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A Relaxed Two-Inertial Subgradient Extragradient Method for Solving Equilibrium and Fixed Point Problems with Applications

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Abstract. In this research, we introduce an improved pseudomonotone subgradient extragradient algorithm for finding common solutions to equilibrium and fixed-point problems in real Hilbert spaces. Under mild and suitable assumptions on the control parameters, we establish strong convergence results for the proposed method. Unlike many existing approaches that depend on contraction mappings or Mann-type techniques requiring heavy computations, our method employs a standard Mann iteration scheme without additional complexity. Moreover, the algorithm integrates a relaxed two-inertial technique, which enhances the convergence speed. We further demonstrate the applicability of our results to variational inequality problems and image recovery tasks. Finally, numerical experiments are presented to validate the theoretical findings and to illustrate the superiority of the proposed method compared with several well-known algorithms in the literature. The results obtained in this paper improve, extend, and unify numerous existing contributions in this research direction.

2020 Mathematics Subject Classifications: 65K15, 47J25

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1. Introduction

The equilibrium problem (EP) is a wide concept that contains several mathematical models, such as optimization problems, variational inequality problems (VIPs), image recovery

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problems, signal processing problems, Nash-equilibrium problems, inverse optimization problems and complementary problems (see for e.g, [1–4]). The equilibrium problem is formulated as follows:

$$\text{find } s^{\dagger} \in C \text{ such that } g(s^{\dagger}, v) \geq 0, \forall v \in C, \tag{1}$$

where C is a nonempty closed convex subset of a real Hilbert space H, and $g : H \times H \to \mathbb{R}$ is a bifunction. We denote by $EP(g)$ the solution set of EP (1). The EP (1) which has been studied in [5] is well known as the Ky Fan inequality. This concept has broadly been studied in recent years by several authors, for example, see [6–8]. This attention stems from the fact that, it clearly incorporates all the above mentioned specific problems. Two well known methods are widely used in solving EPs. These methods are the auxiliary problem principle [9] and the proximal point method (PPM) [10, 11]. The PPM was introduced by Martinet [12] to solve VIPs. The PPM was further used by Moudafi [10] to solve monotone equilibrium problem. It is well known that the convergence of the PPM is not guaranteed if the bifunction g is pseudomonotone. In an attempt to overcome this limitation, Flam et al. [13] and Tran etal. [14] proposed a proximal-like method also known as extragradient method (EGM). The algorithm of Tran etal. [14] is precisely defined as follows:

$$\begin{cases} s_0 \in C \\ y_m = \text{argmin}_{u \in C}\{\lambda g(s_m, u) + \frac{1}{2}\|u - s_m\|^2\} \\ s_{m+1} = \text{argmin}_{u \in C}\{\lambda g(y_m, u) + \frac{1}{2}\|u - s_m\|^2\}, \end{cases} \tag{2}$$

where the bifunction g is pseudomonotone and $\lambda > 0$ is an appropriate parameter. This approach requires the calculation of two strongly convex programming problems in each iteration step. One of the disadvantages of this method is that, in situations where there is a complex structure in the two-valued function or the feasible set, the calculation of the subprograms involved in the algorithm can be expensive. In [15], Lyashko et al. considered the slack projection approach, where the feasible set in the second projection is replaced with a half space. This method which is also known as the subgradient extragradient method is given as follows:

$$\begin{cases} s_0 \in C \\ y_m = \text{argmin}_{u \in C}\{\lambda g(s_m, u) + \frac{1}{2}\|u - s_m\|^2\} \\ s_{m+1} = \text{argmin}_{u \in T_m}\{\lambda g(y_m, u) + \frac{1}{2}\|u - s_m\|^2\}, \end{cases} \tag{3}$$

where T_m is a half space. It is worthy to note that the results of Lyashko et al. [15] for the method (3) only guarantee weak convergence. But in terms of applicability, the strong convergence is more desirable. Another drawback of the method (3) is that, it requires the fixed step size that is completely dependent on the Lipschitz-type constants of the bifunction g. Many modified forms of the method (3) have been studied in recent years, see [16–18].

In recent years, the inertial techniques have been considered significant in improving the numerical efficiency of various iteration methods. According to some existing results in the literature, the application of inertial extrapolation terms improve numerical performance in terms of total number of iteration and execution time. Several inertial-type methods for solving various optimization problems have introduced and studied, see [3, 6, 19–25] and the references in them. It is important to note that the above results utilize a single inertial parameter to speed up the convergence of the methods in them. However, research has it that the incorporation of two inertial parameters improves motion modeling, enhances robustness and stability, increases fault tolerance and redundancy, offers adaptability and flexibility in algorithm design, and expands the range of applicability [26]. It is shown in [27] that one parameter inertial term, expressed as $w_m = s_m + \phi_m(s_m - s_{m-1})$ such that $\phi_m \in [0, 1)$, may produce less acceleration. However, acceleration can be improved by incorporating more than two points, such as s_m and s_{m-1}, in the the inertial term [28]. For instance, acceleration can be enhanced from the following two-step inertial extrapolation:

$$y_m = s_m + \phi(s_m - s_{m-1}) + \psi(s_{m-1} - s_{m-2}),$$

where $\phi > 0$ and $\psi < 0$. In [29], the authors discussed the shortcomings of using one-step inertial in the alternating direction method of multipliers (ADMM), and this led to the suggestion of adaptive acceleration as an alternative solution. Additionally, Polyak [30] outlined the advantages of multi-step inertial methods for improving the speed of convergence optimization technique.

On the other hand, the fixed point problem (FPP) remains an interesting area of research that has attracted a lot of researchers due to its numerous applications to applied sciences and engineering. The FPP is defined as follows:

$$\text{find } s \in H \text{ such that } s \in F(T),$$

where $F(T) = \{s \in H : s = Ts\}$ is the set of fixed points of the mapping T. One of the main focuses of this research is to find a common solution to FPP and EP in real Hilbert space. The idea and motivation for finding common solution problems stems from their possible applicability to some mathematical models such as network-resource allocation, image restoration problem and signal processing problem [20]. Recently, many algorithms with single inertial terms for finding the common solution of EP and FPP have established, for example, see [19–21, 31–33] and the references in them.

Motivated and inspired by the above results, in this research, we introduce an improved pseudomonotone subgradient extragradient algorithm for finding common solutions of EP and FPP in real Hilbert spaces. We obtain the strong convergence results of the proposed method under some mild and suitable assumptions on the control parameters. Unlike many existing methods that rely on contraction, and Mann-like techniques to obtain strong convergence, our method employs the typical Mann iteration technique which does requires complex computations. Furthermore, our method incorporates a relaxed two-inertial technique which enhances its speed of convergence. Additionally, we demonstrate the applicability of our findings to variational inequality problems, and image recovery problems. Finally, we present some numerical experiments to validate our theoretical results and show the superiority of our method over some well known results in the literature. The obtained results in this paper improve, extend and unify many existing results in this research direction.

This paper is structured as follows: Section 2 presents some fundamental definitions and lemmas. Section 3 provides the main theoretical findings. The applications of our findings to VIPs, and image restoration problems are shown in Section 4 and Section 5, respectively. Numerical experiments to demonstrated the superiority of results over some existing methods is provided in Section 6. Finally, the conclusion of our work is given in Chapter 7.

2. Preliminaries

In this section, we recall some important concepts and results that will be useful in this research.
Let C be a nonempty subset of a real Hilbert space H. For any $s, v \in H$, it is well-known that

$$\|s - v\|^2 = \|s\|^2 - 2\langle s, v\rangle + \|v\|^2. \tag{4}$$

$$\|s + v\|^2 = \|s\|^2 + 2\langle s, v\rangle + \|v\|^2. \tag{5}$$

$$\|s - v\|^2 \leq \|s\|^2 + 2\langle v, s - v\rangle. \tag{6}$$

$$\|\alpha s + (1-\alpha)v\|^2 = \alpha\|s\|^2 + (1-\alpha)\|v\|^2 - \alpha(1-\alpha)\|s - v\|^2, \tag{7}$$

where $\alpha \in [0, 1]$.

Definition 1. *A bifunction $g : C \times C \to \mathbb{R}$ is said to be:*

(a) Monotone on C if

$$g(s, v) + g(v, s) \leq 0, \forall s, v \in C; \tag{8}$$

(b) Pseudomonotone on C if

$$g(s, v) \geq 0 \Rightarrow g(v, s) \leq 0, \quad \forall\ s, v \in C;$$

(e) satisfying a Lipschitz-like condition on C if there exist two positive constant ℓ_1, ℓ_2 such that

$$g(s, v) + g(v, w) \geq g(s, w) - \ell_1\|s - v\| - \ell_2\|v - w\|^2, \quad \forall\ s, v, w \in C. \tag{9}$$

For any $v \in H$, a unique nearest point $P_C v \in C$ exists and satisfies the following inequality:

$$\|v - P_C v\| \leq \|v - w\|, \quad \forall w \in C.$$

P_C is well known be a nonexpansive operator and is called the metric projection of H onto C. The operator P_C satisfies

$$\langle s - v, P_C s - P_C v\rangle \geq \|P_C s - P_C v\|^2, \tag{10}$$

for all $s, v \in H$. In addition,the following inequalities hold:

$$\|s - v\|^2 \geq \|s - P_C s\|^2 + \|v - P_C s\|^2$$

and

$$\langle s - P_C s, v - P_C s\rangle \leq 0, \tag{11}$$

for all $s \in H$ and $v \in C$.
For any $s, v \in H$, the subdifferential $\partial_2 g(s, v)$ of $g(s, \cdot)$

$$\partial_2 g(s, v) = \{s \in H : g(s, u) - g(s, v) \geq \langle s, u - v\rangle, \forall u \in H\}. \tag{12}$$

Suppose T is a self mapping with a nonempty fixed point set $F(T)$. Then, T is called

(i) nonexpansive if

$$\|Ts - Tv\| \leq \|s - v\|, \forall s, v \in H.$$

(ii) quasinonexpansive if

$$\|Ts - s^\dagger\| \leq \|s - s^\dagger\|, \forall s \in H, s^\dagger \in F(T).$$

(iii) $I - T$ is demiclosed at zero if $\{s_m\} \subset H$, $s_m \rightharpoonup p^\dagger$ and $\|Ts_m - s_m\| \to 0$ implies $p^\dagger \in F(T)$.

Lemma 1. *[34] Let $t : C \to \mathbb{R}$ be a subdifferential function on C, where C be a convex subset of a real Hilbert space H. Then s^* is a solution to the convex problem: minimize$\{t(s) : s \in C\}$ if and only if $0 \in \partial\phi(s^*) + N_C(s^*)$, where $\partial\phi(s^*)$ denotes the subdifferential of t and $N_C(s^*)$ is the normal cone of C at s^*.*

Lemma 2. *[35] Let $\{c_m\}$ be a sequence of nonnegative real numbers, $\{d_d\}$ a sequence of real numbers in $(0, 1)$ with $\sum_{m=1}^{\infty} d_m = \infty$ and $\{e_m\}$ be a sequence of real numbers. Assume that*

$$c_{m+1} \leq (1 - d_m)c_m + d_m e_m, m \geq 1.$$

Suppose $\limsup_{i\to\infty} e_{m_i} \leq 0$ *for all subsequences* $\{c_{m_i}\}$ *of* $\{c_m\}$ *fulfilling*

$$\liminf_{i\to\infty}\{c_{m_i+1} - c_{m_i}\} \geq 0,$$

then, $\lim_{m\to\infty} c_m = 0.$

3. Main Results

In this section, we introduce an improved subgradient extragradient method with two-inertial for finding the common solution of EP and FPP in real Hilbert spaces. Furthermore, we establish the strong convergence results of the proposed method under the following mild assumptions:

Assumption 1. *Let C is a nonempty closed convex subset of a real Hilbert space H, $g : C \times C \to \mathbb{R}$ be a bifunction and $T : C \to C$ be a mapping such that the following conditions are satisfied:*

(i) g is pseudomonotone on C with $g(s, s) = 0$ for all $s \in H$ and satisfies the Lipschitz-type condition on H with positive constants ℓ_1, ℓ_2

(ii) $g(s, \cdot)$ is subdifferentiable on H for any $s \in H$;

(iii) T is a quasinonexpansive mapping such that $I - T$ is demiclosed at zero;

(iv) The solution set $EP(g) \cap F(T) \neq \emptyset$.

Assumption 2. *Condition on the inertial and relaxation factors.*

(i) Let $\{\alpha_m\} \subset (0, 1)$ and $\{\beta_m\} \subset (0, 1)$ such that $\lim\limits_{m\to\infty} \beta_m = 0$ and $\sum_{m=1}^{\infty} \beta_m = \infty$.

(ii) $\{\alpha_m\} \subset [a, b] \subset (0, 1]$.

(iii) Let $\{\tau_m\} \subset \mathbb{R}_+$ such that $\lim\limits_{m\to\infty} \frac{\tau_m}{\beta_m} = 0$.

Algorithm 3. *Relaxed Two-Inertial Subgradient Extragradient Algorithm for Solving EP and FPP.*

Step 0: *Choose $k \in (0, 1], \delta_1 > 0$, $\mu \in (0, 1)$ and let $\{\mu_m\} \subset (0, \infty), \{\rho_m\} \subset (-\infty, 0)$ be bounded sequences. Take $s_{-1}, s_0, s_1 \in H$ and set $m = 1$.*

Step 1: *Given s_{-2}, s_{-1} and $\{s_m\}$, compute*

$$w_m = (1 - \beta_m)[s_m + \psi_m(s_m - s_{m-1}) + \theta_m(s_{m-1} - s_{m-2})], \tag{13}$$

where

$$\psi_m = \begin{cases} \min\left\{\mu_m, \frac{\tau_m}{\|s_m - s_{m-1}\|}\right\}, & \text{if } s_m \neq s_{m-1}, \\ \mu_m, & \text{otherwise.} \end{cases} \tag{14}$$

$$\theta_m = \begin{cases} \min\left\{\rho_m, \frac{-\tau_m}{\|s_{m-1} - s_{m-2}\|}\right\}, & \text{if } s_{m-1} \neq s_{m-2}, \\ \rho_m, & \text{otherwise.} \end{cases} \tag{15}$$

Step 2: *Compute*

$$y_m = \text{argmin}_{u\in C}\{\delta_m g(w_m, u) + \frac{1}{2}\|u - w_m\|^2\},$$

if $y_m = w_m$*, then stop and* y_m *is a solution. Otherwise, go to step 3.*
Step 3: *Select* $z_m \in \partial_2 g(w_m, y_m)$ *such that* $w_m - \delta_m z_m - y_m \in N_C(y_m)$ *and compute*

$$v_m = \text{argmin}_{u\in T_m}\{k\delta_m g(y_m, u) + \frac{1}{2}\|u - w_m\|^2\}, \tag{16}$$

where

$$T_m = \{w \in H : \langle w_m - \delta_m z_m - y_m, w - y_m\rangle \leq 0\}.$$

Step 4: *Compute*

$$s_{m+1} = (1 - \alpha_m)v_m + \alpha_m T v_m \tag{17}$$

and

$$\delta_{m+1} = \begin{cases} \min\left\{\delta_m, \frac{\mu[\|w_m-y_m\|^2+\|v_m-y_m\|^2]}{2[g(w_m,v_m)-g(w_m,y_m)-g(y_m,v_m)]}\right\}, & \text{if } g(w_m, v_m) - g(w_m, y_m) - g(y_m, v_m) > 0 \\ \tau_m, & \text{otherwise.} \end{cases} \tag{18}$$

Set $m + 1 \leftarrow m$ *and continue again from **step 1**.*

Lemma 3. *A sequence* $\{\delta_m\}$ *generated by Algorithm 3 is non-increasing and* $\lim_{m\to\infty} \delta_m \geq \min\{\frac{\mu}{2\max\{\ell_1,\ell_2\}}, \delta_1\}$.

Proof. By (18), it is obvious that the sequence $\{\delta_m\}$ is non-increasing. Furthermore, due to the Lipschitz-type condition of g on H, when $g(w_m, v_m)-g(w_m, y_m)-g(y_m, v_m) > 0$, we get

$$\begin{aligned} \frac{\mu[\|w_m - y_m\|^2 + \|v_m - y_m\|^2]}{2[g(w_m, v_m) - g(w_m, y_m) - g(y_m, v_m)]} &\geq \frac{\mu[\|w_m - y_m\|^2 + \|v_m - y_m\|^2]}{2(\ell_1\|w_m - y_m\|^2 + \ell_2\|y_m - v_m\|^2)} \\ &\geq \frac{\mu[\|w_m - y_m\|^2 + \|v_m - y_m\|^2]}{2\max\{\ell_1, \ell_2\}(\|w_m - y_m\|^2 + \|y_m - v_m\|^2)} \\ &= \frac{\mu}{2\max\{\ell_1, \ell_2\}}. \end{aligned}$$

Thus, $\{\delta_m\}$ is lower bounded and non-increasing. Furthermore, there exists $\lim_{m\to\infty} \delta_m = \delta \geq \min\{\frac{\mu}{2\max\{\ell_1,\ell_2\}}, \delta_1\}$.

Lemma 4. *Let $\{s_m\}$ be the sequence generated by Algorithm 3 that satisfies Assumption 1. Then, we have*

$$\|v_m - p^\dagger\|^2 \leq \|w_m - p^\dagger\|^2 - (1-k)\|w_m - v_m\| \\ - k\left(1 - \mu\frac{\delta_m}{\delta_{m+1}}\right)\|w_m - y_m\|^2 - k\left(1 - \mu\frac{\delta_m}{\delta_{m+1}}\right)\|v_m - y_m\|^2,$$

for all $p^\dagger \in EP(g)$.

Proof. By Lemma 1 and (16), we have

$$0 \in \partial\left\{k\delta_m g(y_m, u) + \frac{1}{2}\|u - w_m\|^2\right\}(v_m) + N_{T_m}(v_m), \ \forall u \in T_m.$$

It implies that there exist $t_m \in \partial_2 g(w_m, v_m)$ and $\overline{t_m} \in N_{T_m}(v_m)$ such that

$$k\delta_m t_m + v_m - w_m + \overline{t_m} = 0.$$

It follows that

$$\langle w_m - v_m, u - v_m\rangle = k\delta_m\langle t_m, u - v_m\rangle + \langle\overline{t_m}, u - v_m\rangle, \ \forall u \in T_m.$$

From $\overline{t_m} \in N_{T_m}(v_m)$, we have $\langle\overline{t_m}, u - v_m\rangle \leq 0$. Thus,

$$k\delta_m\langle t_m, u - v_m\rangle \geq \langle w_m - v_m, u - v_m\rangle, \ \forall u \in T_m. \tag{19}$$

Furthermore, due to the subdifferentiability of g and $t_m \in \partial_2 g(w_m, v_m)$, we have

$$g(y_m, u) - g(y_m, v_m) \geq \langle t_m, u - v_m\rangle, \ \forall u \in T_m. \tag{20}$$

Using (19) and (20), we have

$$k\delta_m[g(y_m, u) - g(y_m, v_m)] \geq \langle w_m - v_m, u - v_m\rangle, \ \forall u \in T_m. \tag{21}$$

Suppose $u = p^\dagger \in EP(g) \subset C \subset T_m$, then

$$k\delta_m[g(y_m, p^\dagger) - g(y_m, v_m)] \geq \langle w_m - v_m, p^\dagger - v_m\rangle. \tag{22}$$

Since $y_m \in C$, we obtain $g(u, y_m) \geq 0$. From the pseudomonotonicity of g, we have $g(y_m, p^\dagger) \geq 0$. Thus, (22) reduces to

$$\langle w_m - v_m, v_m - p^\dagger\rangle \geq k\delta_m g(y_m, v_m). \tag{23}$$

Also, since $z_m \in \partial_2 g(w_m, y_m)$, we get

$$g(w_m, z) - g(w_m, y_m) \geq \langle z_m, z - y_m\rangle, \ \forall z \in H. \tag{24}$$

If $z = v_m$, then we have

$$g(w_m, v_m) - g(w_m, y_m) \geq \langle z_m, v_m - y_m \rangle. \tag{25}$$

Since $v_m \in T_m$, we have $\langle w_m - \delta_m z_m - y_m, v_m - y_m \rangle \leq 0$. This means that

$$\delta_m \langle z_m, v_m - y_m \rangle \geq \langle w_m - y_m, v_m - y_m \rangle. \tag{26}$$

Using (25) and (26), we have

$$\delta_m [g(w_m, v_m) - g(w_m, y_m)] \geq \langle w_m - y_m, v_m - y_m \rangle. \tag{27}$$

By (18), we have

$$\delta_{m+1}[g(w_m, v_m) - g(w_m, y_m) - g(y_m, v_m)] \leq \frac{\mu}{2}[\|w_m - y_m\|^2 + \|v_m - y_m\|^2],$$

or equivalently,

$$\delta_m [g(w_m, v_m) - g(w_m, y_m) - g(y_m, v_m)] \leq \frac{\delta_m}{\delta_{m+1}} \frac{\mu}{2}[\|w_m - y_m\|^2 + \|v_m - y_m\|^2], \tag{28}$$

Putting (28) into (27), we obtain

$$\langle w_m - y_m, v_m - y_m \rangle \leq \delta_m g(y_m, v_m) + \frac{\delta_m}{\delta_{m+1}} \frac{\mu}{2}[\|w_m - y_m\|^2 + \|v_m - y_m\|^2]. \tag{29}$$

Combining (23) and (29), we have

$$\langle w_m - y_m, v_m - y_m \rangle \leq \frac{1}{k}\langle w_m - v_m, v_m - p^\dagger \rangle + \frac{\delta_m}{\delta_{m+1}} \frac{\mu}{2}[\|w_m - y_m\|^2 + \|v_m - y_m\|^2]. \tag{30}$$

Moreover,

$$\begin{cases} 2\langle w_m - y_m, v_m - y_m \rangle & = \|w_m - y_m\|^2 + \|v_m - y_m\|^2 - \|w_m - v_m\|^2 \\ 2\langle w_m - v_m, v_m - p^\dagger \rangle & = \|w_m - p^\dagger\|^2 - \|v_m - p^\dagger\|^2 - \|w_m - v_m\|^2. \end{cases} \tag{31}$$

From (30) and (31), we have

$$\begin{aligned} \|v_m - p^\dagger\|^2 \leq \|w_m - p^\dagger\|^2 - (1-k)\|w_m - v_m\|^2 \\ - k\left(1 - \mu\frac{\delta_m}{\delta_{m+1}}\right)\|w_m - y_m\|^2 - k\left(1 - \mu\frac{\delta_m}{\delta_{m+1}}\right)\|v_m - y_m\|^2. \end{aligned} \tag{32}$$

Theorem 4. *Suppose that Assumptions 1 and 2 hold. Then, the sequence $\{s_m\}$ generated by Algorithm 3 converges strongly to $p^\dagger \in EP(g) \cap F(T)$, where $p^\dagger = P_{EP(g)\cap F(T)}(0)$.*

Proof. **Claim 1:** The sequence $\{s_m\}$ is bounded.
Indeed, since $k \in (0,1]$, $\mu \in (0,1)$ and by Lemma 3, $\lim_{m\to\infty} \delta_m = \delta$. Then we have

$$1 - k \geq 0, \quad \lim_{m\to\infty} k\left(1 - \mu\frac{\delta_{m.}}{\delta_{m+1}}\right) > 0. \tag{33}$$

By Lemma 4 and (33), for all $F(S) \cap EP(g)$, we get

$$\|v_m - p^\dagger\| \leq \|w_m - p^\dagger\|. \tag{34}$$

From (13), we have

$$\begin{aligned}\|w_m - p^\dagger\| &= \|(1-\beta_m)[s_m + \psi_m(s_m - x_{m-1}) + \theta_m(s_{m-1} - s_{m-2})] - p^\dagger\| \\ &= \|(1-\beta_m)(s_m - p^\dagger) + (1-\beta_m)[\psi_m(s_m - x_{m-1}) + \theta_m(s_{m-1} - s_{m-2})] - \beta_m p^\dagger\| \\ &\leq (1-\beta_m)\|s_m - p^\dagger\| + (1-\beta_m)\psi_m\|s_m - x_{m-1}\| \\ &+ (1-\beta_m)|\theta_m|\|s_{m-1} - s_{m-2}\| + \beta_m\|p^\dagger\| \\ &= (1-\beta_m)\|s_m - p^\dagger\| + \beta_m[(1-\beta_m)\frac{\psi_m}{\beta_m}\|s_m - x_{m-1}\| \\ &+ (1-\beta_m)\frac{|\theta_m|}{\beta_m}\|s_{m-1} - s_{m-2}\| + \|p^\dagger\|]. \end{aligned} \tag{35}$$

In accordance to Assumption 2 and the definition of ψ_m and θ_m, we have

$$\lim_{m\to\infty} \frac{\psi_m}{\beta_m}\|s_m - x_{m-1}\| = 0 \text{ and } \lim_{m\to\infty} \frac{|\theta_m|}{\beta_m}\|s_{m-1} - s_{m-2}\| = 0, \tag{36}$$

this means that

$$\lim_{m\to\infty}\left[(1-\beta_m)\frac{\psi_m}{\beta_m}\|s_m - x_{m-1}\| + (1-\beta_m)\frac{|\theta_m|}{\beta_m}\|s_{m-1} - s_{m-2}\| + \|p^\dagger\|\right] = \|p^\dagger\|,$$

thus, a positive constant M_1 exists such that

$$(1-\beta_m)\frac{\psi_m}{\beta_m}\|s_m - x_{m-1}\| + (1-\beta_m)\frac{|\theta_m|}{\beta_m}\|s_{m-1} - s_{m-2}\| + \|p^\dagger\| \leq M_1. \tag{37}$$

Combining (35) and (37), we have

$$\|w_m - p^\dagger\| \leq (1-\beta_m)\|s_m - p^\dagger\| + \beta_m M_1. \tag{38}$$

Next, from (17), (38), and (34), we have

$$\begin{aligned}\|s_{m+1} - p^\dagger\| &\leq \|(1-\alpha_m)v_m + \alpha_m S v_m - p^\dagger\| \\ &\leq \|(1-\alpha_m)\|v_m - p^\dagger\| + \alpha_m\|Sv_m - p^\dagger\| \\ &\leq (1-\alpha_m)\|v_m - p^\dagger\| + \alpha_m\|v_m - p^\dagger\|\end{aligned}$$

$$\begin{aligned} &= \|v_m - p^\dagger\| \\ &\leq \|w_m - p^\dagger\| \\ &\leq (1-\beta_m)\|s_m - p^\dagger\| + \beta_m M_1 \\ &\leq \max\{\|s_m - p^\dagger\|, M_1\} \\ &\vdots \\ &\leq \max\{\|s_{m_0} - u^*\|, M_1\}, \end{aligned}$$

Hence, the sequence $\{s_m\}$ is bounded.

Claim 2:

$$\begin{aligned} &(1-k)\|w_m - v_m\|^2 + k\left(1 - \mu\frac{\delta_m}{\delta_{m+1}}\right)\|w_m - y_m\|^2 \\ &+ k\left(1 - \mu\frac{\delta_m}{\delta_{m+1}}\right)\|v_m - y_m\|^2 + \alpha_m(1-\alpha_m)\|Sv_m - v_m\| \\ &\leq \|s_m - p^\dagger\|^2 - \|s_{m+1} - p\|^2 + \beta_m M_2, \end{aligned} \tag{39}$$

for some $M_2 > 0$.

Indeed, from (38), we have

$$\begin{aligned} \|w_m - p^\dagger\|^2 &\leq (1-\beta_m)^2\|s_m - p^\dagger\|^2 + 2\beta_m(1-\beta_m)^2 M_1\|s_m - p^\dagger\|^2 + \beta_m^2 M_1^2 \\ &\leq \|s_m - p^\dagger\|^2 + \beta_m[2(1-\beta_m)^2 M_1\|s_m - p^\dagger\|^2 + \beta_m M_1^2] \\ &\leq \|s_m - p^\dagger\|^2 + \beta_m M_2, \end{aligned} \tag{40}$$

where $M_2 = \max\{2(1-\beta_m)^2 M_1\|s_m - p^\dagger\|^2 + \beta_m M_1^2 : m \in \mathbb{N}\}$.

Next from (17), (32) and (40), we have

$$\begin{aligned} \|s_{m+1} - p\|^2 &= \|(1-\alpha_m)v_m + \alpha_m Sv_m - p^\dagger\|^2 \\ &= (1-\alpha_m)\|v_m - p^\dagger\|^2 + \alpha_m\|Sv_m - p^\dagger\|^2 - \alpha_m(1-\alpha_m)\|Sv_m - v_m\| \\ &\leq (1-\alpha_m)\|v_m - p^\dagger\|^2 + \alpha_m\|v_m - p^\dagger\|^2 - \alpha_m(1-\alpha_m)\|Sv_m - v_m\| \\ &= \|v_m - p^\dagger\|^2 - \alpha_m(1-\alpha_m)\|Sv_m - v_m\| \\ &\leq \|w_m - p^\dagger\|^2 - (1-k)\|w_m - v_m\|^2 - k\left(1 - \mu\frac{\delta_m}{\delta_{m+1}}\right)\|w_m - y_m\|^2 \\ &- k\left(1 - \mu\frac{\delta_m}{\delta_{m+1}}\right)\|v_m - y_m\|^2 - \alpha_m(1-\alpha_m)\|Sv_m - v_m\| \qquad (41) \\ &\leq \|s_m - p^\dagger\|^2 + \beta_m M_2 - (1-k)\|w_m - v_m\|^2 - k\left(1 - \mu\frac{\delta_m}{\delta_{m+1}}\right)\|w_m - y_m\|^2 \\ &- k\left(1 - \mu\frac{\delta_m}{\delta_{m+1}}\right)\|v_m - y_m\|^2 - \alpha_m(1-\alpha_m)\|Sv_m - v_m\|. \end{aligned}$$

Thus,

$$(1-k)\|w_m - v_m\|^2 + k\left(1 - \mu\frac{\delta_m}{\delta_{m+1}}\right)\|w_m - y_m\|^2$$

$$
\begin{aligned}
&+ k\left(1-\mu\frac{\delta_m}{\delta_{m+1}}\right)\|v_m-y_m\|^2+\alpha_m(1-\alpha_m)\|Sv_m-v_m\| \\
&\leq \|s_m-p^\dagger\|^2-\|s_{m+1}-p\|^2+\beta_m M_2. \qquad (42)
\end{aligned}
$$

Claim 3:

$$
\begin{aligned}
\|s_{m+1}-p^\dagger\|^2 &\leq (1-\beta_m)\|s_m-p^\dagger\|^2+\beta_m\Bigg[2(1-\beta_m)^2\frac{\psi_m}{\beta_m}\|s_m-x_{m-1}\|\|s_m-p^\dagger\| \\
&+2(1-\beta_m)^2\frac{|\phi_m|}{\beta_m}\|s_{m-1}-s_{m-2}\|\|s_m-p^\dagger\| \\
&+\psi_m^2\|s_m-x_{m-1}\|^2+2\psi_m|\phi_m|\|s_m-x_{m-1}\|\|s_{m-1}-s_{m-2}\| \\
&+|\theta_m|\|s_{m-1}-s_{m-2}\|\frac{|\phi_m|}{\beta_m}\|s_{m-1}-s_{m-2}\|+2\|p^\dagger\|\|w_m-s_{m+1}\|+2\langle -p^\dagger, s_{m+1}-p^\dagger\rangle\Bigg].
\end{aligned}
$$

Indeed, by (13), (17), (41) and (6), we have

$$
\begin{aligned}
\|s_{m+1}-p^\dagger\|^2 &\leq \|w_m-p^\dagger\|^2 \\
&\leq \|(1-\beta_m)[s_m+\psi_m(s_m-x_{m-1})+\theta_m(s_{m-1}-s_{m-2})]-p\|^2 \\
&= \|(1-\beta_m)(s_m-p^\dagger)+(1-\beta_m)[\psi_m(s_m-x_{m-1})+\theta_m(s_{m-1}-s_{m-2})]-\beta_m p^\dagger\|^2 \\
&\leq \|(1-\beta_m)(s_m-p^\dagger)+(1-\beta_m)[\psi_m(s_m-x_{m-1})+\theta_m(s_{m-1}-s_{m-2})]\|^2+2\beta_m\langle -p^\dagger, w_m-p^\dagger\rangle \\
&\leq (1-\beta_m)\|s_m-p^\dagger\|^2+2(1-\beta_m)^2\|s_m-p^\dagger\|\|\psi_m(s_m-x_{m-1})+\theta_m(s_{m-1}-s_{m-2})\| \\
&+\|\psi_m(s_m-x_{m-1})+\theta_m(s_{m-1}-s_{m-2})\|^2+2\beta_m\langle -p^\dagger, w_m-p^\dagger\rangle \\
&\leq (1-\beta_m)\|s_m-p^\dagger\|^2+2(1-\beta_m)^2\psi_m\|s_m-p^\dagger\|\|s_m-x_{m-1}\| \\
&+2(1-\beta_m)^2|\theta_m|\|s_m-p^\dagger\|\|s_{m-1}-s_{m-2}\| \\
&+\psi_m^2\|s_m-x_{m-1}\|^2+2\psi_m|\theta_m|\|s_m-x_{m-1}\|\|s_{m-1}-s_{m-2}\| \\
&+\theta_m^2\|s_{m-1}-s_{m-2}\|^2+2\beta_m\langle -p^\dagger, w_m-p^\dagger\rangle. \\
&= (1-\beta_m)\|s_m-p^\dagger\|^2+2(1-\beta_m)^2\psi_m\|s_m-p^\dagger\|\|s_m-x_{m-1}\| \\
&+2(1-\beta_m)^2|\theta_m|\|s_m-p^\dagger\|\|s_{m-1}-s_{m-2}\| \\
&+\psi_m^2\|s_m-x_{m-1}\|^2+2\psi_m|\theta_m|\|s_m-x_{m-1}\|\|s_{m-1}-s_{m-2}\| \\
&+\theta_m^2\|s_{m-1}-s_{m-2}\|^2+2\beta_m\langle -p^\dagger, w_m-s_{m+1}\rangle+2\beta_m\langle -p^\dagger, s_{m+1}-p^\dagger\rangle \\
&\leq (1-\beta_m)\|s_m-p^\dagger\|^2+2(1-\beta_m)^2\psi_m\|s_m-p^\dagger\|\|s_m-x_{m-1}\| \\
&+2(1-\beta_m)^2|\theta_m|\|s_m-p^\dagger\|\|s_{m-1}-s_{m-2}\| \\
&+\psi_m^2\|s_m-x_{m-1}\|^2+2\psi_m|\theta_m|\|s_m-x_{m-1}\|\|s_{m-1}-s_{m-2}\| \\
&+\theta_m^2\|s_{m-1}-s_{m-2}\|^2+2\beta_m\|p^\dagger\|\|w_m-s_{m+1}\|+2\beta_m\langle -p^\dagger, s_{m+1}-p^\dagger\rangle \\
&\leq (1-\beta_m)\|s_m-p^\dagger\|^2+\beta_m\Bigg[2(1-\beta_m)^2\frac{\psi_m}{\beta_m}\|s_m-x_{m-1}\|\|s_m-p^\dagger\| \\
&+2(1-\beta_m)^2\frac{|\theta_m|}{\beta_m}\|s_{m-1}-s_{m-2}\|\|s_m-p^\dagger\|
\end{aligned}
$$

$$+ \psi_m \|s_m - x_{m-1}\| \frac{\psi_m}{\beta_m} \|s_m - x_{m-1}\| + 2\frac{\psi_m}{\beta_m} \|s_m - x_{m-1}\| \frac{|\theta_m|}{\beta_m} \|s_{m-1} - s_{m-2}\|$$
$$+ |\theta_m| \|s_{m-1} - s_{m-2}\| \frac{|\phi_m|}{\beta_m} \|s_{m-1} - s_{m-2}\| + 2\|p^\dagger\| \|w_m - s_{m+1}\| + 2\langle -p^\dagger, s_{m+1} - p^\dagger \rangle \Big].$$

Claim 4: The sequence $\|s_m - p^\dagger\|^2$ converges to zero.
Set

$$c_m = \|s_m - p^\dagger\|.$$

and

$$e_m = \Big[2(1-\beta_m)^2 \frac{\psi_m}{\beta_m} \|s_m - x_{m-1}\| \|s_m - p^\dagger\|$$
$$+ 2(1-\beta_m)^2 \frac{|\theta_m|}{\beta_m} \|s_{m-1} - s_{m-2}\| \|s_m - p^\dagger\|$$
$$+ \psi_m \|s_m - x_{m-1}\| \frac{\psi_m}{\beta_m} \|s_m - x_{m-1}\| + 2\frac{\psi_m}{\beta_m} \|s_m - x_{m-1}\| \frac{|\theta_m|}{\beta_m} \|s_{m-1} - s_{m-2}\|$$
$$+ |\theta_m| \|s_{m-1} - s_{m-2}\| \frac{|\phi_m|}{\beta_m} \|s_{m-1} - s_{m-2}\| + 2\|p^\dagger\| \|w_m - s_{m+1}\| + 2\langle -p^\dagger, s_{m+1} - p^\dagger \rangle \Big].$$

Then, **Claim 4** can rewritten as follows:

$$c_{m+1} \leq (1-\beta_m)c_m + \beta_m e_m.$$

Indeed, by Lemma 2, it is enough to show that $\limsup_{i\to\infty} e_{m_i} \leq 0$ for every subsequence $\{c_{m_i}\}$ of $\{c_m\}$ fulfilling

$$\liminf_{i\to\infty} (p_{m_i+1} - p_{m_i}) \geq 0.$$

To see this, we have to show that $\limsup_{i\to\infty} \langle -p^\dagger, s_{m_i+1} - p^\dagger \rangle \leq 0$ and $\lim_{i\to\infty} \|w_{m_i} - s_{m_i+1}\| = 0$, for every subsequence $\|s_{m_i} - p^\dagger\|$ of $\|s_m - p^\dagger\|$ satisfying

$$\liminf_{i\to\infty} (\|s_{m_i+1} - p^\dagger\| - \|s_{m_i} - p^\dagger\|) \geq 0. \tag{43}$$

Suppose $\{\|s_{m_i} - p^\dagger\|\}$ is a subsequence of $\{\|s_m - p^\dagger]\|\}$ such that (43) holds. By **Claim 4** and Assumption 2, we have

$$\limsup_{i\to\infty} \Big[(1-k)\|w_{m_i} - v_{m_i}\|^2 + k\left(1 - \mu\frac{\delta_{m_i}}{\delta_{m_i+1}}\right) \|w_{m_i} - y_{m_i}\|^2$$
$$+ k\left(1 - \mu\frac{\delta_{m_i}}{\delta_{m_i+1}}\right) \|v_{m_i} - y_{m_i}\|^2 + \alpha_{m_i}(1-\alpha_{m_i})\|Sv_{m_i} - v_{m_i}\| \Big]$$

$$\begin{aligned}&\le \limsup_{i\to\infty}\Big(\|s_{m_i}-p^{\dagger}\|^2-\|s_{m_i+1}-p\|^2\Big)+\limsup_{i\to\infty}\beta_{m_i}M_2\\&=-\liminf_{i\to\infty}\big(\|s_{m_i+1}-^{*}\|^2-\|s_{m_j}-u^{*}\|^2\big)\\&\le 0.\end{aligned}$$

It follows that

$$\lim_{i\to\infty}\|w_{m_i}-v_{m_i}\|=0,\ \ \lim_{i\to\infty}\|w_{m_i}-y_{m_i}\|=0,\ \ \lim_{i\to\infty}\|v_{m_i}-y_{m_i}\| \text{ and } \lim_{i\to\infty}\|Sv_{m_i}-v_{m_i}\|=0. \tag{44}$$

From (17), (44) and due to the boundedness of $\{\alpha_{m_i}\}$, we have

$$\lim_{i\to\infty}\|s_{m_i+1}-v_{m_i}\|=\lim_{i\to\infty}\alpha_{m_i}\|Sv_{m_i}-v_{m_i}\|=0. \tag{45}$$

From (13), we have

$$\begin{aligned}\|w_{m_i}-s_{m_i}\|&=\|(1-\beta_{m_i})[s_{m_i}+\psi_{m_i}(s_{m_i}-s_{m_i-1})+\theta_{m_i}(s_{m_i-1}-s_{m_i-2})]-s_{m_i}\|\\&\le \psi_{m_i}\|s_{m_i}-s_{m_i-1}\|+|\theta_{m_i}|\|s_{m_i-1}-s_{m_i-2}\|\\&+\beta_{m_i}\psi_{m_i}\|s_{m_i}-s_{m_i-1}\|+\beta_{m_i}|\theta_{m_i}|\|s_{m_i-1}-s_{m_i-2}\|\\&=\beta_{m_i}\frac{\psi_{m_i}}{\beta_{m_i}}\|s_{m_i}-s_{m_i-1}\|+\beta_{m_i}\frac{|\theta_{m_i}|}{\beta_{m_i}}\|s_{m_i-1}-s_{m_i-2}\|\\&+\beta_{m_i}^2\frac{\psi_{m_i}}{\beta_{m_i}}\|s_{m_i}-s_{m_i-1}\|+\beta_{m_i}^2\frac{|\theta_{m_i}|}{\beta_{m_i}}\|s_{m_i-1}-s_{m_i-2}\|.\end{aligned}$$

By Assumption 40 (iii), we obtain

$$\lim_{i\to\infty}\|w_{m_i}-s_{m_i}\|=0. \tag{46}$$

Using (44), (45) and (46), we have

$$\|s_{m_i+1}-s_{m_i}\|\le\|s_{m_i+1}-v_{m_i}\|+\|v_{m_i}-w_{m_i}\|+\|w_{m_i}-s_{m_i}\|\to 0 \text{ as } i\to\infty. \tag{47}$$

By (46) and (47), we have

$$\|s_{m_i+1}-w_{m_i}\|\le\|s_{m_i+1}-s_{m_i}\|+\|s_{m_i}-w_{m_i}\|\to 0 \text{ as } i\to\infty. \tag{48}$$

Since $\{s_m\}$ is bounded, a subsequence $\{s_{m_i}\}$ exists such that $\{s_{m_i}\}\subset\{s_m\}$ with $s_{m_i}\rightharpoonup p^*$ as $i\to\infty$. Next, we show that $p^*\in EP(g)\cap F(T)$. In fact, from (44) and (46), we have

$$\|s_{m_i}-y_{m_i}\|\le\|s_{m_i}-w_{m_i}\|+\|w_{m_i}-y_{m_i}\|\to 0 \text{ as } i\to\infty.$$

This implies that $y_{m_i}\rightharpoonup p^*$. From (21) and (29), we have

$$k\delta_{m_i}g(y_{m_i},u)\ge k\delta_{m_i}g(y_{m_i},v_{m_i})+\langle w_{m_i}-v_{m_i},u-v_{m_i}\rangle$$

$$\geq k\left(\langle w_{m_i}-y_{m_i}, v_{m_i}-y_m\rangle - \frac{\delta_{m_i}}{\delta_{m+1}}\frac{\mu}{2}[\|w_{m_i}-y_{m_i}\|^2+\|v_{m_i}-y_m\|^2]\right)$$
$$+\langle w_{m_i}-v_{m_i}, u-v_{m_i}\rangle.$$

Since $k>0$ and $\lim_{i\to\infty}\delta_{m_i}=\delta$, we have

$$0\leq \limsup_{i\to\infty} g(y_{m_i},u)=g(p^*,u),\ \forall u\in C.$$

This implies that $p^*\in EP(g)$. Due to the demiclosedness of the mapping $I-T$, we have $p^*\in F(T)$. Hence, $p^*\in EP(g)\cap F(T)$.
Moreover, since $\{s_{m_i}\}$ is a bounded sequence, then it has a subsequence $\{s_{m_{i_j}}\}$ such that $s_{m_{i_j}}\rightharpoonup p^*\in H$ as $j\to\infty$ and

$$\lim_{j\to\infty}\langle -p^\dagger, s_{m_{i_j}}-p^\dagger\rangle = \limsup_{i\to\infty}\langle -p^\dagger, s_{m_i}-p^\dagger\rangle. \tag{49}$$

Since $p^\dagger=P_{EP(g)\cap F(T)}(0)$, we get

$$\limsup_{i\to\infty}\langle -p^\dagger, s_{m_i}-p^\dagger\rangle = \lim_{j\to\infty}\langle -p^\dagger, s_{m_{i_j}}-p^\dagger\rangle = \langle -p^\dagger, p^*-p^\dagger\rangle\leq 0. \tag{50}$$

From (47) and (50), we have

$$\limsup_{i\to\infty}\langle -p^\dagger, s_{m_i+1}-p^\dagger\rangle = \limsup_{i\to\infty}\langle -p^\dagger, s_{m_i}-p^\dagger\rangle = \langle -p^\dagger, p^*-p^\dagger\rangle\leq 0. \tag{51}$$

By **Claim 3**, Lemma 2 and (48), we have that

$$\lim_{m\to\infty}\|s_m-p^\dagger\|=0.$$

This completes the proof.

4. Application to Variational Inequality Problem

In this section, we consider the application of our main results to (VIP). The classical VIP for an operator $B:H\to H$ is formulated as follows: find $w^*\in C$ such that

$$\langle Bw^*, y-w^*\rangle\geq 0, \forall y\in H. \tag{52}$$

The solution set of the VIP (52) is denoted by $VI(C,B)$. Now, we consider the following condition for solving the VIP (52):

(A_1) $B:H\to H$ is a pseudomonotone operator, i.e.

$$\langle Bw, y-w\rangle\geq 0 \implies \langle By, w-y\rangle\leq 0, \forall w,y\in H.$$

(A_2) $B : H \to H$ is a L–Lipschitz continuous operator, i.e. there exist $L > 0$ such that

$$\|Bw - By\| \leq L\|w - y\|, \forall w, y \in H.$$

(A_3) $B : H \to H$ is a sequentially weakly continuous operator.

Set $g(w, y) = \langle Bw, y - w\rangle$, $\forall w, y \in C$, then the (EP) becomes the (VIP) with $L = 2\ell_1 = 2\ell_2$. Moreover, we have

$$y_m = \operatorname{argmin}_{y\in C}\{\delta_m g(w_m, u) + \frac{1}{2}\|u - w_m\|^2\} = P_C(w_m - \delta_m Bw_m).$$

Hence, we obtain the following corollary from Theorem 4:

Corollary 5. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Suppose Assumption 1 (3)–(4) and Assumption 2 hold such that the solution set $VI(C, B) \cap F(T) \neq \emptyset$. Then the sequence generated by Algorithm 6 strongly converges to an element $p^\dagger \in VI(C, B) \cap F(T)$, where $p^\dagger = P_{VI(C,B)\cap F(T)}(0)$.*

Algorithm 6. *Relaxed Two-Inertial Subgradient Extragradient Algorithm for Solving VIP and FPP.*

Step 0: *Choose $k \in (0, 1], \delta_1 > 0$, $\mu \in (0, 1)$ and let $\{\mu_m\} \subset (0,\infty), \{\rho_m\} \subset (-\infty, 0)$ be bounded sequences. Take $s_{-1}, s_0, s_1 \in H$ and set $m = 1$.*
Step 1: *Given s_{-2}, s_{-1} and $\{s_m\}$, compute*

$$w_m = (1 - \beta_m)[s_m + \psi_m(s_m - s_{m-1}) + \theta_m(s_{m-1} - s_{m-2})], \tag{53}$$

where

$$\psi_m = \begin{cases} \min\left\{\mu_m, \frac{\tau_m}{\|s_m - s_{m-1}\|}\right\}, & \text{if } s_m \neq s_{m-1}, \\ \mu_m, & \text{otherwise.} \end{cases} \tag{54}$$

$$\theta_m = \begin{cases} \min\left\{\rho_m, \frac{-\tau_m}{\|s_{m-1} - s_{m-2}\|}\right\}, & \text{if } s_{m-1} \neq s_{m-2}, \\ \rho_m, & \text{otherwise.} \end{cases} \tag{55}$$

Step 2: *Compute*

$$y_m = P_C(w_m - \delta_m Bw_m),$$

if $y_m = w_m$, then stop and y_m is a solution. Otherwise, go to step 3.
Step 3: *Compute*

$$v_m = P_{T_m}(w_m - k\delta_m By_m), \tag{56}$$

where

$$T_m = \{w \in H : \langle w_m - \delta_m z_m - y_m, w - y_m\rangle \leq 0\}.$$

Step 3: *Compute*

$$s_{m+1} = (1-\alpha_m)v_m + \alpha_m T v_m \tag{57}$$

and

$$\delta_{m+1} = \begin{cases} \min\left\{\delta_m, \frac{\mu[\|w_m-y_m\|^2+\|v_m-y_m\|^2]}{2\langle Bw_m-By_m, v_m-y_m\rangle}\right\}, & \text{if } \langle Bw_m - By_m, v_m - y_m\rangle > 0 \\ \tau_m, & \text{otherwise.} \end{cases} \tag{58}$$

*Set $m+1 \leftarrow m$ and continue again from **step 1**.*

5. Application to Image restoration problem

In this section, the suggested Algorithm 3 is utilized to solve image recovery problem. For all images, it is well known that $D = M \times N$ pixels, and each pixel is known to be in the range $[0, 255]$. Thus, the underlying real Hilbert space $\mathbb{R}^D$ is endowed with the standard Euclidean norm $\|\cdot\|$, and we put $C = [0, 255]^D$. The degraded image $\bar{y}$ is defined by

$$\bar{y} = Fx^* + \xi,$$

where x^* is the original image, ξ is a norm term and F is a blurring matrix. The aim is to restore the original image x^* based on F and $\bar{y}$. The following model which produces the recovered image given by the following minimization problem will be considered:

$$\min_{x\in C} \frac{1}{2}\|Fx - \bar{y}\|^2.$$

The point spread function (or convolution matrix) is denoted by F and let $\phi(x) = \frac{1}{2}\|Fx - \bar{y}\|^2$. By the linearity of F and the convexity of $\|\cdot\|^2$, it not hard to verify that the function ϕ is convex. In what follows, this constrained convex minimization problem can be transformed as an (EP) with $f(x, y) = \phi(y) - \phi(x)$ for all $x, y \in C$. The quality of the recovered image is measured by the signal-to-noise ratio (SNR) in decibel (DB) as

$$SNR = 30 \log \frac{\|x^*\|}{\|x - x^*\|},$$

where x is the restored image and x^* is the original image. It is known that better restorations are achieved with higher SNR values. The starting points x_0 and x_1 are taken to be $1 \in \mathbb{R}^D$ and $0 \in \mathbb{R}^D$, respectively.

In this numerical test, our target is to compare the image recovery efficiency of our Algorithm 3 with Algorithm 3 of Xie et al. [19] (shortly, XCT Alg. 3), Algorithm 3.1 of Yang and Liu [36] (shortly, YL Alg.3.1) and Algorithm 2.1 of Yekini el al. [37] (shortly, SSTT Alg. 2.1). For Algorithm 3, we choose the following parameters: $Ts = \frac{s}{2}$, $\tau_m = \frac{1}{(2m+1)^3}$, $\alpha_m = \beta_m = \frac{1}{(2m+1)}$, $k = 0.5$, $\delta_1 = 1.2$, $\mu_m = \frac{.99m}{m+0.001}$, $\rho_m = \frac{1}{m^2}$, $\mu = 0.6$, $\tau = 0.4$ and $\lambda_1 = 2.5$. For XCT Alg. 3, choose $\alpha_m = \frac{1}{(2m+1)}$, $\beta_m = \frac{1}{2}(1-\alpha_m)$, $\mu = 0.6$, $\lambda_1 = 2.5$, $Ts = \frac{s}{2}$, $f(x) = \frac{x}{3}$ and $k = 0.8$. For YL Alg. 3.1, choose $\lambda_1 = 2.5$, $\mu = 0.6$, $Ss = \frac{s}{2}$, $\beta_m = \frac{1}{2}$. For SSTT Alg. 2.1, choose $\lambda_1 = 2.5$, $\mu = 0.6$, $\alpha = 0.1$ and $\tau = 0.1$. The test image is Hand X-ray and the stopping criterion for all the algorithms is $E_m = \|s_{m+1} - s_m\| < 10^{-8}$. Thus, we obtain the following figures and tables.

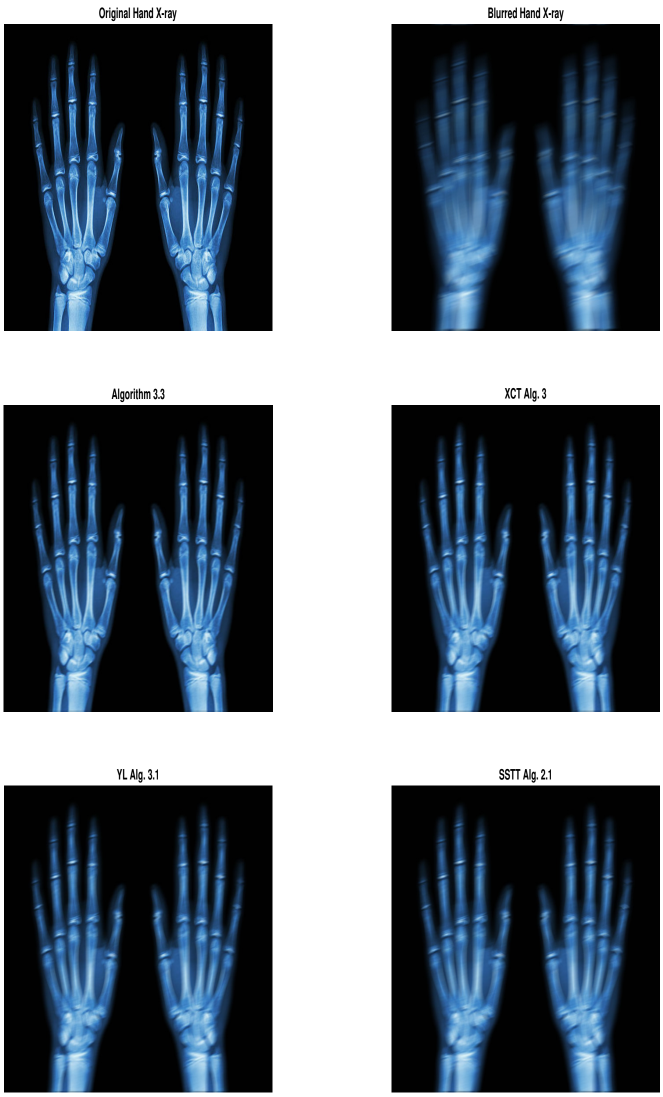

Figure 1: Comparison of restored images via various methods when the number of iterations is 2500 of Hand X-ray image

Table 1: Numerical comparison for Algorithm 3.3, XCT Alg. 3, YL Alg. 3.1 and SSTT Alg. 2.1.

Image	m	Algorithm 3.3	XCT Alg. 3	YL Alg. 3.1	SSTT Alg. 2.1
Hand X-ray	1000	38.3164	29.8141	28.4283	23.8513
Size=720 × 630	1500	38.3532	30.5151	28.4556	24.4121
	2500	38.3782	31.3882	28.4782	25.1106

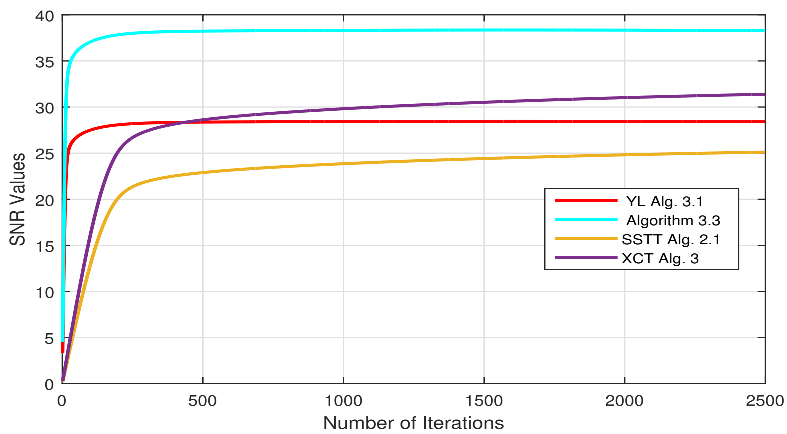

Figure 2: Graphs of SNR for the methods Algorithm 3, XCT Alg. 3, YL Al and SSTT Alg. 2.1 of Hand X-ray image.

6. Numerical Example

In this section, we present two numerical examples to further test the computational advantage of the proposed Algorithm 3 with some single inertial methods such as Algorithm 3 of Xie et al. [19] (shortly, XCT Alg. 3), Algorithm 3.1 of Yang and Liu [36] (shortly, YL Alg.3.1) and Algorithm 2.1 of Yekini el al. [37] (shortly, SSTT Alg. 2.1). All the computations are performed using Matlab R2023b which is running on a personal computer with an Intel(R) Core(TM) i5-10210U CPU at 2.11GHz and 8.00 Gb-RAM.

Example 1. *Let the feasible set C be defined by $C = \{s \in \mathbb{R}^n : -5 \leq s_j \leq 5, j = 1, 2 \cdots, n\}$, and $g : C \times C \to \mathbb{R}$ be a bifunction defined by*

$$g(s, v) = \langle Ps + Qv + r, v - s\rangle, \forall s, v \in C,$$

where $r \in \mathbb{R}^n$ and $P, Q \in \mathbb{R}^{n\times n}$. The matrix P is symmetric positive semi-definite and the matrix $(Q - P)$ is symmetric negative semi-definite with Lipschitz constant $\ell_1 = \ell_2 = \frac{\|P-Q\|}{2}$ (for more details, see [19]). In this experiment, we use the same control parameters given in Section 5. We consider the stopping criterion $E_m = \|s_{m+1} - s_m\| < 10^{-7}$ and for $n = 40, 80, 120$, we obtain the following table and figure.

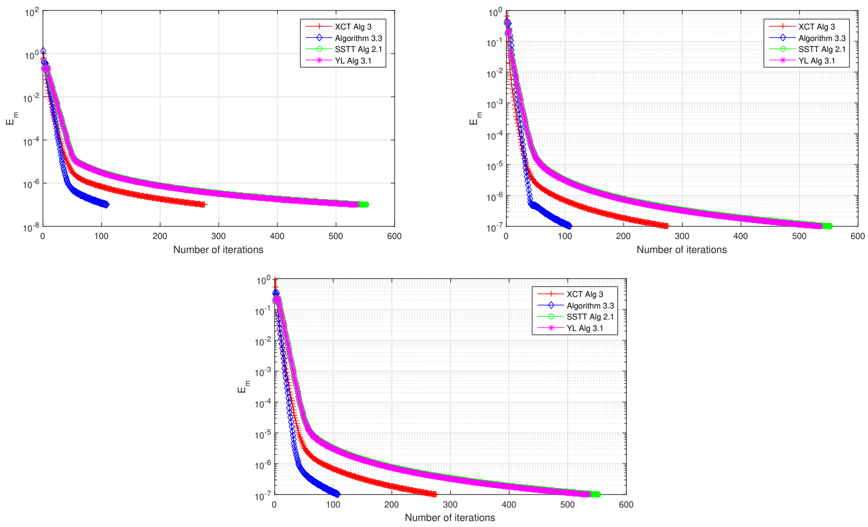

Figure 3: Example 1, $m = 40$ (top left); $m = 80$ (top right); $m = 120$ (middle).

Table 2: Results of the Numerical Simulations for Different Dimensions

Numerical Results for $m = 40, 80$ and 120 in Example 1								
	Algorithm 3.3		XCT Alg. 3		YL Alg.3.1		SSTT Alg. 2.1	
n	Iter	CPU time (sec.)	Iter	CPU time (sec.)	Iter	CPU time (sec.)	Iter	CPU time (sec.)
$n = 40$	120	0.0414	280	0.7027	540	1.5032	580	1.9495
$n = 80$	105	0.0495	270	0.6727	535	1.4025	575	1.8218
$n = 120$	112	0.0396	278	0.6894	538	1.4894	579	1.8127

Example 2. *Let $H = L_2([0,1])$ be the infinite dimensional Hilbert space endowed with the inner product $\langle s, v\rangle = \int_0^1 s(z)v(z)dz$, $\forall s, v \in H$ and norm $\|s\| = \left(\int_0^1 |s(z)|^2 dz\right)$, $\forall s \in H$. Let q, V be two real numbers with $\frac{V}{e+1} < \frac{q}{e} < q < V$ for some $q >$. Let the set C be given by $C = \{s \in H : \|s\| \leq q\}$ and the operator B be given by $B(s) = (V - \|s\|)$, $\forall s \in H$. It known that B is pseudomonotone instead of monotone [19]. In this experiment, we take $V = 1.9$, $q = 1.1$ and $e = 1.3$. The defined (VIP) for B and C has the solution $s^{\dagger}(z) = 0$. We choose the stopping criteria $E_m = \|s_{m+1} - s_m\| < 10^{-8}$ and the following initial values:*
Case 1: $s_0(z) = s_1(z) = 100z$;
Case 2: $s_0(z) = s_1(z) = 300e^z$;
Case 3: $s_0(z) = s_1(z) = 300\log(z)$;
Case 4: $s_0(z) = s_1(z) = 100\sin(z)$.
Using the same control parameters given in Section 5, we have the following figures and tables

Table 3: Numerical results of Example 2

Cases		Algorithm 3.3	XCT Alg. 3	YL Alg.3.1	SSTT Alg. 2.1
Case 1	CPU time (sec.)	0.0007	0.0013	0.0017	0.0024
	No of Iter.	7	9	11	12
Case 2	CPU time (sec.)	0.0016	0.0018	0.0024	0.0026
	No of Iter.	9	10	12	14
Case 3	CPU time (sec.)	0.0019	0.0021	0.0028	0.0036
	No of Iter.	6	10	10	11

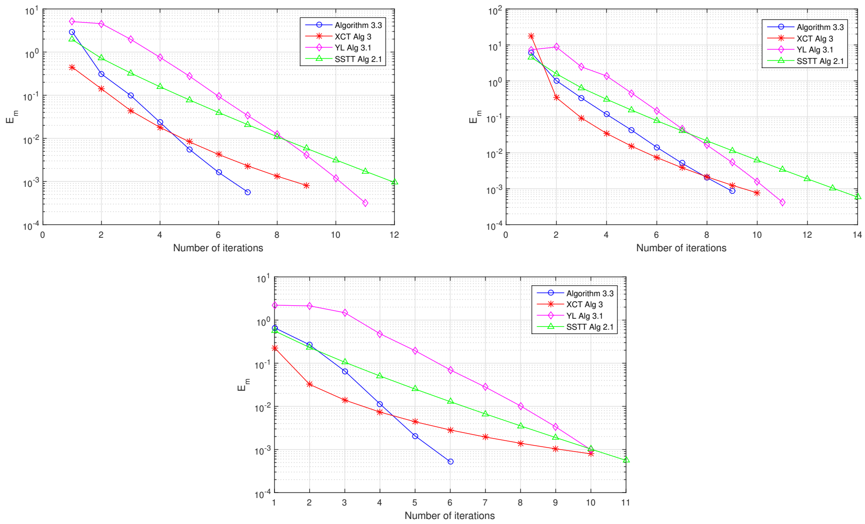

Figure 4: Example 2, **Case 1** (top left); **Case 2** (top right); **Case 3** (bottom).

7. Conclusion

In this work, we studied a new improve subgradient extragradient method for approximating the common solutions of EP and FPP in real Hilbert spaces. The fastness in convergence of the introduced method is enhanced with two-inertial technique. We prove the strong convergence of the new method under some standard assumptions. We showed that our method outperformed some notable methods when applied in solving some real life related problems such VIPs and image recovery problems. Finally, numerical experiments are carried to show the relevance of findings over many existing results in the literature.

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Data Availability Data sharing is not applicable for this article as no datasets were generated or analyzed during the current study.

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