset)$, then $\circlearrowleft$ and $\circlearrowleft'$ are disjoint. If $\mathfrak{W} \subseteq \mathfrak{W}'$ and $\mathfrak{U} \subseteq \mathfrak{U}'$, then $\circlearrowleft'$ is a subgraph of $\circlearrowleft$ and $\circlearrowleft$ is a supergraph of $\circlearrowleft'$, written as $\circlearrowleft' \subseteq \circlearrowleft$. A graph $\circlearrowleft$ with no loops and no multiple edges is called a simple graph.

Definition 1. *[5] For* $\circlearrowleft = (\mathfrak{W}(\circlearrowleft), \mathfrak{U}(\circlearrowleft))$*, the collection* $\Im = \{\circlearrowleft' : \circlearrowleft' = (\mathfrak{W}', \mathfrak{U}')$*, where* $\mathfrak{W}' \subseteq \mathfrak{W}, \mathfrak{U}' \subseteq \mathfrak{U}\}$ *is said to be a graph ideal on a graph topological space* $(\mathfrak{W}(\circlearrowleft), \tau)$ *if it satisfies the following three conditions:*

(1) $(\emptyset, \emptyset) \in \Im$.

(2) If $\circlearrowleft'$ *is a subgraph of* $\circlearrowleft''$ *and the graph* $\circlearrowleft'' \in \Im$*, then* $\circlearrowleft' \in \Im$.

(3) If $\circlearrowleft'$ *and* $\circlearrowleft'' \in \Im$*, then* $\circlearrowleft' \cup \circlearrowleft'' \in \Im$.

Definition 2. *[1] On any non-empty universal set* E*, the operator* $\sum : 2^E \to 2^E$ *is a Kuratowski closure operator provided:*

(1) $\sum(\emptyset) = \emptyset$;

(2) $\Game \subseteq \sum(\Game)$ *for every* $\Game \in 2^E$;

(3) $\sum(\Game \cup \Omega) = \sum(\Game) \cup \sum(\Omega)$ *for any* $\Game, \Omega \in 2^E$;

(4) $\sum(\sum(\Game)) = \sum(\Game)$ *for every* $\Game \in 2^E$.

3. Graph proximity and graph ideal-proximity spaces

The section outlines the requirements that must be met in order to construct a graph proximity relation on a graph's vertices. The idea of graph closeness is demonstrated with examples and a discussion of the ramifications of these requirements. An important development in the subject is the introduction of novel graph proximity using ideals, which makes it possible to generate unique topological space using graph ideals and improve our knowledge of their features. Using a few counter-examples, a number of conclusions pertaining to the graph ideal-proximity spaces were thoroughly examined.

Definition 3. *Consider a graph* $\circlearrowleft = (\mathfrak{W}(\circlearrowleft), \mathfrak{U}(\circlearrowleft))$ *and* $\Game, \Omega$ *two subgraphs of* $\circlearrowleft$*. A binary relation* $\delta \subseteq \mathfrak{W}(\circlearrowleft) \times \mathfrak{W}(\circlearrowleft)$ *is said to be graph proximity on* $\circlearrowleft$*, if* δ *satisfies the following conditions:*

(i) if $\mathfrak{W}(\Game)\delta\mathfrak{W}(\Omega)$*, then* $\mathfrak{W}(\Omega)\delta\mathfrak{W}(\Game)$,

(ii) $\mathfrak{W}(\Game)\delta(\mathfrak{W}(\Omega) \cup \mathfrak{W}(\mathfrak{C})) \Leftrightarrow \mathfrak{W}(\Game)\delta\mathfrak{W}(\Omega)$ *or* $\mathfrak{W}(\Game)\delta\mathfrak{W}(\mathfrak{C})$,

(iii) if $\mathfrak{W}(\Game)\delta\mathfrak{W}(\Omega)$*, then* $\mathfrak{W}(\Game) \neq \emptyset$ *and* $\mathfrak{W}(\Omega) \neq \emptyset$,

(iv) if $\mathfrak{W}(\Game) \cap \mathfrak{W}(\Omega) \neq \emptyset$*, then* $\mathfrak{W}(\Game)\delta\mathfrak{W}(\Omega)$,

(v) if $\mathfrak{W}(\Game) \not\delta \mathfrak{W}(\Omega)$*, then there exist* $\mathfrak{W}(\mathfrak{C}) \subseteq \mathfrak{W}(\circlearrowleft)$ *and* $\mathfrak{W}(\mathfrak{D}) \subseteq \mathfrak{W}(\circlearrowleft)$ *such that* $\mathfrak{W}(\Game) \not\delta \mathfrak{W}(\mathfrak{C})^c$, $\mathfrak{W}(\mathfrak{D})^c \not\delta \mathfrak{W}(\Omega)$ *and* $\mathfrak{W}(\mathfrak{C}) \cap \mathfrak{W}(\mathfrak{D}) = \emptyset$.

A graph proximity space is a pair $(\circlearrowleft, \delta)$ *consisting of a graph* $\circlearrowleft$ *and a graph proximity relation on* $\mathfrak{W}(\circlearrowleft)$. *We shall write* $\mathfrak{W}(\circlearrowright)\delta\mathfrak{W}(\Omega)$ *if the sets* $\mathfrak{W}(\circlearrowright) \subseteq \mathfrak{W}(\circlearrowleft)$ *and* $\mathfrak{W}(\Omega) \subseteq \mathfrak{W}(\circlearrowleft)$ *are* δ*-related, otherwise we shall write* $\mathfrak{W}(\circlearrowright) \not\delta\, \mathfrak{W}(\Omega)$.

The following proposition is straightforward.

Proposition 1. *Let* $\circlearrowright, \Omega$ *be two subgraphs of* $\circlearrowleft$. *We define a binary relation* δ *on* $2^{\mathfrak{W}(\circlearrowleft)}$ *as:*

$$\mathfrak{W}(\circlearrowright)\delta\mathfrak{W}(\Omega) \Leftrightarrow \mathfrak{W}(\circlearrowright) \neq \emptyset \text{ and } \mathfrak{W}(\Omega) \neq \emptyset.$$

Then, δ *is a graph proximity relation.*

Remark 1. *Note that, Proposition 1 states that any two non-empty sets of vertices are considered proximal (near) to each other. The relation* δ *ignores the structure of the graph (edges) entirely; it only cares that the involved vertex sets are non-empty. We provide the following example for more explanation.*

Example 1. *Let* $\circlearrowleft$ *be a graph* $(\mathfrak{W}(\circlearrowleft), \mathfrak{U}(\circlearrowleft))$, *where* $\mathfrak{W}(\circlearrowleft) = \{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4, \varsigma_5, \varsigma_6\}$ *and* $\mathfrak{U}(\circlearrowleft) = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$. *A drawing of the graph* $\circlearrowleft$ *is shown in Figure 1.*

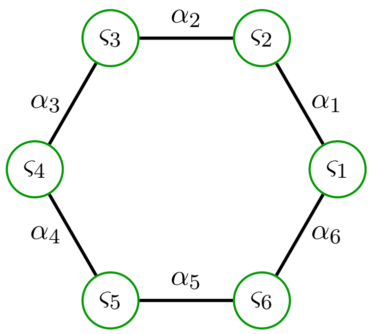


Figure 1: Graph defined in Example 1.

The initial graphical representation of the produced graph proximity relation δ *according to Proposition 1 is shown in Figure 2.*

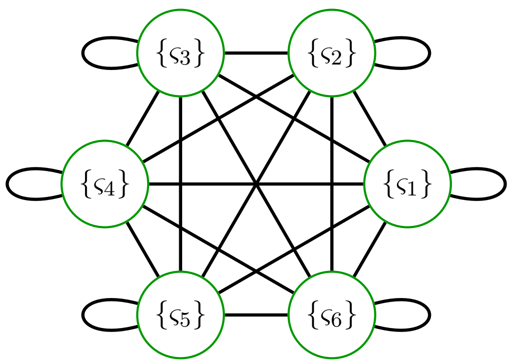


Figure 2: Initial graph of δ in Example 1.

Remark 2. *From Definition 3, on a non empty graph $\circlearrowleft$, we deduce that the graph proximity relation δ on the vertices set of the graph $\circlearrowleft$ is a complete graph with self-loops. The degree of each vertex of this graph is n, where n is the number of vertices in the graph of δ. The graphical representation of δ have the following properties:*

(1) δ is a regular graph with a degree of each vertex equal n, where n is the number of vertices in δ.

(2) If $\mathfrak{W}(\hexagon)$ is adjacent to $\mathfrak{W}(\Omega)$, then $\mathfrak{W}(\Omega)$ is adjacent to $\mathfrak{W}(\hexagon)$.

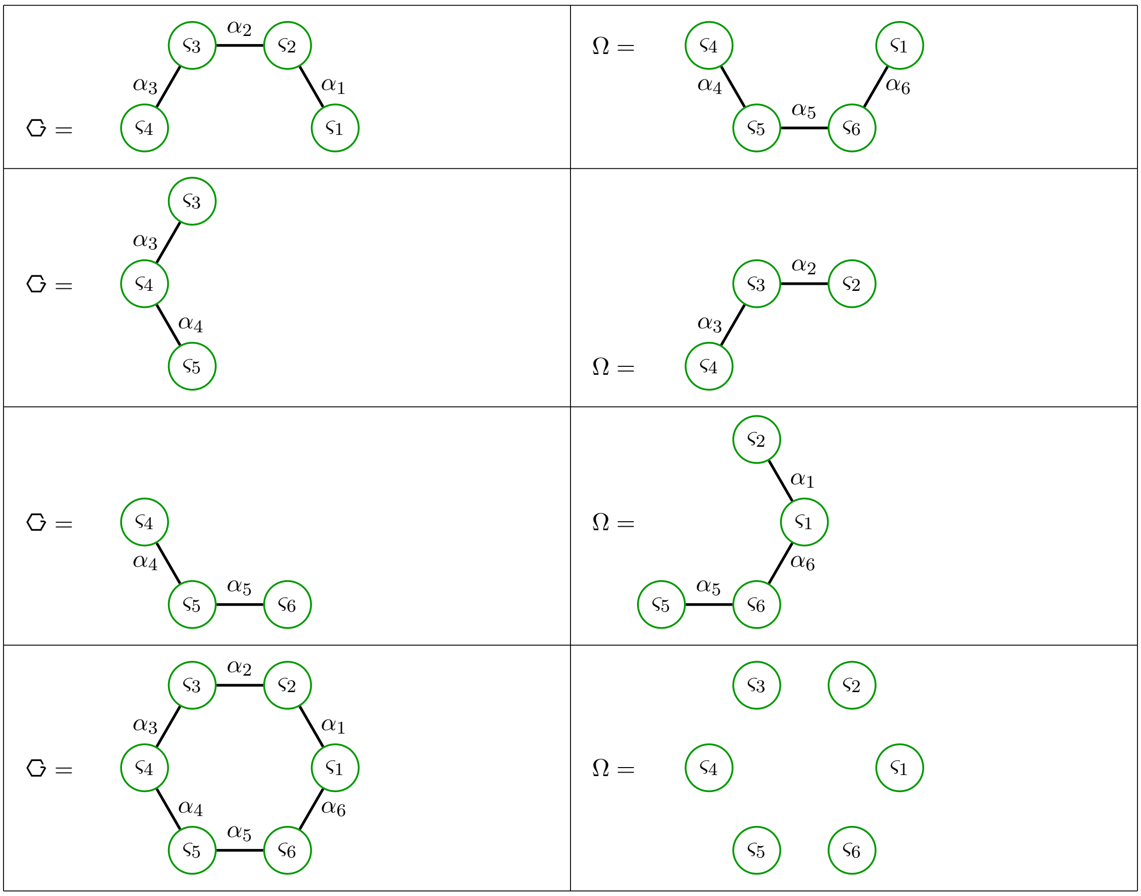
(3) If $\mathfrak{W}(\hexagon)$ is adjacent to $(\mathfrak{W}(\Omega) \cup \mathfrak{W}(\mathfrak{C}))$, then $\mathfrak{W}(\hexagon)$ is adjacent to $\mathfrak{W}(\Omega)$ or $\mathfrak{W}(\hexagon)$ is adjacent to $\mathfrak{W}(\mathfrak{C})$.

(4) If $\mathfrak{W}(\hexagon)$ is adjacent to $\mathfrak{W}(\Omega)$, then $\mathfrak{W}(\hexagon)$ and $\mathfrak{W}(\Omega)$ are not empty sets.

(5) If $\mathfrak{W}(\hexagon)$ and $\mathfrak{W}(\Omega)$ are not separated sets, then they are adjacent.

Remark 3. *Consider a graph $\circlearrowleft = (\mathfrak{W}(\circlearrowleft), \mathfrak{U}(\circlearrowleft))$. From the graph proximity definition, which is considered to be built on the graph vertices $\mathfrak{W}(\circlearrowleft)$ of the graph $\circlearrowleft$. The meaning of the graph proximity definition can be extended to cover the nearness between subgraphs of the graph $\circlearrowleft$. This means that, we can say that to subgraphs $\hexagon$ and Ω of $\circlearrowleft$ are near to each other if there vertices $\mathfrak{W}(\hexagon)$ and $\mathfrak{W}(\Omega)$ are near to each other. In other words, $\mathfrak{W}(\hexagon)\delta\mathfrak{W}(\Omega) \Rightarrow \hexagon\delta^{\circlearrowleft}\Omega$, where $\delta^{\circlearrowleft}$ is the corresponding graph proximity relation of the subgraphs $\hexagon$ and Ω. The conditions in Definition 3, are true for $\delta^{\circlearrowleft}$. For more explanation, consider Example 1. In Table 1, each pair of subgraphs of the graph $\circlearrowleft$ are $\delta^{\circlearrowleft}$ related. We can see that when the graph $\circlearrowleft$ is empty graph ($\mathfrak{U}(\circlearrowleft) = \emptyset$), then the graph proximity $\delta^{\circlearrowleft}$, coincides with the definition of δ. Further, this meaning of this remark is correct in following graph ideal-proximity.*

Table 1: Some near subgraphs according the meaning of graph proximity δ discussed in Remark 3.



Definition 4. *Let $\Im$ be a graph ideal on a non empty graph $\circlearrowleft$ and $\hexagon, \Omega$ two subgraphs of $\circlearrowleft$. A binary relation $\delta_{\Im}$ on the vertices set of the graph $\circlearrowleft$ is said to be a graph ideal-proximity relation on $\circlearrowleft$ if $\delta_{\Im}$ satisfies the following conditions:-*

($\Im_{P_1}$) *if* $\mathfrak{W}(\hexagon)\delta_{\Im}\mathfrak{W}(\Omega)$, *then* $\mathfrak{W}(\Omega)\delta_{\Im}\mathfrak{W}(\hexagon)$,

($\Im_{P_2}$) $\mathfrak{W}(\hexagon)\delta_{\Im}(\mathfrak{W}(\Omega) \cup \mathfrak{W}(\mathfrak{C})) \Leftrightarrow \mathfrak{W}(\hexagon)\delta_{\Im}\mathfrak{W}(\Omega)$ *or* $\mathfrak{W}(\hexagon)\delta_{\Im}\mathfrak{W}(\mathfrak{C})$,

($\Im_{P_3}$) $\mathfrak{W}(\hexagon)\overline{\delta}_{\Im}\mathfrak{W}(\Omega)$ *for all* $[(\mathfrak{W}(\hexagon) = \mathfrak{W}(\circlearrowleft')$ *for some* $\circlearrowleft' \in \Im), (\mathfrak{W}(\Omega) \in P(\mathfrak{W}(\circlearrowleft)))]$,

($\Im_{P_4}$) *if* $\mathfrak{W}(\hexagon) \cap \mathfrak{W}(\Omega) = \mathfrak{W}(\circlearrowleft')$ *for some* $\circlearrowleft' \notin \Im$, *then* $\mathfrak{W}(\hexagon)\delta_{\Im}\mathfrak{W}(\Omega)$,

($\Im_{P_5}$) *if* $\mathfrak{W}(\hexagon)\overline{\delta}_{\Im}\mathfrak{W}(\Omega)$, *then* $\exists \mathfrak{W}(\mathfrak{C}), \mathfrak{W}(\mathfrak{D}) \subseteq \mathfrak{W}(\circlearrowleft)$ *such that* $\mathfrak{W}(\hexagon)\overline{\delta}_{\Im}\mathfrak{W}(\mathfrak{C})^c, \mathfrak{W}(\mathfrak{D})^c\overline{\delta}_{\Im}\mathfrak{W}(\Omega)$ *and*
$\mathfrak{W}(\mathfrak{C}) \cap \mathfrak{W}(\mathfrak{D}) = \mathfrak{W}(\circlearrowleft')$ *for some* $\circlearrowleft' \in \Im$.

a graph ideal-proximity space is a pair $(\circlearrowleft, \delta_{\Im})$ consisting of a graph $\circlearrowleft$ and a graph ideal-proximity relation $\delta_{\Im}$ on $\circlearrowleft$. We shall write $\mathfrak{W}(\hexagon)\delta_{\Im}\mathfrak{W}(\Omega)$ if the vertices $\mathfrak{W}(\hexagon), \mathfrak{W}(\Omega) \subseteq$

$\mathfrak{W}(\circlearrowleft)$ *are* $\delta_{\Im}$*-related, otherwise we shall write* $\mathfrak{W}(\circlearrowright)\bar{\delta}_{\Im}\mathfrak{W}(\Omega)$*. The graph ideal-proximity relation* $\delta_{\Im}$ *is separated, if* $\varsigma_1\delta_{\Im}\varsigma_2 \Rightarrow \varsigma_1 = \varsigma_2$*.*

The following proposition is straightforward.

Proposition 2. *Let* $\Im = \{(\emptyset, \emptyset)\}$*. Then the graph ideal-proximity relation is a graph proximity relation. That is* $\delta_{\Im} = \delta$*.*

Example 2. *Consider a graph ideal* $\Im$ *on a non empty graph* $\circlearrowleft$ *and* $\delta_{\Im}$ *be a graph ideal-proximity relation on* $P(\mathfrak{W}(\circlearrowleft))$ *defined as:*

$$\mathfrak{W}(\circlearrowright)\delta_{\Im}\mathfrak{W}(\Omega) \Leftrightarrow \mathfrak{W}(\circlearrowright) = \mathfrak{W}(\circlearrowleft') \text{ and } \mathfrak{W}(\Omega) = \mathfrak{W}(\circlearrowleft'') \text{ for some } \circlearrowleft', \circlearrowleft'' \notin \Im.$$

Then $\delta_{\Im}$ *is a graph ideal-proximity relation. Indeed, one easily sees that* $\delta_{\Im}$ *satisfies conditions* $(\Im_{P_1}) - (\Im_{P_4})$*. To prove that* $\delta_{\Im}$ *also satisfies condition* $(\Im_{P_5})$*, let* $\mathfrak{W}(\circlearrowright)\bar{\delta}_{\Im}\mathfrak{W}(\Omega)$*. Therefore,* $\mathfrak{W}(\circlearrowright) = \mathfrak{W}(\circlearrowleft')$ *for some* $\circlearrowleft' \in \Im$ *or* $\mathfrak{W}(\Omega) = \mathfrak{W}(\circlearrowleft'')$ *for some* $\circlearrowleft'' \in \Im$*. If* $\mathfrak{W}(\circlearrowright) = \mathfrak{W}(\circlearrowleft')$ *for some* $\circlearrowleft' \in \Im$*, by taking* $\mathfrak{W}(\mathfrak{C}) = \mathfrak{W}(\circlearrowright)$ *and* $\mathfrak{W}(\mathfrak{D}) = \mathfrak{W}(\circlearrowright)^c$ *have the required properties. If* $\mathfrak{W}(\Omega) = \mathfrak{W}(\circlearrowleft'')$ *for some* $\circlearrowleft'' \in \Im$*, by taking* $\mathfrak{W}(\mathfrak{C}) = \mathfrak{W}(\Omega)^c$ *and* $\mathfrak{W}(\mathfrak{D}) = \mathfrak{W}(\Omega)$*. Hence, we complete the proof.*

According to the above example, we can generate a graph ideal-proximity as shown in the following example.

Example 3. *Let* $\circlearrowleft$ *be a graph* $(\mathfrak{W}(\circlearrowleft), \mathfrak{U}(\circlearrowleft))$*, where* $\mathfrak{W}(\circlearrowleft) = \{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4, \varsigma_5, \varsigma_6\}$ *and* $\mathfrak{U}(\circlearrowleft) = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$*. A drawing of the graph* $\circlearrowleft$ *is shown in Figure 3.*

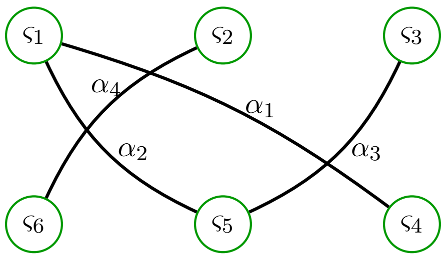


Figure 3: Graph defined in Example 3.

Let $\Im = \{(\emptyset, \emptyset), (\{\varsigma_1\}, \emptyset), (\{\varsigma_4\}, \emptyset), (\{\varsigma_5\}, \emptyset), (\{\varsigma_1, \varsigma_4\}, \emptyset), (\{\varsigma_1, \varsigma_5\}, \emptyset), (\{\varsigma_4, \varsigma_5\}, \emptyset), (\{\varsigma_1, \varsigma_4\}, \{\alpha_1\}), (\{\varsigma_1, \varsigma_5\}, \{\alpha_2\}), (\{\varsigma_1, \varsigma_4, \varsigma_5\}, \emptyset), (\{\varsigma_1, \varsigma_4, \varsigma_5\}, \{\alpha_1, \alpha_2\}), (\{\varsigma_2\}, \emptyset), (\{\varsigma_1, \varsigma_2\}, \emptyset), (\{\varsigma_1, \varsigma_2, \varsigma_4\}, \emptyset), (\{\varsigma_1, \varsigma_2, \varsigma_5\}, \emptyset\}), (\{\varsigma_1, \varsigma_2, \varsigma_4, \varsigma_5\}, \emptyset), (\{\varsigma_1, \varsigma_2, \varsigma_5\}, \{\alpha_2\}), (\{\varsigma_1, \varsigma_2, \varsigma_4, \varsigma_5\}, \{\alpha_1, \alpha_2\})\}$ *be a graph ideal.*

The corresponding graph ideal-proximity relation is given by,

$\delta_{\Im} = \{(\{\varsigma_6\}, \{\varsigma_6\}), (\{\varsigma_6\}, \{\varsigma_2\}), (\{\varsigma_6\}, \{\varsigma_3\}), (\{\varsigma_2\}, \{\varsigma_2\}), (\{\varsigma_2\}, \{\varsigma_6\}), (\{\varsigma_2\}, \{\varsigma_3\}), (\{\varsigma_3\}, \{\varsigma_3\}), (\{\varsigma_3\}, \{\varsigma_6\}), (\{\varsigma_3\}, \{\varsigma_2\}), (\{\varsigma_1, \varsigma_2\}, \{\varsigma_3\}), (\{\varsigma_1, \varsigma_2\}, \{\varsigma_6\}), (\{\varsigma_1, \varsigma_2\}, \{\varsigma_3, \varsigma_6\}), (\{\varsigma_1, \varsigma_2\}, \{\varsigma_3, \varsigma_4\}), (\{\varsigma_1, \varsigma_2\}, \{\varsigma_3, \varsigma_5\}), (\{\varsigma_1, \varsigma_2\}, \{\varsigma_3, \varsigma_4, \varsigma_5\}), (\{\varsigma_1, \varsigma_2\}, \{\varsigma_3, \varsigma_4, \varsigma_6\}), (\{\varsigma_1, \varsigma_2\}, \{\varsigma_1, \varsigma_3, \varsigma_4, \varsigma_5\}), (\{\varsigma_1, \varsigma_2\}, \{\varsigma_2, \varsigma_3, \varsigma_4, \varsigma_5\}), (\{\varsigma_1, \varsigma_2\}, \circlearrowleft), (\{\varsigma_3\}, \{\varsigma_1, \varsigma_2\}), (\{\varsigma_6\}, \{\varsigma_1, \varsigma_2\}), (\{\varsigma_3, \varsigma_6\}, \{\varsigma_1, \varsigma_2\}), (\{\varsigma_1, \varsigma_6\}, \{\varsigma_1, \varsigma_2\}), (\{\varsigma_1, \varsigma_2\}, \{\varsigma_1, \varsigma_2\}),$

$(\{\varsigma_1,\varsigma_3,\varsigma_6\},\{\varsigma_1,\varsigma_2\})$, $(\{\varsigma_2,\varsigma_3,\varsigma_6\},\{\varsigma_1,\varsigma_2\})$, $(\{\varsigma_3,\varsigma_4,\varsigma_6\},\{\varsigma_1,\varsigma_2\})$, $(\{\varsigma_3,\varsigma_5,\varsigma_6\},\{\varsigma_1,\varsigma_2\})$, $(\{\varsigma_1,\varsigma_2,\varsigma_3\},\{\varsigma_1,\varsigma_2\})$, $(\{\varsigma_1,\varsigma_2,\varsigma_4\},\{\varsigma_1,\varsigma_2\})$, $(\{\varsigma_1,\varsigma_2,\varsigma_5\},\{\varsigma_1,\varsigma_2\})$, $(\{\varsigma_1,\varsigma_2,\varsigma_6\},\{\varsigma_1,\varsigma_2\})$, $(\{\varsigma_1,\varsigma_3,\varsigma_4\},\{\varsigma_1,\varsigma_2\})$, $(\{\varsigma_1,\varsigma_3,\varsigma_5\},\{\varsigma_1,\varsigma_2\})$, $(\{\varsigma_1,\varsigma_3,\varsigma_6\},\{\varsigma_1,\varsigma_2\})$, $(\{\varsigma_1,\varsigma_5,\varsigma_6\},\{\varsigma_1,\varsigma_2\})$, $(\{\varsigma_1,\varsigma_2,\varsigma_3,\varsigma_4\},\{\varsigma_1,\varsigma_2\})$, $(\{\varsigma_1,\varsigma_2,\varsigma_3,\varsigma_5\},\{\varsigma_1,\varsigma_2\})$, $(\{\varsigma_1,\varsigma_2,\varsigma_3,\varsigma_6\},\{\varsigma_1,\varsigma_2\})$, $(\circlearrowleft,\{\varsigma_1,\varsigma_2\},\dots)\}$.

The initial graphical representation of the produced graph ideal-proximity-relation according to Example 2 is shown in Figure 4.

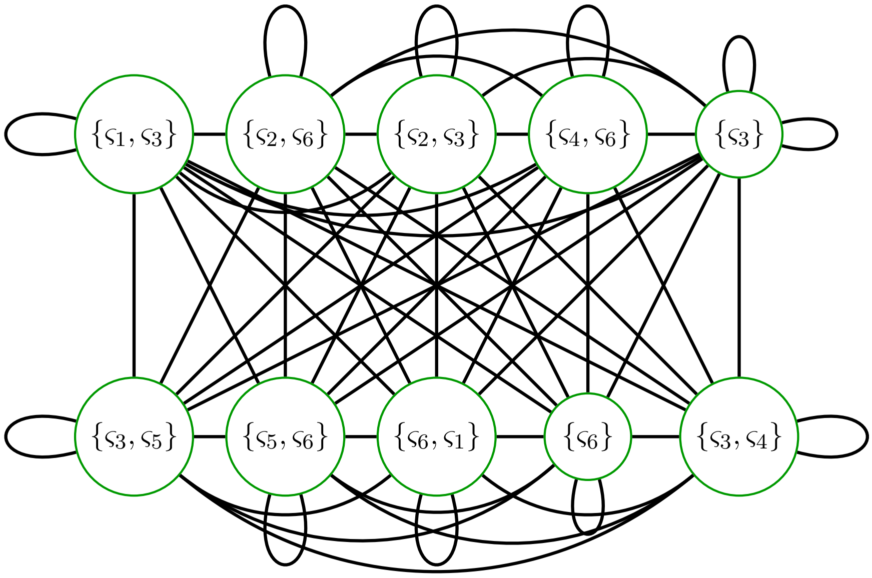


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Figure 4: Initial graph of $\delta_{\Im}$ in Example 3.

Example 4. *Let $\Im$ be a graph ideal on a nonempty graph $\circlearrowleft$. For any two subgraphs $\circlearrowright,\Omega\subseteq\circlearrowleft$, let us define*

$$\mathfrak{W}(\circlearrowright)\delta_{\mathscr{G}}\mathfrak{W}(\Omega)\Leftrightarrow\mathfrak{W}(\circlearrowright)\cap\mathfrak{W}(\Omega)=\mathfrak{W}(\circlearrowleft')\ \textit{for some}\ \circlearrowleft'\notin\mathscr{G}.$$

We shall illustrate that $\delta_{\mathscr{G}}$ is a graph ideal-proximity relation on the graph $\circlearrowleft$. According to Definition 4, we have $\delta_{\mathscr{G}}$ satisfies conditions $(\mathscr{G}P_1)-(\mathscr{G}P_4)$. To prove that $\delta_{\mathscr{G}}$ satisfies condition $(\mathscr{G}P_5)$, let $\mathfrak{W}(\circlearrowright)\overline{\delta}_{\mathscr{G}}\mathfrak{W}(\Omega)$. Therefore $\mathfrak{W}(\circlearrowright)\cap\mathfrak{W}(\Omega)=\mathfrak{W}(\circlearrowleft')$ for some $\circlearrowleft'\in\mathscr{G}$. Taking $\mathfrak{W}(\mathfrak{C})=\mathfrak{W}(\Omega)^c$ and $\mathfrak{W}(\mathfrak{D})=\mathfrak{W}(\Omega)$, completes the required proof.

The following lemma is straightforward.

Lemma 1. *If $\mathfrak{W}(\circlearrowright)\delta_{\Im}\mathfrak{W}(\Omega),\mathfrak{W}(\circlearrowright)\subseteq\mathfrak{W}(\mathfrak{C})$, and $\mathfrak{W}(\Omega)\subseteq\mathfrak{W}(\mathfrak{D})$, then $\mathfrak{W}(\mathfrak{C})\delta_{\Im}\mathfrak{W}(\mathfrak{D})$.*

Remark 4. *From Definition 4, on a non-empty graph $\circlearrowleft$, we deduce that the graph ideal-proximity binary relation $\delta_{\Im}$ is a complete graph with self-loops. The degree of each vertex of this graph is n, where n is the number of vertices in the graph of $\delta_{\Im}$. The graphical representation of $\delta_{\Im}$ have the following properties:*

(1) $\delta_{\Im}$ *is a regular graph with a degree of each vertex equal* n*, where* n *is the number of vertices in* $\delta_{\Im}$*.*

(2) If $\mathfrak{W}(\hexagon)$ *is adjacent to* $\mathfrak{W}(\Omega)$*, then* $\mathfrak{W}(\Omega)$ *is adjacent to* $\mathfrak{W}(\hexagon)$*.*

(3) If $\mathfrak{W}(\hexagon)$ *is adjacent to* $(\mathfrak{W}(\Omega) \cup \mathfrak{W}(\mathfrak{C}))$*, then* $\mathfrak{W}(\hexagon)$ *is adjacent to* $\mathfrak{W}(\Omega)$ *or* $\mathfrak{W}(\hexagon)$ *is adjacent to* $\mathfrak{W}(\mathfrak{C})$*.*

(4) If $\mathfrak{W}(\hexagon)$ *is adjacent to* $\mathfrak{W}(\Omega)$*, then* $\mathfrak{W}(\hexagon) = \mathfrak{W}(\circlearrowleft')$ *for some* $\circlearrowleft' \notin \Im$ *and* $\mathfrak{W}(\Omega) = \mathfrak{W}(\circlearrowleft'')$ *for some* $\circlearrowleft'' \notin \Im$*.*

(5) If $\mathfrak{W}(\hexagon) \cap \mathfrak{W}(\Omega) = \mathfrak{W}(\circlearrowleft')$ *for some* $\circlearrowleft' \notin \Im$*, then they are adjacent.*

Theorem 1. *Let* $(\circlearrowleft, \delta_{\Im})$ *be a graph ideal-proximity space. Then the* $\delta_{\Im}$*- operator*

$$\delta_{\Im} : P(\mathfrak{W}(\circlearrowleft)) \to P(\mathfrak{W}(\circlearrowleft))$$

defined by:

$$\mathfrak{W}(\hexagon)^{\delta_{\Im}} = \{\varsigma \in \mathfrak{W}(\circlearrowleft) : \varsigma \delta_{\Im} \mathfrak{W}(\hexagon)\}$$

satisfies the following:-

(1) $\mathfrak{W}(\hexagon) \subseteq \mathfrak{W}(\Omega) \Rightarrow \mathfrak{W}(\hexagon)^{\delta_{\Im}} \subseteq \mathfrak{W}(\Omega)^{\delta_{\Im}}$*,*

(2) $(\mathfrak{W}(\hexagon) \cup \mathfrak{W}(\Omega))^{\delta_{\Im}} = \mathfrak{W}(\hexagon)^{\delta_{\Im}} \cup \mathfrak{W}(\Omega)^{\delta_{\Im}}$*,*

(3) $(\mathfrak{W}(\hexagon) \cap \mathfrak{W}(\Omega))^{\delta_{\Im}} \subseteq \mathfrak{W}(\hexagon)^{\delta_{\Im}} \cap \mathfrak{W}(\Omega)^{\delta_{\Im}}$*,*

(4) $\mathfrak{W}(\hexagon)^{\delta_{\Im}} - \mathfrak{W}(\Omega)^{\delta_{\Im}} \subseteq (\mathfrak{W}(\hexagon) - \mathfrak{W}(\Omega))^{\delta_{\Im}}$*,*

(5) $\mathfrak{W}(\hexagon) = \mathfrak{W}(\circlearrowleft')$ *for some* $\circlearrowleft' \in \Im \Rightarrow \mathfrak{W}(\hexagon)^{\delta_{\Im}} = \emptyset$*,*

(6) $\mathfrak{W}(\Omega) = \mathfrak{W}(\circlearrowleft')$ *for some* $\circlearrowleft' \in \Im \Rightarrow (\mathfrak{W}(\hexagon) \cup \mathfrak{W}(\Omega))^{\delta_{\Im}} = \mathfrak{W}(\hexagon)^{\delta_{\Im}} = (\mathfrak{W}(\hexagon) - \mathfrak{W}(\Omega))^{\delta_{\Im}}$*,*

(7) $\mathfrak{W}(\hexagon) \Delta \mathfrak{W}(\Omega) = \mathfrak{W}(\circlearrowleft^{\star})$ *for some* $\circlearrowleft^{\star} \in \Im \Rightarrow \mathfrak{W}(\hexagon)^{\delta_{\Im}} = \mathfrak{W}(\Omega)^{\delta_{\Im}}$*, where* $\mathfrak{W}(\hexagon) \Delta \mathfrak{W}(\Omega) = (\mathfrak{W}(\hexagon) - \mathfrak{W}(\Omega)) \cup (\mathfrak{W}(\Omega) - \mathfrak{W}(\hexagon))$*,*

(8) $\mathfrak{W}(\hexagon)^{\delta_{\Im}} - \left(\mathfrak{W}(\Omega)^{\delta_{\Im}}\right)^{\delta_{\Im}} \subseteq \left(\mathfrak{W}(\hexagon) - \mathfrak{W}(\Omega)^{\delta_{\Im}}\right)^{\delta_{\Im}}$*.*

Proof.

(1) Assume that $\varsigma \in \mathfrak{W}(\hexagon)^{\delta_{\Im}}$. Then, the definition of $\mathfrak{W}(\hexagon)^{\delta_{\Im}}$ implies that $\varsigma \delta_{\Im} \mathfrak{W}(\hexagon)$ and
Lemma 1 implies that $\varsigma \delta_{\Im} \mathfrak{W}(\Omega)$. As a result, $\varsigma \in \mathfrak{W}(\Omega)^{\delta_{\Im}}$.

(2) According to part (1), we have $\mathfrak{W}(\hexagon)^{\delta_{\Im}} \cup \mathfrak{W}(\Omega)^{\delta_{\Im}} \subseteq (\mathfrak{W}(\hexagon) \cup \mathfrak{W}(\Omega))^{\delta_{\Im}}$. To prove the other inclusion, let $\varsigma \in (\mathfrak{W}(\hexagon) \cup \mathfrak{W}(\Omega))^{\delta_{\Im}}$. Then $\varsigma \delta_{\Im} (\mathfrak{W}(\hexagon) \cup \mathfrak{W}(\Omega))$. Hence $(\Im_{P_2})$ means that $\varsigma \delta_{\Im} \mathfrak{W}(\hexagon)$ or $\varsigma \delta_{\Im} \mathfrak{W}(\Omega)$, consequently $\varsigma \in \left(\mathfrak{W}(\hexagon)^{\delta_{\Im}} \cup \mathfrak{W}(\Omega)^{\delta_{\Im}}\right)$. Hence the result.

(3) The result is straightforward from part (1).

(4) $\forall \mathfrak{W}(\circlearrowleft), \mathfrak{W}(\Omega) \subseteq \mathfrak{W}(\circlearrowright), \mathfrak{W}(\circlearrowleft) = (\mathfrak{W}(\circlearrowleft) - \mathfrak{W}(\Omega)) \cup (\mathfrak{W}(\circlearrowleft) \cap \mathfrak{W}(\Omega))$. According to part (2), $\mathfrak{W}(\circlearrowleft)^{\delta_\Im} = (\mathfrak{W}(\circlearrowleft) - \mathfrak{W}(\Omega))^{\delta_\Im} \cup (\mathfrak{W}(\circlearrowleft) \cap \mathfrak{W}(\Omega))^{\delta_\Im}$, also part (3) implies that
$(\mathfrak{W}(\circlearrowleft) \cap \mathfrak{W}(\Omega))^{\delta_\Im} \subseteq \mathfrak{W}(\Omega)^{\delta_\Im}$. This implies that
$\mathfrak{W}(\circlearrowleft)^{\delta_\Im} - \mathfrak{W}(\Omega)^{\delta_\Im} \subseteq \left[(\mathfrak{W}(\circlearrowleft) - \mathfrak{W}(\Omega))^{\delta_\Im} - \mathfrak{W}(\Omega)^{\delta_\Im}\right] \subseteq (\mathfrak{W}(\circlearrowleft) - \mathfrak{W}(\Omega))^{\delta_\Im}$.

(5) Suppose that $\mathfrak{W}(\circlearrowleft) = \mathfrak{W}(\circlearrowright')$ for some $\circlearrowright' \in \Im$. Then $(\Im_{P_3})$ implies that $\varsigma\bar{\delta}_\Im \mathfrak{W}(\circlearrowleft) \forall \varsigma \in \mathfrak{W}(\circlearrowright)$. As a result, $\mathfrak{W}(\circlearrowleft)^{\delta_\Im} = \emptyset$.

(6) Let $\mathfrak{W}(\Omega) = \mathfrak{W}(\circlearrowright')$ for some $\circlearrowright' \in \Im$. According to parts (2), (5) and (4) of this theorem, we have the required result.

(7) Assume that $\mathfrak{W}(\circlearrowleft) \triangle \mathfrak{W}(\Omega) = (\mathfrak{W}(\circlearrowleft) - \mathfrak{W}(\Omega)) \cup (\mathfrak{W}(\Omega) - \mathfrak{W}(\circlearrowleft)) = \mathfrak{W}(\circlearrowright^\star)$ for some $\circlearrowright^\star \in \Im$, then $(\mathfrak{W}(\circlearrowleft) - \mathfrak{W}(\Omega)) = \mathfrak{W}(\circlearrowright')$ for some
$\circlearrowright' \in \Im, (\mathfrak{W}(\Omega) - \mathfrak{W}(\circlearrowleft)) = \mathfrak{W}(\circlearrowright'')$ for some $\circlearrowright'' \in \Im$. Since $\mathfrak{W}(\circlearrowleft)^{\delta_\Im} = ((\mathfrak{W}(\circlearrowleft) - \mathfrak{W}(\Omega)) \cup (\mathfrak{W}(\circlearrowleft) \cap \mathfrak{W}(\Omega)))^{\delta_\Im}$ and
$(\mathfrak{W}(\circlearrowleft) - \mathfrak{W}(\Omega)) = \mathfrak{W}(\circlearrowright')$ for some $\circlearrowright' \in \Im$, based on part (6), $\mathfrak{W}(\circlearrowleft)^{\delta_\Im} = (\mathfrak{W}(\circlearrowleft) \cap \mathfrak{W}(\Omega))^{\delta_\Im} \subseteq \mathfrak{W}(\Omega)^{\delta_\Im}$. It implies that $\mathfrak{W}(\circlearrowleft)^{\delta_\Im} \subseteq \mathfrak{W}(\Omega)^{\delta_\Im}$. Similarly, since $\mathfrak{W}(\Omega)^{\delta_\Im} = ((\mathfrak{W}(\Omega) - \mathfrak{W}(\circlearrowleft)) \cup (\mathfrak{W}(\circlearrowleft) \cap \mathfrak{W}(\Omega)))^{\delta_\Im}$ and $(\mathfrak{W}(\Omega) - \mathfrak{W}(\circlearrowleft)) = \mathfrak{W}(\circlearrowright'')$ for some $\circlearrowright'' \in \Im$. According to part (6), $\mathfrak{W}(\Omega)^{\delta_\Im} = (\mathfrak{W}(\circlearrowleft) \cap \mathfrak{W}(\Omega))^{\delta_\Im} \subseteq \mathfrak{W}(\circlearrowleft)^{\delta_\Im}$. As a result, $\mathfrak{W}(\Omega)^{\delta_\Im} \subseteq \mathfrak{W}(\circlearrowleft)^{\delta_\Im}$. Consequentially, $\mathfrak{W}(\circlearrowleft)^{\delta_\Im} = \mathfrak{W}(\Omega)^{\delta_\Im}$.

(8) Straightforward by using part (4).

Remark 5. *The following example shows that* $\mathfrak{W}(\circlearrowleft) \not\subseteq \mathfrak{W}(\circlearrowleft)^{\delta_\Im}$, *in general.*

Example 5. *Let* $\circlearrowright$ *be a graph* $(\mathfrak{W}(\circlearrowright), \mathfrak{U}(\circlearrowright))$, *where* $\mathfrak{W}(\circlearrowright) = \{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4\}$ *and* $\mathfrak{U}(\circlearrowright) = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$. *A drawing of the graph* $\circlearrowright$ *is shown in Figure 5.*

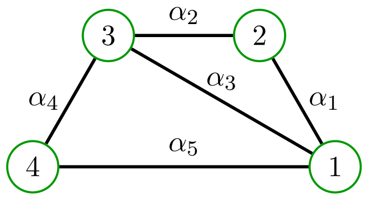

Figure 5: Graph defined in Example 5.

Define a graph ideal $\Im$ *on* $\circlearrowright$ *as:* $\Im = \{(\emptyset, \emptyset)\}, (\{\varsigma_1\}, \emptyset), (\{\varsigma_2\}, \emptyset), (\{\varsigma_1, \varsigma_2\}, \emptyset), (\{\varsigma_1 \varsigma_2\}, \{\alpha_1\})\}$. *Let* $\mathfrak{W}(\circlearrowleft) = \{\varsigma_2\}$ *and* $\delta_\Im$ *is defined as Example 4. Then* $\mathfrak{W}(\circlearrowleft)^{\delta_\Im} = \emptyset$.

Lemma 2. *Let* $(\circlearrowright, \delta_\Im)$ *be a graph ideal-proximity space and* $\circlearrowleft, \Omega$ *be two subgraphs of the graph* $\circlearrowright$.

$$\textit{If } \mathfrak{W}(\Omega)\bar{\delta}_\Im \mathfrak{W}(\circlearrowleft), \ \textit{then } \mathfrak{W}(\circlearrowleft)^{\delta_\Im} \subseteq \mathfrak{W}(\Omega)^c$$

Proof. Assume that $\hexagon^{\delta_\Im} \cap \Omega \neq \emptyset$. Then $\exists \varsigma \in \mathfrak{W}(\hexagon)^{\delta_\Im}$ and $\varsigma \in \mathfrak{W}(\Omega)$. Thus, $\varsigma \delta_\Im \mathfrak{W}(\hexagon)$ and $\varsigma \in \mathfrak{W}(\Omega)$. Lemma 1 implies $\mathfrak{W}(\hexagon)\delta_\Im\mathfrak{W}(\Omega)$. Then the result.

Theorem 2. *For every graph ideal-proximity relation $\delta_\Im$ on $\circlearrowleft$ and any tow subgraphs $\hexagon$, Ω of $\circlearrowleft$.*

$$\textit{If } \mathfrak{W}(\Omega)\overline{\delta}_\Im\mathfrak{W}(\hexagon), \ \textit{then } \mathfrak{W}(\Omega)\overline{\delta}_\Im\mathfrak{W}(\hexagon)^{\delta_\Im}.$$

Proof. Let $\mathfrak{W}(\Omega)\overline{\delta}_\Im\mathfrak{W}(\hexagon)$. Then $(\Im_{P_5})$ implies that $\exists \mathfrak{W}(\mathfrak{C}), \mathfrak{W}(\mathfrak{D}) \subseteq \mathfrak{W}(\circlearrowleft)$ such that

$$\mathfrak{W}(\Omega)\overline{\delta}_\Im\mathfrak{W}(\mathfrak{C})^c, \mathfrak{W}(\mathfrak{D})^c\overline{\delta}_\Im\mathfrak{W}(\hexagon) \text{ and } \mathfrak{W}(\mathfrak{C}) \cap \mathfrak{W}(\mathfrak{D}) = \mathfrak{W}(\circlearrowleft') \text{ for some } \circlearrowleft' \in \Im \tag{2.8}$$

This result, and using Lemma 2, implies $\mathfrak{W}(\hexagon)^{\delta_\Im} \subseteq \mathfrak{W}(\mathfrak{D})$.
Now, we want to prove that $\mathfrak{W}(\hexagon)^{\delta_\Im} \subseteq \mathfrak{W}(\mathfrak{C})^c$. Let $\varsigma \in \mathfrak{W}(\hexagon)^{\delta_\Im}$, then $\varsigma\delta_\Im\mathfrak{W}(\hexagon)$. If $\varsigma \in \mathfrak{W}(\mathfrak{C})$, then $\varsigma \in \mathfrak{W}(\mathfrak{C}) \cap \mathfrak{W}(\mathfrak{D})$. According to the proprieties of a graph ideal, we have $\{\varsigma\} = \mathfrak{W}(\circlearrowleft^\star)$ for some $\circlearrowleft^\star \in \Im$. Thus by $(\Im_{P_3})$ $\varsigma\overline{\delta}_\Im\mathfrak{W}(\hexagon)$, which is contradiction. Therefore, $\varsigma \in \mathfrak{W}(\mathfrak{C})^c$. Then, $\mathfrak{W}(\hexagon)^{\delta_\Im} \subseteq \mathfrak{W}(\mathfrak{C})^c$.
This results combined to Lemma 1 imply $\mathfrak{W}(\Omega)\overline{\delta}_\Im\mathfrak{W}(\hexagon)^{\delta_\Im}$.

The following corollary is straightforward from (IP P_1) and Theorem 2.

Corollary 1. *For every graph ideal-proximity relation $\delta_\Im$ on $\mathfrak{W}(\circlearrowleft)$ and any two subgraphs $\hexagon$, Ω of $\circlearrowleft$.*

$$\textit{If } \mathfrak{W}(\Omega)\overline{\delta}_\Im\mathfrak{W}(\hexagon), \ \textit{then } \mathfrak{W}(\Omega)^{\delta_\Im}\overline{\delta}_\Im\mathfrak{W}(\hexagon)^{\delta_\Im}.$$

Lemma 3. *Let $(\circlearrowleft, \delta_\Im)$ be a graph ideal-proximity space. Then*

$$\left(\mathfrak{W}(\hexagon)^{\delta_\Im}\right)^{\delta_\Im} \subseteq \mathfrak{W}(\hexagon)^{\delta_\Im}.$$

Proof. Let $\varsigma \notin \mathfrak{W}(\hexagon)^{\delta_\Im}$. Then $\varsigma\overline{\delta}_\Im\mathfrak{W}(\hexagon)$. Thus, Theorem 2 implies that $\varsigma\overline{\delta}_\Im\mathfrak{W}(\hexagon)^{\delta_\Im}$. Therefore, $\varsigma \notin \left(\mathfrak{W}(\hexagon)^{\delta_\Im}\right)^{\delta_\Im}$.

Proposition 3. *Let $(\circlearrowleft, \delta_\Im)$ be a graph ideal-proximity space, $\mathfrak{W}(\hexagon) \subseteq \mathfrak{W}(\circlearrowleft)$ and $\Im \subseteq P(\circlearrowleft)$. Then $\mathfrak{W}(\hexagon)^{\delta_\Im} = \emptyset$.*

Proof. Let $\Im = P(\circlearrowleft)$, then $\{\varsigma\} = \mathfrak{W}(\circlearrowleft')$ for some $\circlearrowleft' \in \Im$ for all $\varsigma \in \mathfrak{W}(\circlearrowleft)$. The axiom $(\Im_{P_3})$ implies that $\varsigma\overline{\delta}_\Im\mathfrak{W}(\hexagon)$. As a result, $\mathfrak{W}(\hexagon)^{\delta_\Im} = \emptyset$.

Theorem 3. *For a subgraph $\mathfrak{W}(\hexagon)$ of a n graph ideal-proximity space $(\circlearrowleft, \delta_\Im)$, the following statements are valid:-*

(1) $\mathfrak{W}(\hexagon) \cap \mathfrak{W}(\Omega)^{\delta_\Im} = \emptyset, \forall \mathfrak{W}(\hexagon) = \mathfrak{W}(\circlearrowleft')$ *for some* $\circlearrowleft' \in \Im$ *and* $\mathfrak{W}(\Omega) \subseteq \mathfrak{W}(\circlearrowleft)$,

(2) $\varsigma\delta_\Im\mathfrak{W}(\circlearrowleft), \forall \varsigma \in \mathfrak{W}(\circlearrowleft) \Leftrightarrow \Im = \{(\emptyset, \emptyset)\}$.

Proof.

(1) Assume that $\mathfrak{W}(\circlearrowright) \cap \mathfrak{W}(\Omega)^{\delta_\Im} \neq \emptyset$ and $\mathfrak{W}(\circlearrowright) = \mathfrak{W}(\circlearrowleft')$ for some $\circlearrowleft' \in \Im$. It follows that, $\exists \varsigma \in \mathfrak{W}(\circlearrowleft)$ such that $\varsigma \in \mathfrak{W}(\circlearrowright)$ and $\varsigma \delta_\Im \mathfrak{W}(\Omega)$. According to Lemma 1, we have $\mathfrak{W}(\circlearrowright)\delta_\Im \mathfrak{W}(\Omega)$ which is contradiction. Therefore, $\mathfrak{W}(\circlearrowright) \cap \mathfrak{W}(\Omega)^{\delta_\Im} = \emptyset$.

(2) If $\varsigma \delta_\Im \mathfrak{W}(\circlearrowleft), \forall \varsigma \in \mathfrak{W}(\circlearrowleft).(\Im_{P_3})$ implies that $\{\varsigma\} = \mathfrak{W}(\circlearrowleft')$ for some $\circlearrowleft' \notin \Im$ for all $\varsigma \in \mathfrak{W}(\circlearrowleft)$. Thus, $\Im = \{(\emptyset, \emptyset)\}$. Conversely, $\Im = \{(\emptyset, \emptyset)\}$ and axiom $(\Im_{P_4})$ imply the result.

4. Graph ideal-proximizable spaces and $\tau_{\delta_\Im}$ topology

The section focuses on the introduction of new graph proximity topological spaces utilizing ideals marks a significant advancement in the field and enhancing the understanding of their properties. Several results related to the graph ideal-proximity topological spaces, were discussed in details with the help of some counter-examples. The properties and relationships between the newly proposed topologies and existing structures were discussed.

Theorem 4. *Assume that $(\circlearrowleft, \delta_\Im)$ be a graph ideal-proximity space. Then, the operator*

$$\mathfrak{CL}^{\delta_\Im} : P(\mathfrak{W}(\circlearrowleft)) \to P(\mathfrak{W}(\circlearrowleft))$$

given by

$$\mathfrak{CL}^{\delta_\Im}(\mathfrak{W}(\circlearrowright)) = \mathfrak{W}(\circlearrowright) \cup \mathfrak{W}(\circlearrowright)^{\delta_\Im}$$

satisfies Kuratowski axioms and induces a topology on $\mathfrak{W}(\circlearrowleft)$ called $\tau_{\delta_\Im}$ given by:

$$\boldsymbol{\tau}_{\delta_\Im} = \left\{ \mathfrak{W}(\circlearrowright) \subseteq \mathfrak{W}(\circlearrowleft) : \mathfrak{CL}^{\delta_\Im}(\mathfrak{W}(\circlearrowright)^c) = \mathfrak{W}(\circlearrowright)^c \right\}.$$

Proof.

(1) According to axiom $(\Im_{P_3})$ $\emptyset^{\delta_\Im} = \emptyset$, and hence $\mathfrak{CL}^{\delta_\Im}(\emptyset) = \emptyset$.

(2) The definition of $\mathfrak{CL}^{\delta_\Im}(\mathfrak{W}(\circlearrowright))$, implies that $\mathfrak{W}(\circlearrowright) \subseteq \mathfrak{CL}^{\delta_\Im}(\mathfrak{W}(\circlearrowright))$.

(3) According to Theorem 1 (2), we have $\mathfrak{CL}^{\delta_\Im}(\mathfrak{W}(\circlearrowright) \cup \mathfrak{W}(\Omega)) = \mathfrak{CL}^{\delta_\Im}(\mathfrak{W}(\circlearrowright)) \cup \mathfrak{CL}^{\delta_\Im}(\mathfrak{W}(\Omega))$.

(4) According to Theorem 1 (1), we have

$$\mathfrak{CL}^{\delta^{\delta_\Im}}(\mathfrak{W}(\circlearrowright)) \subseteq \mathfrak{CL}^{\delta_\Im}\left(\mathfrak{CL}^{\delta_\Im}(\mathfrak{W}(\circlearrowright))\right).$$

Therefore, it suffices to show that $\forall \mathfrak{W}(\circlearrowright) \subseteq \mathfrak{W}(\circlearrowleft)$, we have $\mathfrak{CL}^{\delta_\Im}\left(\mathfrak{CL}^{\delta_\Im}(\mathfrak{W}(\circlearrowright))\right) \subseteq \mathfrak{CL}^{\delta_\Im}(\mathfrak{W}(\circlearrowright))$ or equivalently that

$$\text{if } \varsigma \notin \mathfrak{CL}^{\delta_\Im}(\mathfrak{W}(\circlearrowright)), \text{ then } \varsigma \notin \mathfrak{CL}^{\delta_\Im}\left(\mathfrak{CL}^{\delta_\Im}(\mathfrak{W}(\circlearrowright))\right).$$

Assume that, $\varsigma \notin \mathfrak{CL}^{\delta_\Im}(\mathfrak{W}(\circlearrowright))$. Hence, $\varsigma \notin \mathfrak{W}(\circlearrowright)$ and $\varsigma \overline{\delta}_\Im \mathfrak{W}(\circlearrowright)$. According to Theorem 1 , we have $\varsigma \overline{\delta}_\Im \mathfrak{W}(\circlearrowright)^{\delta_\Im}$ and axiom ($\Im P_2$) implies that $\varsigma \overline{\delta}_\Im \left(\mathfrak{W}(\circlearrowright) \cup \mathfrak{W}(\circlearrowright)^{\delta_\Im}\right)$. So, $\varsigma \overline{\delta}_\Im \mathfrak{CL}^{\delta_\Im}(\mathfrak{W}(\circlearrowright))$. This equations, combined with $\varsigma \overline{\delta}_\Im \mathfrak{W}(\circlearrowright)$, completes the proof.

Example 6. *Let* $\circlearrowleft$ *be a graph* $(\mathfrak{W}(\circlearrowleft), \mathfrak{U}(\circlearrowleft))$, *where* $\mathfrak{W}(\circlearrowleft) = \{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4\}$ *and* $\mathfrak{U}(\circlearrowleft) = \{\alpha_1, \alpha_2, \alpha_3,$
$\alpha_4\}$. *A drawing of the graph* $\circlearrowleft$ *is shown in Figure 6.*

$\circlearrowleft =$

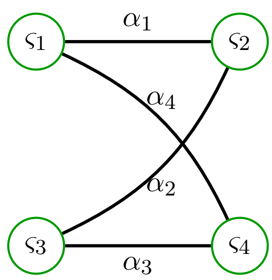

Figure 6: Graph defined in Example 6.

$\circlearrowright \subseteq \mathfrak{W}(\circlearrowleft)$	$\mathfrak{W}(\circlearrowright)^c$	$\mathfrak{W}(\circlearrowright)^{\delta_{\Im}}$	$\mathfrak{W}(\circlearrowright) \cup \mathfrak{W}(\circlearrowright)^{\delta_{\Im}}$	$\mathfrak{CL}^{\delta_{\Im}}(\mathfrak{W}(\circlearrowright)^c) = \mathfrak{W}(\circlearrowright)^c$?
$\emptyset$	$\{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4\}$	$\emptyset$	$\emptyset$	Yes
$\mathfrak{W}(\circlearrowleft)$	$\emptyset$	$\{\varsigma_2, \varsigma_3\}$	$\{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4\}$	Yes
$\{\varsigma_1\}$	$\{\varsigma_2, \varsigma_3, \varsigma_4\}$	$\emptyset$	$\{\varsigma_1\}$	Yes
$\{\varsigma_2\}$	$\{\varsigma_1, \varsigma_3, \varsigma_4\}$	$\{\varsigma_2, \varsigma_3\}$	$\{\varsigma_2, \varsigma_3\}$	No
$\{\varsigma_3\}$	$\{\varsigma_1, \varsigma_2, \varsigma_4\}$	$\{\varsigma_2, \varsigma_3\}$	$\{\varsigma_2, \varsigma_3\}$	No
$\{\varsigma_4\}$	$\{\varsigma_1, \varsigma_2, \varsigma_3\}$	$\emptyset$	$\{\varsigma_4\}$	Yes
$\{\varsigma_1, \varsigma_2\}$	$\{\varsigma_3, \varsigma_4\}$	$\{\varsigma_2, \varsigma_3\}$	$\{\varsigma_1, \varsigma_2, \varsigma_3\}$	No
$\{\varsigma_1, \varsigma_3\}$	$\{\varsigma_2, \varsigma_4\}$	$\{\varsigma_2, \varsigma_3\}$	$\{\varsigma_1, \varsigma_2, \varsigma_3\}$	No
$\{\varsigma_1, \varsigma_4\}$	$\{\varsigma_2, \varsigma_3\}$	$\emptyset$	$\{\varsigma_1, \varsigma_4\}$	Yes
$\{\varsigma_2, \varsigma_3\}$	$\{\varsigma_1, \varsigma_4\}$	$\{\varsigma_2, \varsigma_3\}$	$\{\varsigma_2, \varsigma_3\}$	Yes
$\{\varsigma_2, \varsigma_4\}$	$\{\varsigma_1, \varsigma_3\}$	$\{\varsigma_2, \varsigma_3\}$	$\{\varsigma_2, \varsigma_3, \varsigma_4\}$	No
$\{\varsigma_3, \varsigma_4\}$	$\{\varsigma_1, \varsigma_2\}$	$\{\varsigma_2, \varsigma_3\}$	$\{\varsigma_2, \varsigma_3, \varsigma_4\}$	No
$\{\varsigma_1, \varsigma_2, \varsigma_3\}$	$\{\varsigma_4\}$	$\{\varsigma_2, \varsigma_3\}$	$\{\varsigma_1, \varsigma_2, \varsigma_3\}$	Yes
$\{\varsigma_1, \varsigma_2, \varsigma_4\}$	$\{\varsigma_3\}$	$\{\varsigma_2, \varsigma_3\}$	$\{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4\}$	No
$\{\varsigma_1, \varsigma_3, \varsigma_4\}$	$\{\varsigma_2\}$	$\{\varsigma_2, \varsigma_3\}$	$\{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4\}$	No
$\{\varsigma_2, \varsigma_3, \varsigma_4\}$	$\{\varsigma_1\}$	$\{\varsigma_2, \varsigma_3\}$	$\{\varsigma_2, \varsigma_3, \varsigma_4\}$	Yes

Table 2: Illustration of Theorem 4

Define a graph ideal $\Im$ *on* $\circlearrowleft$ *as:* $\Im = \{(\emptyset, \emptyset)\}, (\{\varsigma_1\}, \emptyset), (\{\varsigma_4\}, \emptyset), (\{\varsigma_1, \varsigma_4\}, \emptyset), (\{\varsigma_1\, \varsigma_4\}, \{\alpha_4\})\}$.

The initial graphical representation of the graph ideal-proximity relation according to Example 2 is shown in Figure 7.

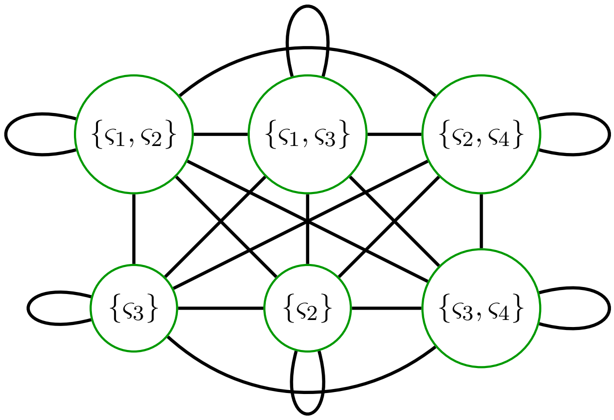

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Figure 7: Initial graph of $\delta_{\Im}$ in Example 6.

The computations of the graph local function associated to the defined graph ideal-proximity relation $\delta_{\Im}$ *are given in Table 2. According to Table 2,* $\tau_{\delta_{\Im}} = \{\emptyset, \mathfrak{W}(\circlearrowleft), \{\varsigma_1\}, \{\varsigma_4\}, \{\varsigma_1, \varsigma_4\}, \{\varsigma_2, \varsigma_3\}, \{\varsigma_1, \varsigma_2, \varsigma_3\}, \{\varsigma_2, \varsigma_3, \varsigma_4\}\}$.

The following theorem is straightforward from Theorem 2 and $(\Im_{P_2})$.

Theorem 5. *Let* $(\circlearrowleft, \delta_{\Im})$ *be a graph ideal-proximity space. Then the closure operator defined in Theorem 4, has the following property:*

$$\mathfrak{W}(\Omega)\bar{\delta}_{\Im}\mathfrak{W}(\circlearrowright) \Leftrightarrow \mathfrak{W}(\Omega)\bar{\delta}_{\Im}\mathfrak{CL}^{\delta_{\Im}}(\mathfrak{W}(\circlearrowright)).$$

Theorem 6. *Let* $(\circlearrowleft, \delta_{\Im})$ *be a graph ideal-proximity space. Then*

$$\mathfrak{CL}^{\delta_{\Im}}\left(\mathfrak{W}(\circlearrowright)^{\delta_{\Im}}\right) = \mathfrak{W}(\circlearrowright)^{\delta_{\Im}}.$$

Thus, $\mathfrak{W}(\circlearrowright)^{\delta_{\Im}}$ *is* $\tau_{\delta_{\Im}}-$ *closed set.*

Proof. We want to prove that $\mathfrak{CL}^{\delta_{\Im}}\left(\mathfrak{W}(\circlearrowright)^{\delta_{\Im}}\right) \subseteq \mathfrak{W}(\circlearrowright)^{\delta_{\Im}}$. Let $\varsigma \in \mathfrak{CL}^{\delta_{\Im}}\left(\mathfrak{W}(\circlearrowright)^{\delta_{\Im}}\right)$. Then $\varsigma \in \mathfrak{W}(\circlearrowright)^{\delta_{\Im}}$ or $\varsigma\delta_{\Im}\mathfrak{W}(\circlearrowright)^{\delta_{\Im}}$. It follows that $\varsigma \in \left(\mathfrak{W}(\circlearrowright)^{\delta_{\Im}}\right)^{\delta_{\Im}}$. Thus, according to Lemma 3, we get $\varsigma \in \mathfrak{W}(\circlearrowright)^{\delta_{\Im}}$.

The following proposition is straightforward from Proposition 3.

Proposition 4. *Let* $(\circlearrowleft, \delta_{\Im})$ *be a graph ideal-proximity space,* $\mathfrak{W}(\circlearrowright) \subseteq \mathfrak{W}(\circlearrowleft)$ *and* $\Im = P(\circlearrowleft)$*. Then* $\tau(\delta_{\Im}) = P(\circlearrowleft)$*.*

Definition 5. *A topological space* $(\mathfrak{W}(\circlearrowleft), \tau)$ *is called* $\Im$*-normal space if for all* $\mathfrak{W}(\mathfrak{F})_1, \mathfrak{W}(\mathfrak{F})_2 \in \tau^c$ *such that* $\mathfrak{W}(\mathfrak{F})_1 \cap \mathfrak{W}(\mathfrak{F})_2 = \mathfrak{W}(\circlearrowleft')$ *for some* $\circlearrowleft' \in \Im$*, then* $\exists H, G \in \tau$ *such that* $\mathfrak{W}(\mathfrak{F})_1 \subseteq H, \mathfrak{W}(\mathfrak{F})_2 \subseteq G$ *and* $H \cap G = \mathfrak{W}(\circlearrowleft')$ *for some* $\circlearrowleft' \in \Im$*, where* τ^c *is the collection of all* τ*-closed sets.*

Theorem 7. *Assume that,* $(\mathfrak{W}(\circlearrowleft), \tau)$ *be a normal space and* $\delta_{\Im}$ *be a graph ideal-proximity relation on* $\circlearrowleft$ *defined as:*

$$\mathfrak{W}(\circlearrowright)\delta_{\Im}\mathfrak{W}(\Omega) \Leftrightarrow \mathfrak{CL}(\mathfrak{W}(\circlearrowright)) \cap \mathfrak{CL}(\mathfrak{W}(\Omega)) = \mathfrak{W}(\circlearrowleft') \text{ for some } \circlearrowleft' \notin \Im \forall \mathfrak{W}(\circlearrowright), \mathfrak{W}(\Omega) \subseteq \mathfrak{W}(\circlearrowleft).$$

Then $\delta_{\Im}$ *is a graph ideal-proximity relation on* $\circlearrowleft$.

Proof. It follows directly from definition of the graph ideal-proximity relation $\delta_{\Im}$ in Theorem 7 that $\delta_{\Im}$ satisfies conditions $(\Im_{P_1}) - (\Im_{P_4})$. To prove that $\delta_{\Im}$ satisfies axiom $(\Im_{P_5})$. Assume that, $\mathfrak{W}(\circlearrowright)\bar{\delta}_{\Im}\mathfrak{W}(\Omega)$, then $\mathfrak{CL}(\mathfrak{W}(\circlearrowright)) \cap \mathfrak{CL}(\mathfrak{W}(\Omega)) = \mathfrak{W}(\circlearrowleft')$ for some $\circlearrowleft' \in \Im$. Since $(\mathfrak{W}(\circlearrowleft), \tau)$ is $\Im$-normal space, it follows that $\exists H, G \in \tau$ such that $\mathfrak{CL}(\mathfrak{W}(\circlearrowright)) \subseteq H, \mathfrak{CL}(\mathfrak{W}(\Omega)) \subseteq G$ and $H \cap G = \mathfrak{W}(\circlearrowleft'')$ for some $\circlearrowleft'' \in \Im$. Hence $\exists H, G \subseteq \mathfrak{W}(\circlearrowleft)$ such that $\mathfrak{W}(\circlearrowright)\delta_{\Im}H^c, G^c\bar{\delta}_{\Im}\mathfrak{W}(\Omega)$ and $H \cap G = \mathfrak{W}(\circlearrowleft'')$ for some $\circlearrowleft'' \in \Im$.

Definition 6. *A topological space* $(\mathfrak{W}(\circlearrowleft), \tau)$ *is called a graph ideal-proximizable space if there exists graph ideal-proximity relation* $\delta_{\Im}$ *such that* $\tau_{\delta_{\Im}} = \tau$.

Theorem 8. *(Main Theorem). Let* $\Im$ *be a graph ideal on a non empty graph* $\circlearrowleft$, $(\mathfrak{W}(\circlearrowleft), \tau)$ *be a* $\Im$*-normal* T_1 *space and* $\delta_{\Im}$ *is the formula given in Theorem 7. Then* $(\mathfrak{W}(\circlearrowleft), \tau)$ *is a graph ideal-proximizable space.*

Proof. To prove the theorem, it suffices to illustrate that the topology τ, generated by the closure operator $\mathfrak{CL}$ coincide with the topology $\tau_{\delta_{\Im}}$, generated by $\mathfrak{CL}^{\delta_{\Im}}$. In other words, we show that $\forall \mathfrak{W}(\circlearrowright) \subseteq \mathfrak{W}(\circlearrowleft)$, $\mathfrak{CL}(\mathfrak{W}(\circlearrowright)) = \mathfrak{CL}^{\delta_{\Im}}(\mathfrak{W}(\circlearrowright))$. Let $\varsigma \in \mathfrak{CL}^{\delta_{\Im}}(\mathfrak{W}(\circlearrowright))$. Then $\varsigma \in \mathfrak{W}(\circlearrowright)$ or $\varsigma \in \mathfrak{W}(\circlearrowright)^{\delta_{\Im}}$. If $\varsigma \in \mathfrak{W}(\circlearrowright)$, hence the result. Now, if $\varsigma \in \mathfrak{W}(\circlearrowright)^{\delta_{\Im}}$, then $\varsigma\delta_{\Im}\mathfrak{W}(\circlearrowright)$, and hence $\mathfrak{CL}(\{\varsigma\}) \cap \mathfrak{CL}(\mathfrak{W}(\circlearrowright)) = \mathfrak{W}(\circlearrowleft')$ for some $\circlearrowleft' \notin \Im$. Since $(\mathfrak{W}(\circlearrowleft), \tau)$ is T_1 space, then $\{\varsigma\} \cap \mathfrak{CL}(\mathfrak{W}(\circlearrowright)) = \mathfrak{W}(\circlearrowleft'')$ for some $\circlearrowleft'' \notin \Im$. Consequently, $\varsigma \in \mathfrak{CL}(\mathfrak{W}(\circlearrowright))$. Hence, $\mathfrak{CL}^{\delta_{\Im}}(\mathfrak{W}(\circlearrowright)) \subseteq \mathfrak{CL}(\mathfrak{W}(\circlearrowright))$.

Now, we want to prove that $\mathfrak{CL}(\mathfrak{W}(\circlearrowright)) \subseteq \mathfrak{CL}^{\delta_{\Im}}(\mathfrak{W}(\circlearrowright))$ or equivalently, if $\varsigma \notin \mathfrak{CL}^{\delta_{\Im}}(\mathfrak{W}(\circlearrowright))$, then $\varsigma \notin \mathfrak{CL}(\mathfrak{W}(\circlearrowright))$. Let $\varsigma \notin \mathfrak{CL}^{\delta_{\Im}}(\mathfrak{W}(\circlearrowright))$, then $\varsigma \notin \mathfrak{W}(\circlearrowright)$ and $\varsigma \notin \mathfrak{W}(\circlearrowright)^{\delta_{\Im}}$. It implies that, $\varsigma\bar{\delta}_{\Im}\mathfrak{W}(\circlearrowright)$ and hence, the graph ideal-proximity relation in Theorem 7 implies that $\mathfrak{CL}(\{\varsigma\}) \cap \mathfrak{CL}(\mathfrak{W}(\circlearrowright)) = \mathfrak{W}(\circlearrowleft'')$ for some $\circlearrowleft'' \in \Im$. Since $(\mathfrak{W}(\circlearrowleft), \tau)$ is $\Im$-normal T_1 space, then $\exists H, G \in \tau$ such that

$$\{\varsigma\} \subseteq H, \mathfrak{CL}(\mathfrak{W}(\circlearrowright)) \subseteq G \text{ and } H \cap G = \mathfrak{W}(\circlearrowleft''') \text{ for some } \circlearrowleft''' \in \Im.$$

According to the definition of a graph ideal, we get $H \cap \mathfrak{W}(\circlearrowright) = \mathfrak{W}(\circlearrowleft'''')$ for some $\circlearrowleft'''' \in \Im$. Therefore, $\exists H \in \tau, \varsigma \in H$ such that $H \cap \mathfrak{W}(\circlearrowright) = \mathfrak{W}(\circlearrowleft''''')$ for some $\circlearrowleft''''' \in \Im$. As a result, $\varsigma \notin \mathfrak{CL}(\mathfrak{W}(\circlearrowright))$. It follows that $\mathfrak{CL}(\mathfrak{W}(\circlearrowright)) \subseteq \mathfrak{CL}^{\delta_{\Im}}(\mathfrak{W}(\circlearrowright))$. This result, completes the proof of the theorem.

5. An application of nearness via graph ideals

Applications of graph theory to real-world issues have shown a great deal of interest in recent decades. These mathematical methods may be used to simulate a wide range of systems where intricate interactions between the system's constituents are crucial, including networks, biological networks, data structures, process scheduling, computations, and more. Here, we try to propose an applicable illustration graph ideal-proximities utilizing a suitable given graph ideal. Similar to the classical idea of near sets, two graphs are near, as long as the vertices of these graphs possess joint elements. Further, nearness among vertices of graphs means that these graphs have the same properties as shown in the following example.

Example 7. ***Selection of a house:***
Considering $\{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4, \varsigma_5, \varsigma_6\}$ *is a collection of six houses where* H *=*{*expensive, beautiful, cheap, in green surroundings, wooden modern, in good repair, in bad repair*} *be a set of parameters. The data is given as in Table 3. From the data, we can deduce that* $\circlearrowright$ *be the graph* $(\mathfrak{W}(\circlearrowright), \mathfrak{U}(\circlearrowright))$, *where* $\mathfrak{W}(\circlearrowright) = \{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4, \varsigma_5, \varsigma_6\}$ *and* $\mathfrak{U}(\circlearrowright) = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}, \alpha_{11}, \alpha_{12}\}$. *We represent the graph* $\circlearrowright$ *as in Figure 8.*

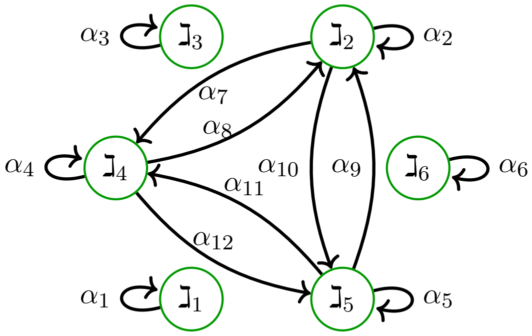

Figure 8: Graph defined in Example 7.

Table 3: Decision system of six houses in Example 7.

House	Expensive	Beautiful	Cheap	In Green surroundings	Wooden modern	In good repair	In bad repair
ς_1	Medium	Medium	Medium	Low	High	High	Low
ς_2	Low	Medium	Medium	High	High	High	High
ς_3	Medium	High	Low	Low	High	High	High
ς_4	Low	Medium	Medium	High	High	High	High
ς_5	Low	Medium	Medium	High	High	High	Low
ς_6	Low	High	Medium	Low	High	High	High

From Table3, we have:

- *The set of vertices is the set of houses:* $\mathfrak{V}(\circlearrowright) = \{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4, \varsigma_5, \varsigma_6\}$.
- *The set of attributes:* AT *=*{*expensive, beautiful, cheap, in green surroundings, wooden modern, in good repair, in bad repair*}.

- *The family of edges are those connect vertices of the same value of attributes.*

We suppose that ***Mr.Z*** *wants to purchase a house based on the following conditions* {*beautiful, cheap, in green surroundings, wooden, in good repair*}. *Consequently, anyone can propose a suitable ideal* $\Im$ *and a graph ideal-proximity relation* $\delta_{\Im}$ *to illustrate the nearness of vertices.*
For example, let $\Im = \{\circlearrowright' = (\mathfrak{W}(\circlearrowright'), \mathfrak{U}(\circlearrowright') \subseteq \circlearrowright : \circlearrowright'$ *is a subgraph over the set of vertices* $\{\varsigma_1, \varsigma_3, \varsigma_5\}\}$ *and a graph ideal-proximity relation defined as:*

$$\mathfrak{W}(\mathcal{G})\delta_{\Im}\mathfrak{W}(\Omega) \Leftrightarrow \mathfrak{W}(\mathcal{G}) = \mathfrak{W}(\circlearrowright'), \mathfrak{W}(\Omega) = \mathfrak{W}(\circlearrowright'') \text{ for some } \circlearrowright', \circlearrowright'' \notin \Im.$$

Then $\delta_{\Im}$ *is a graph ideal-proximity relation. In simpler terms: Two sets of vertices,* $\mathfrak{W}(\mathcal{G})$ *and* $\mathfrak{W}(\Omega)$*, are in graph ideal-proximity* ($\delta_{\Im}$*-related) if and only if neither of them can be generated by a subgraph belonging to the graph ideal* $\Im$ *(i.e., they are both "large" or "non-negligible" subsets of vertices). The above condition states that* $\delta_{\Im}$ *is a graph ideal-proximity relation because it satisfies conditions* $(\Im_{P_1}) - (\Im_{P_5})$. *These conditions are the axioms for a graph ideal-proximity relation (which is an extension of the general graph proximity relation* δ. *As a result we can construct* $\delta_{\Im}$ *as follows:*
$\delta_{\Im} = \{(\{\varsigma_2\}, \{\varsigma_2\}), (\{\varsigma_2\}, \{\varsigma_4\}), (\{\varsigma_2\}, \{\varsigma_6\}), (\{\varsigma_2\}, \{\varsigma_2, \varsigma_4\}), (\{\varsigma_2\}, \{\varsigma_2, \varsigma_6\}), (\{\varsigma_2\}, \{\varsigma_2, \varsigma_4, \varsigma_6\}),$
$(\{\varsigma_4\}, \{\varsigma_2\}), (\{\varsigma_4\}, \{\varsigma_4\}), (\{\varsigma_2\}, \{\varsigma_6\}), (\{\varsigma_4\}, \{\varsigma_2, \varsigma_4\}), (\{\varsigma_4\}, \{\varsigma_2, \varsigma_6\}), (\{\varsigma_4\}, \{\varsigma_2, \varsigma_4, \varsigma_6\}), (\{\varsigma_6\}, \{\varsigma_2\}),$
$(\{\varsigma_6\}, \{\varsigma_4\}), (\{\varsigma_6\}, \{\varsigma_6\}), (\{\varsigma_6\}, \{\varsigma_2, \varsigma_4\}), (\{\varsigma_6\}, \{\varsigma_2, \varsigma_6\}), (\{\varsigma_6\}, \{\varsigma_2, \varsigma_4, \varsigma_6\}), (\{\varsigma_2, \varsigma_2\}, \{\varsigma_2\}), (\{\varsigma_2, \varsigma_2\}, \{\varsigma_4\}),$
$(\{\varsigma_2, \varsigma_2\}, \{\varsigma_6\}), (\{\varsigma_2, \varsigma_2\}, \{\varsigma_2, \varsigma_4\}), (\{\varsigma_2, \varsigma_2\}, \{\varsigma_2, \varsigma_6\}), (\{\varsigma_2, \varsigma_2\}, \{\varsigma_2, \varsigma_4, \varsigma_6\})(\{\varsigma_4, \varsigma_4\}, \{\varsigma_2\}), (\{\varsigma_4, \varsigma_4\}, \{\varsigma_4\}),$
$(\{\varsigma_4, \varsigma_4\}, \{\varsigma_6\}), (\{\varsigma_4, \varsigma_4\}, \{\varsigma_2, \varsigma_4\}), (\{\varsigma_4, \varsigma_4\}, \{\varsigma_2, \varsigma_6\}), (\{\varsigma_4, \varsigma_4\}, \{\varsigma_2, \varsigma_4, \varsigma_6\}), (\{\varsigma_6, \varsigma_6\}, \{\varsigma_2\}), (\{\varsigma_6, \varsigma_6\}, \{\varsigma_4\}),$
$(\{\varsigma_6, \varsigma_6\}, \{\varsigma_6\}), (\{\varsigma_6, \varsigma_6\}, \{\varsigma_2, \varsigma_4\}), (\{\varsigma_6, \varsigma_6\}, \{\varsigma_2, \varsigma_6\}), (\{\varsigma_6, \varsigma_6\}, \{\varsigma_2, \varsigma_4, \varsigma_6\})(\{\varsigma_2, \varsigma_4\}, \{\varsigma_2\}), (\{\varsigma_2, \varsigma_4\}, \{\varsigma_4\}),$
$(\{\varsigma_2, \varsigma_4\}, \{\varsigma_6\}), (\{\varsigma_2, \varsigma_4\}, \{\varsigma_2, \varsigma_4\}), (\{\varsigma_2, \varsigma_4\}, \{\varsigma_2, \varsigma_6\}), (\{\varsigma_2, \varsigma_4\}, \{\varsigma_2, \varsigma_4, \varsigma_6\}), (\{\varsigma_2, \varsigma_6\}, \{\varsigma_2\}), (\{\varsigma_2, \varsigma_6\}, \{\varsigma_4\}),$
$(\{\varsigma_2, \varsigma_6\}, \{\varsigma_6\}), (\{\varsigma_2, \varsigma_6\}, \{\varsigma_2, \varsigma_6\}), (\{\varsigma_2, \varsigma_6\}, \{\varsigma_2, \varsigma_6\}), (\{\varsigma_2, \varsigma_6\}, \{\varsigma_2, \varsigma_4, \varsigma_6\}), \{\varsigma_4, \varsigma_6\}, \{\varsigma_2\}), (\{\varsigma_4, \varsigma_6\}, \{\varsigma_4\}),$
$(\{\varsigma_4, \varsigma_6\}, \{\varsigma_6\}), (\{\varsigma_4, \varsigma_6\}, \{\varsigma_2, \varsigma_4\}), (\{\varsigma_4, \varsigma_6\}, \{\varsigma_2, \varsigma_6\}), (\{\varsigma_4, \varsigma_6\}, \{\varsigma_2, \varsigma_4, \varsigma_6\}), (\{\varsigma_2, \varsigma_2, \varsigma_4\}, \{\varsigma_2\}), (\{\varsigma_2, \varsigma_2, \varsigma_4\},$
$\{\varsigma_4\}), (\{\varsigma_2, \varsigma_2, \varsigma_4\}, \{\varsigma_6\}), (\{\varsigma_2, \varsigma_2, \varsigma_4\}, \{\varsigma_2, \varsigma_4\}), (\{\varsigma_2, \varsigma_2, \varsigma_4\}, \{\varsigma_2, \varsigma_6\}), (\{\varsigma_2, \varsigma_2, \varsigma_4\}, \{\varsigma_2, \varsigma_4, \varsigma_6\}), (\{\varsigma_2, \varsigma_2, \varsigma_6\},$
$\{\varsigma_2\}), (\{\varsigma_2, \varsigma_2, \varsigma_6\}, \{\varsigma_4\}), (\{\varsigma_2, \varsigma_2, \varsigma_6\}, \{\varsigma_6\}), (\{\varsigma_2, \varsigma_2, \varsigma_6\}, \{\varsigma_2, \varsigma_4\}), (\{\varsigma_2, \varsigma_2, \varsigma_6\}, \{\varsigma_2, \varsigma_6\}), (\{\varsigma_2, \varsigma_2, \varsigma_6\},$
$\{\varsigma_2, \varsigma_4, \varsigma_6\}), (\{\varsigma_2, \varsigma_4, \varsigma_6\}, \{\varsigma_2\}), (\{\varsigma_2, \varsigma_4, \varsigma_6\}, \{\varsigma_4\}), (\{\varsigma_2, \varsigma_4, \varsigma_6\}, \{\varsigma_6\}), (\{\varsigma_2, \varsigma_4, \varsigma_6\}, \{\varsigma_2, \varsigma_4\}),$
$(\{\varsigma_2, \varsigma_4, \varsigma_6\}, \{\varsigma_2, \varsigma_6\}), (\{\varsigma_2, \varsigma_4, \varsigma_6\}, \{\varsigma_2, \varsigma_4, \varsigma_6\}) \ldots\}.$

This method is the best tool to help ***Mr.Z*** *in his decision-making about selecting the house that is most suitable to his choice of parameters. For example, take* $\{\varsigma_2, \varsigma_3, \varsigma_4\}$. *Then, from the computations of the graph ideal-proximity relation* $\delta_{\Im}$*, we find that* $\mathfrak{CL}^{\delta_{\Im}}(\mathfrak{W}(\mathcal{G})) = \{\varsigma_2, \varsigma_3, \varsigma_4\}$ *and only the vertex* ς_3 *is open. One can see that* ***Mr.Z*** *will decide to buy the house* ς_3 *according to his choice parameters in* H. *It should be noticed that, all applications related to decision makings are only restricted to the information of individuals only, not their interactions. graph ideal-proximity relations, we are able to involve the interactions (edges) of individuals with each other that aim to enhancing the accuracy in decisions. This simple example shows how to apply the nearness between subgraphs of a given graph, which would be extended to many real-life applications. One may define a graph ideal-proximity relation as* $\mathfrak{W}(\mathcal{G})\delta_{\Im}\mathfrak{W}(\Omega)$ *if and only if either there is a path from* $\mathfrak{W}(\mathcal{G})$ *to* $\mathfrak{W}(\Omega)$ *or* $d(\mathfrak{W}(\mathcal{G}), \mathfrak{W}(\Omega)) = 0$. *Notice that if there is a path from* $\mathfrak{W}(\mathcal{G})$ *to* $\mathfrak{W}(\Omega)$ *there is also a path from* $\mathfrak{W}(\Omega)$ *to* $\mathfrak{W}(\mathcal{G})$. *In this case we say that an edge connects*

$\mathfrak{W}(\hexagon)$ and $\mathfrak{W}(\Omega)$. The distance $d(\mathfrak{W}(\hexagon), \mathfrak{W}(\Omega))$ between two vertices sets is the minimum length which connects $\mathfrak{W}(\hexagon)$ and $\mathfrak{W}(\Omega)$. The path length corresponds the number of edges in the path. The distance between two vertices $\varsigma_1 \in \mathfrak{W}(\hexagon)$ and $\varsigma_2 \in \mathfrak{W}(\Omega)$ is the length of the shortest path between ς_1 and ς_2, which is denoted by $d(\varsigma_1, \varsigma_2)$. We say that $d(\varsigma_1, \varsigma_2) = 0$ if and only if $\varsigma_1 = \varsigma_2$. As a result, we say that $d(\mathfrak{W}(\hexagon), \mathfrak{W}(\Omega)) = 0$ if and only if $\mathfrak{W}(\hexagon) = \mathfrak{W}(\Omega)$.

6. Conclusions

In this paper, we have made a new contribution to the field of graph theory by introducing the notion of "graph proximity", which is extension of the proximity on a set. We considered that two subgraphs of a given graph are near to each other if the vertices of these graphs are near. We proposed some possible graph proximity relations with proofs and suitable examples. Also, graph ideal-proximity relation was proposed and studied with the aid of suitable examples. We generated a new topological space over the vertices of a given graph using a closure operator generated on these vertices corresponding to the proposed graph ideal-proximity. Near set theory supplies a major for the classification of members of a set in classes depending on their closeness. We follow the same idea in the graph theory. So, our definitions of graph proximities depend on the nearness of vertices of these graphs. that is we say that two graphs are near if there vertices are near. Based on the idea of nearness between vertices, a real-life application is provided to demonstrate the significance of this research. All these points seem to be much promising for further interesting research.

The results of this study are preliminary, and future research may yield more information by looking at other graph proximity relations aspects like fuzzy and soft graph proximities, among others. We'll also look at the results of directed graphs with loops. The creation of generalized rough approximation spaces that improve the accuracy of lower and upper approximations, on the other hand, is the prospective goal of using the graph proximity relations. This approach is especially useful for decision-making tasks where accurate estimates are necessary to produce useful results. The method aims to enhance the entire decision-making process by applying sophisticated analytical tools and principles from graph theory. The future of this work will be fascinating.

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Conflicts of interest: The authors declare that they have no conflicts of interest.

Nomenclature: The following symbols were used in this paper:

Symbol	**Description**
$\circlearrowleft$	Simple undirected graph
$\mathfrak{W}(\circlearrowleft)$	Set of vertices (nodes) over $\circlearrowleft$
$\mathfrak{U}(\circlearrowleft)$	Set of edges over $\circlearrowleft$
$P(\circlearrowleft)$ or $2^{\circlearrowleft}$	Family of all subgraphs of $\circlearrowleft$
$P(\mathfrak{W}(\circlearrowleft)), P(\mathfrak{U}(\circlearrowleft))$	Family of all subsets of $\mathfrak{W}(\circlearrowleft), \mathfrak{U}(\circlearrowleft)$, respectively
$\hexagon, \Omega, \circlearrowleft^{\star}, \circlearrowleft', \circlearrowleft'', \ldots$	Subgraphs of $\circlearrowleft$
$\mathfrak{W}(\hexagon)^c$	Complement of the vertices set $\mathfrak{W}(\hexagon)$
$\varsigma, \varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4, \varsigma_5, \varsigma_6$	Vertices (nods) of $\circlearrowleft$
$\alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \ldots$	Edges of $\circlearrowleft$
δ	Graph proximity relation on $\circlearrowleft$
$\mathfrak{W}(\hexagon) \not\delta \mathfrak{W}(\Omega)$	Means that the vertices $\mathfrak{W}(\hexagon), \mathfrak{W}(\Omega)$ are not δ-related
$\mathfrak{I}$	Graph ideal
$\delta_{\mathfrak{I}}$	Graph ideal-proximity relation on $\circlearrowleft$
$\mathfrak{W}(\hexagon)\overline{\delta}_{\mathfrak{I}}\mathfrak{W}(\Omega)$	Means that the vertices $\mathfrak{W}(\hexagon), \mathfrak{W}(\Omega)$ are not $\delta_{\mathfrak{I}}$-related
τ, τ^c	Adjacency topological space (ATS) on $\circlearrowleft$ and its complement, respectively
$\mathfrak{CL}(\mathfrak{W}(\hexagon)$ and $\mathfrak{int}(\mathfrak{W}(\hexagon)$	Closure and interior of $\mathfrak{W}(\hexagon)$ with respect to ATS τ, respectively
$\tau_{\delta_{\mathfrak{I}}}$	Topology generated by $\delta_{\mathfrak{I}}$
$\mathfrak{CL}^{\delta_{\mathfrak{I}}}(\mathfrak{W}(\hexagon))$ and $\mathfrak{int}^{\delta_{\mathfrak{I}}}(\mathfrak{W}(\hexagon)$	Closure and interior of $\mathfrak{W}(\hexagon)$ with respect to $\tau_{\delta_{\mathfrak{I}}}$, respectively

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