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On n-Anti-Homoderivations of Prime and Semiprime Rings

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Abstract. We present the notion of n-anti-homoderivations, a novel class of additive mappings on rings. Within this framework, we establish strong necessary conditions for the commutativity of rings and the existence of nontrivial central ideals. Our approach generalizes several classical results as special cases and addresses open problems concerning the interaction between such mappings and the structural properties of rings.

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1. Introduction

The study of derivations and their generalizations has been critical in the evolution of ring theory, particularly in determining ring commutativity. The foundation of this field was laid by Posner [1] (1957), who proved one of the most influential results in the theory of derivations: commutative prime rings are those that admit a nonzero centralizing derivation. Awtar [2, 3] (1973) later provided a simplified proof of Posner's theorem and extended it to Lie and Jordan ideals, showing that under certain conditions, these ideals

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must be central.

The study of commuting maps (where every element commutes with its image under the map) and centralizing maps (where the commutator of every element and its image lies in the ring's center) advanced significantly after Mayne expanded Posner's theorem in multiple directions [4–6] (1976, 1982, 1984). The first key conclusion establishes that for a prime ring to be commutative, it should have a nontrivial centralizing automorphism. The second conclusion states that one of the prime rings $\mathcal{S}$ is commutative if it meets one of the requirements: $\mathcal{S}$ admits a centralizing derivation that preserves a non-zero ideal; or it admits a nontrivial centralizing automorphism that preserves a non-zero ideal. The third result shows that $\mathcal{S}$ is commutative if it is a prime ring with a nontrivial automorphism or derivation that centralizes a quadratic Jordan ideal at its center.

These ideas were extended to semiprime rings in 1987 by Bell and Martindale [7], who showed that a derivation (or an endomorphism) centralizing on a nonzero left ideal $\mathcal{U}$ of $\mathcal{S}$ forces $\mathcal{U}$ to be contained in the center of $\mathcal{S}$.

Herstein [8, 9] (1978, 1979) contributed foundational results on non-zero derivation $\mathcal{D}$ of a prime ring $\mathcal{S}$, including: If $\mathcal{D}^3 \neq 0$, then the subring produced by $\mathcal{D}(\mathcal{S})$ possesses a non-zero ideal of $\mathcal{S}$. In addition, if $char\, \mathcal{S} \neq 2$ such that $[\mathcal{D}(\mathcal{S}), \mathcal{D}(\mathcal{S})] = (0)$, then $\mathcal{S}$ is commutative. Furthermore, if $char\, \mathcal{S} \neq 2$ and $a \in \mathcal{S}$ satisfies $[a, \mathcal{D}(\mathcal{S})] = (0)$, then a is central. These findings were extended by Bell and Daif [10, 11] (1994, 1995) who investigated strong commutativity-preserving (SCP) derivations and demonstrated that semiprime rings must contain central ideals under specific circumstances. For the situation of semiprime rings, Daif [12] (1998) extended a Herstein's result about a derivation $\mathcal{D}$ on a prime ring $\mathcal{S}$ fulfilling $[\mathcal{D}(\mathcal{S}), \mathcal{D}(\mathcal{S})] = (0)$. He demonstrated that this finding can be extended for ideals, but not for one-sided ideals. Recent work by Mouhssine et al. [13] showed the adaptability of $\alpha-$generalized derivation techniques by extending these concepts to near-ring situations.

A major shift in the theory occurred when El-Soufi [14] (2000) introduced the concept of homoderivation map $\hbar$, which is additive and satisfies: $\hbar(cd) = c\hbar(d) + \hbar(c)d + \hbar(c)\hbar(d)$. Over the past decade, the theory of homoderivations and their generalizations has seen considerable advancement. Building on El-Soufi's pioneering work [14], several researchers have extended the scope of these mappings: Tammam et al. [15, 16] made groundbreaking contributions through their work on centrally extended homoderivations and n-homoderivations. Their 2022 paper introduced the comprehensive framework of n-homoderivations which is an additive mapping $\hbar_n$ satisfying $\hbar_n(ab) = a\hbar_n(b) + \hbar_n(a)b + n\hbar_n(a)\hbar_n(b), n \in \mathbb{Z}$. They verified that if $\hbar_n$ is an n-homoderivation of a prime ring $\mathcal{S}$ and $\hbar_n$ is centralizing on a one-sided ideal of $\mathcal{S}$, then $\mathcal{S}$, under specific conditions, is commutative. Several previous concepts were combined in this formulation, which also opened up new research directions (see [17–20]).

The previous discussion makes it clear that the investigation of ring-theoretic structures has been significantly enriched by the study of additive mappings that either preserve or reverse the multiplicative order of elements. Among the order-preserving mappings, notable examples include derivations, homomorphisms, and homoderivations. These operators respect the sequence of multiplication in the sense that they satisfy identities such as $\phi(cd) = \phi(c)\phi(d)$ or $D(cd) = D(c)d + cD(d)$. On the other hand, mappings such as anti-homomorphisms and reverse derivations intentionally reverse the multiplicative order, exhibiting properties like $\psi(cd) = \psi(d)\psi(c)$ or $D_{\text{rev}}(cd) = D_{\text{rev}}(d)c + dD_{\text{rev}}(c)$ [21]. In the prime and semiprime ring configuration, several studies have been undertaken on these mappings.

In light of these advancements, we introduce the new concepts of anti-homoderivations and n-anti-homoderivations as follows:

Definition 1. *An* anti-homoderivation $\mathcal{H}$ *of a ring* $\mathcal{S}$ *is an additive mapping* $\mathcal{H} : \mathcal{S} \to \mathcal{S}$ *which satisfies the condition:*

$$\mathcal{H}(rs) = \mathcal{H}(s)\mathcal{H}(r) + r\mathcal{H}(s) + \mathcal{H}(r)s, \ \textit{for all } r, s \in \mathcal{S}.$$

Definition 2. *Let* $n \in \mathbb{Z}$*. An* n-anti-homoderivation $\mathcal{H}_n$ *of a ring* $\mathcal{S}$ *is an additive mapping* $\mathcal{H}_n : \mathcal{S} \to \mathcal{S}$ *which satisfies the condition:*

$$\mathcal{H}_n(rs) = n\mathcal{H}_n(s)\mathcal{H}_n(r) + r\mathcal{H}_n(s) + \mathcal{H}_n(r)s, \ \textit{for all } r, s \in \mathcal{S}.$$

We will investigate ring commutativity using an n-anti-homoderivation that satisfies specific algebraic identities. Specifically, we show that it is enough for the n-anti-homoderivation to be centralizing and to satisfy a zero power condition to verify the commutativity of a prime ring on a non-zero ideal of the ring. Furthermore, we examine a ring $\mathcal{S}$ equipped with an n-anti-homoderivation $\mathcal{H}_n$, where $\mathcal{H}_n$ satisfies, for instance, one of the following conditions on a suitable subset $\mathcal{U}$:

(1) $[\mathcal{H}_n(r), \mathcal{H}_n(s)] = 0$ *for all* $r, s \in \mathcal{U}$.

(2) $rs - \mathcal{H}_n(rs) = sr - \mathcal{H}_n(sr)$ *for all* $r, s \in \mathcal{U}$.

(3) $\mathcal{H}_n(r)\mathcal{H}_n(s) + \mathcal{H}_n(sr) = \mathcal{H}_n(s)\mathcal{H}_n(r) + \mathcal{H}_n(rs)$ *for all* $r, s \in \mathcal{U}$.

(4) $\mathcal{H}_n(rs) + (n+1)\mathcal{H}_n(r)\mathcal{H}_n(s) + rs = \mathcal{H}_n(sr) + (n+1)\mathcal{H}_n(s)\mathcal{H}_n(r) + sr$ *for all* $r, s \in \mathcal{U}$.

Throughout this study, $\mathcal{S}$ stands for an associative ring not necessarily having unit, whereas $\Upsilon(\mathcal{S})$ designates the center of $\mathcal{S}$. Whenever $nr = 0$ with $r \in \mathcal{S}$, then $r = 0$, the ring $\mathcal{S}$ is referred to as n-torsion free, $n \in \mathbb{Z} - \{0\}$ [22]. The ring $\mathcal{S}$ is said to be prime if the product of two non-zero ideals of $\mathcal{S}$ is not zero. For associative rings, this is equivalent to each of the following conditions (see [23, page 47]):

(i) If $r\mathcal{S}s = \{0\}$, where $r, s \in \mathcal{S}$, then $r = 0$ or $s = 0$.

(ii) The left annihilator of a non-zero left ideal is zero.

For alternative rings, let $\mathcal{S}$ be a 3-torsion free alternative ring. Then $\mathcal{S}$ is prime if and only if $a\mathcal{S} \cdot b = 0 \quad (\text{or } a \cdot \mathcal{S}b = 0)$ implies $a = 0$ or $b = 0$, for all $a, b \in \mathcal{S}$ (see [24, Theorem 1.1]). The ring $\mathcal{S}$ is called semiprime if it contains no nonzero ideal whose square is zero; equivalently, if $r\mathcal{S}r = \{0\}$, then $r = 0$.

Let $\phi \neq \mathcal{A} \subseteq \mathcal{S}$. A mapping $\mathcal{H}$ of $\mathcal{S}$ will be regarded as zero-power valued (ZPV) on $\mathcal{A}$ if for each $a \in \mathcal{A}$ we have $\mathcal{H}(a) \in \mathcal{A}$, and $\mathcal{H}^{k(a)}(a) = 0$ for some positive integer $k(a) > 1$ [14]. In addition, $\mathcal{H}$ is considered commuting (centralizing) on $\mathcal{A}$ whether, for any $r \in \mathcal{A}$, $[r, \mathcal{H}(r)] = 0$ ($[r, \mathcal{H}(r)] \in \Upsilon(\mathcal{S})$) [6].

If an additive map $\mathcal{F}$ on $\mathcal{S}$ fulfills the formula $\mathcal{F}(rt) = \mathcal{F}(r)t + r\mathcal{F}(t)$ for all $r, t \in \mathcal{S}$, then it is a derivation [25].

The facts that follow will be used in our study.

Lemma 1. [5, Lemma 4] *For a prime ring $\mathcal{S}$, if $\xi, \eta\xi \in \Upsilon(\mathcal{S})$, then either $\xi = 0$ or $\eta \in \Upsilon(\mathcal{S})$.*

Lemma 2. [26, Lemma 1] *Let $\mathcal{U} \neq (0)$ represent an ideal in a semiprime ring $\mathcal{S}$. If $t \in \mathcal{S}$, $[t, [\mathcal{U}, \mathcal{U}]] = (0)$, then $[t, \mathcal{U}] = (0)$.*

Lemma 3. [7] *If $\mathcal{U} \neq (0)$ is a one-sided ideal of a semiprime ring $\mathcal{S}$, then $\Upsilon(\mathcal{U}) \subseteq \Upsilon(\mathcal{S})$. In particular, if $\mathcal{U}$ is a commutative one-sided ideal, $\mathcal{U} \subseteq \Upsilon(\mathcal{S})$.*

Lemma 4. [25, Theorem 3.1] *A Jordan derivation of a prime ring with characteristic other than two is an ordinary derivation.*

Lemma 5. [5, Lemma 3] *If a prime ring's non-zero left ideal is commutative, then the ring is commutative.*

Lemma 6. *Assume that $\mathcal{U} \neq (0)$ is an ideal of a prime ring $\mathcal{S}$ and $\xi, \eta \in \mathcal{S}$. If $\xi\mathcal{U}\eta = (0)$, then either $\xi = 0$ or $\eta = 0$.*

Proof. Assume that $\xi\mathcal{U}\eta = (0)$. So, $\xi\mathcal{S}\mathcal{U}\eta = (0)$. By the primeness of $\mathcal{S}$, we get $\xi = 0$ or $\mathcal{U}\eta = (0)$. If $\xi \neq 0$, then $\mathcal{U}\mathcal{S}\eta = (0)$. Again, by primeness of $\mathcal{S}$ and $\mathcal{U} \neq (0)$, then $\eta = 0$.

2. Examples of an n-anti-homoderivations

Examples demonstrating the existence of n-anti-homoderivations are shown below.

Example 1. *Consider the non-commutative ring*

$$\mathcal{S} = \left\{ \begin{bmatrix} \xi & \eta \\ 0 & \gamma \end{bmatrix} \mid \xi, \eta, \gamma \in \mathbb{Z}_m, m > 2 \right\}.$$

Defining $\mathcal{H}_n : \mathcal{S} \to \mathcal{S}$ by $\mathcal{H}_n \left(\begin{bmatrix} \xi & \eta \\ 0 & \gamma \end{bmatrix} \right) = n \begin{bmatrix} 0 & \xi - \gamma \\ 0 & 0 \end{bmatrix}$. We can readily demonstrate that, for every $n \in \mathbb{Z}$, $\mathcal{H}_n$ is an n-anti-homoderivation.

Example 2. *Suppose* $\mathcal{S} = \left\{ \begin{bmatrix} \xi & 0 \\ \eta & 0 \end{bmatrix} : \xi, \eta \in \mathcal{K},\ \text{where } \mathcal{K} \text{ is any ring} \right\}$. *Consider the map* $\mathcal{H}_n : \mathcal{S} \to \mathcal{S}$ *defined by* $\mathcal{H}_n\left(\begin{bmatrix} \xi & 0 \\ \eta & 0 \end{bmatrix}\right) = n \begin{bmatrix} 0 & 0 \\ \eta & 0 \end{bmatrix}$. *It is simply confirmed that, for every* $n \in \mathbb{Z}$, $\mathcal{H}_n$ *is an* n*-anti-homoderivation.*

Example 3. *Consider the infinite non-commutative ring* $R = \mathbb{Z}_7[x] \oplus M_2(\mathbb{Z}_3)$, *equipped with componentwise addition and multiplication, that is,* $(z, A) + (t, B) = (z + t,\ A + B)$ *and* $(z, A)(t, B) = (zt,\ AB)$, *for all* $z, t \in \mathbb{Z}_7[x]$ *and* $A, B \in M_2(\mathbb{Z}_3)$. *Define a map* $H_2(z, A) = \big(3z,\ 2A + 2\,\mathrm{tr}(A)I_2\big)$, *for all* $(z, A) \in R$, *where* $\mathrm{tr}(A)$ *denotes the trace of* A *and* I_2 *is the* 2×2 *identity matrix. Then,* H_2 *is a* 2*-anti-homoderivation. However, it is neither a derivation, a homomorphism, a homoderivation, nor a* 2*-homoderivation on* R.

3. Preliminaries

The purpose of this section is to present certain n-anti-homoderivation features.

Lemma 7. *For an* n*-torsion free semiprime ring* $\mathcal{S}$, *the zero map is the only additive map* $\mathcal{H}_n$ *that is both a derivation and an* n*-anti-homoderivation.*

Proof. Assume that $\mathcal{H}_n$ is an n-anti-homoderivation of an n-torsion free semiprime ring $\mathcal{S}$. If $\mathcal{H}_n$ acts as a derivation on $\mathcal{S}$, then $\mathcal{H}_n(s)\mathcal{H}_n(r) = 0$, for any $r, s \in \mathcal{S}$. By substituting s for st, we gain $\mathcal{H}_n(s)t\mathcal{H}_n(r) = 0$, for any $r, s, t \in \mathcal{S}$. So, for any $r \in \mathcal{S}$, $\mathcal{H}_n(r)\mathcal{S}\mathcal{H}_n(r) = (0)$. Since $\mathcal{S}$ is a semiprime, it follows that $\mathcal{H}_n = 0$.

Lemma 8. *Suppose* $\mathcal{H}_n$ *is an* n*-anti-homoderivation of a prime ring* $\mathcal{S}$ *and* $\mathcal{U}$ *is a non-zero left ideal of* $\mathcal{S}$. *If* $\mathcal{H}_n(\mathcal{U}) = (0)$, *then* $\mathcal{H}_n(\mathcal{S}) = (0)$.

Proof. Suppose $0 \neq u \in \mathcal{U}$ and $r \in \mathcal{S}$. So,

$$0 = \mathcal{H}_n(ru) = n\mathcal{H}_n(u)\mathcal{H}_n(r) + \mathcal{H}_n(r)u + r\mathcal{H}_n(u) = \mathcal{H}_n(r)u.$$

Changing r to rs, $s \in \mathcal{S}$, to obtain

$$0 = \mathcal{H}_n(rs)u = \mathcal{H}_n(r)su.$$

Since $\mathcal{S}$ is prime and $u \neq 0$, we get $\mathcal{H}_n = 0$.

Lemma 9. *Let* $\mathcal{H}_n$ *be an* n*-anti-homoderivation of a prime ring* $\mathcal{S}$ *and* $b \in \mathcal{S}$. *If* $b\mathcal{H}_n(\mathcal{S}) = (0)$, *then either* $b = 0$ *or* $\mathcal{H}_n(\mathcal{S}) = (0)$.

Proof. According to the assumption $b\mathcal{H}_n(r) = 0$ for each $r \in \mathcal{S}$. Switching out r for rs, then

$$b\mathcal{H}_n(rs) = 0 = nb\mathcal{H}_n(s)\mathcal{H}_n(r) + b\mathcal{H}_n(r)s + br\mathcal{H}_n(s) = br\mathcal{H}_n(s),$$

for any $r, s \in \mathcal{S}$. If $\mathcal{H}_n \neq 0$, then $\mathcal{H}_n(s) \neq 0$ for some $s \in \mathcal{S}$. Hence, by primeness of $\mathcal{S}$, $b = 0$.

Theorem 1. *Given a prime ring $\mathcal{S}$ with char $\mathcal{S} \neq 2$, let $\mathcal{H}_n \neq 0$ be an n-anti-homoderivation of $\mathcal{S}$. Any element $b \in \mathcal{S}$ satisfying $[b, \mathcal{H}_n(\mathcal{S})] = (0)$ should be a central element.*

Proof. Suppose that $b \notin \Upsilon(\mathcal{S})$. Using the hypothesis, for all $r, s \in \mathcal{S}$, we possess

$$0 = [b, \mathcal{H}_n(rs)] = [b, n\mathcal{H}_n(s)\mathcal{H}_n(r) + \mathcal{H}_n(r)s + r\mathcal{H}_n(s)].$$

Using again that b commutes with all $\mathcal{H}_n(t)\,\forall t \in \mathcal{S}$, we get

$$[b, r]\mathcal{H}_n(s) + \mathcal{H}_n(r)[b, s] = 0. \tag{1}$$

If $s \in \mathcal{S}$ commutes with b then $[b, s] = 0$. Hence, (1) reduces to $[b, r]\mathcal{H}_n(s) = 0$ for all $r \in \mathcal{S}$. Because $b \notin \Upsilon(\mathcal{S})$, by Lemma 9, $\mathcal{H}_n$ vanishes on the centralizer, $C_{\mathcal{S}}(b) = \{s \in \mathcal{S} : bs = sb\}$, $b \in \mathcal{S}$. But for any $r \in \mathcal{S}$, $\mathcal{H}_n(r) \in C_{\mathcal{S}}(b)$ by hypothesis, hence we get that

$$\mathcal{H}_n^2(r) = 0\ \forall r \in \mathcal{S}. \tag{2}$$

Thus, we have, for all $r, s \in \mathcal{S}$, that

$$0 = \mathcal{H}_n^2(rs) = \mathcal{H}_n(n\mathcal{H}_n(s)\mathcal{H}_n(r) + \mathcal{H}_n(r)s + r\mathcal{H}_n(s))\ \textit{for all}\, r, s \in \mathcal{S}. \tag{3}$$

Again, making use of (2) in (3), we obtain $2\mathcal{H}_n(r)\mathcal{H}_n(s) = 0$. But, $\mathcal{S}$ is prime of characteristic not two, hence $\mathcal{H}_n(r)\mathcal{H}_n(s) = 0$. Replacing s by st yields

$$0 = \mathcal{H}_n(r)\mathcal{H}_n(st) = \mathcal{H}_n(r)s\mathcal{H}_n(t)\,\forall r, s, t \in \mathcal{S}.$$

By the fact that $\mathcal{S}$ is prime, $\mathcal{H}_n(r) = 0$ for each $r \in \mathcal{S}$, which is a contradiction. So, $b \in \Upsilon(\mathcal{S})$

Theorem 2. *Let $\mathcal{S}$ be any ring and $\mathcal{H}_n$ an n-anti-homoderivation of $\mathcal{S}$ with $\mathcal{H}_n^3 \neq 0$. Then, $\Lambda =< \mathcal{H}_n(\mathcal{S}) >$, the subring of $\mathcal{S}$ generated by $\mathcal{H}_n(\mathcal{S})$, contains a nonzero ideal of $\mathcal{S}$.*

Proof. Because $\mathcal{H}_n^3 \neq 0$ and $\mathcal{H}_n(\mathcal{S}) \subseteq \Lambda$, $\mathcal{H}_n^2(\Lambda) \neq (0)$. Pick $y \in \Lambda$ such that $\mathcal{H}_n^2(y) \neq 0$. If $x \in \mathcal{S}$, then

$$\Lambda \ni \mathcal{H}_n(xy) = n\mathcal{H}_n(y)\mathcal{H}_n(x) + x\mathcal{H}_n(y) + \mathcal{H}_n(x)y,$$

and since $y, \mathcal{H}_n(x), \mathcal{H}_n(y) \in \Lambda$, in the end, we obtain $x\mathcal{H}_n(y) \in \Lambda$, which means, $\mathcal{S}\mathcal{H}_n(y) \subseteq \Lambda$. In the same way, $\mathcal{H}_n(y)\mathcal{S} \subseteq \Lambda$. If $r, s \in \mathcal{S}$ then

$$\Lambda \ni \mathcal{H}_n(r\mathcal{H}_n(y)) = n\mathcal{H}_n^2(y)\mathcal{H}_n(r) + r\mathcal{H}_n^2(y) + \mathcal{H}_n(r)\mathcal{H}_n(y).$$

That is, $r\mathcal{H}_n^2(y) \in \Lambda$. Similarly, $\mathcal{H}_n^2(y)s \in \Lambda$. But

$$\begin{aligned}\Lambda \ni \mathcal{H}_n(r\mathcal{H}_n(y)s) \quad = \quad & n^2\mathcal{H}_n(s)\mathcal{H}_n^2(y)\mathcal{H}_n(r) + n\mathcal{H}_n(s)r\mathcal{H}_n^2(y) \\ & + n\mathcal{H}_n(s)\mathcal{H}_n(r)\mathcal{H}_n(y) + n\mathcal{H}_n^2(y)\mathcal{H}_n(r)s \\ & + \mathcal{H}_n(r)\mathcal{H}_n(y)s + r\mathcal{H}_n^2(y)s + r\mathcal{H}_n(y)\mathcal{H}_n(s),\end{aligned}$$

hence we get $r\mathcal{H}_n^2(y)s \in \Lambda$ for all $r, s \in \mathcal{S}$. Since $\mathcal{H}_n^2(y) \neq 0$, $\mathcal{S}\mathcal{H}_n^2(y)\mathcal{S} \subseteq \Lambda$. Thus, the ideal of $\mathcal{S}$ produced by $\mathcal{H}_n^2(y) \neq 0$ must be included in Λ.

Corollary 1. *Let $\mathcal{H}_0$ be a derivation of a ring $\mathcal{S}$, with $\mathcal{H}_0^3 \neq 0$. Hence, the subring Λ of $\mathcal{S}$ generated by $\mathcal{H}_0(\mathcal{S})$ contains a nonzero ideal of $\mathcal{S}$.*

Theorem 3. *If $\mathcal{S}$ is a prime ring with char $\mathcal{S}$ neither two nor n, and $\mathcal{H}_n \neq 0$ is an n-anti-homoderivation of $\mathcal{S}$, then $(\mathcal{H}_n(r))^2 \neq 0$, for some $r \in \mathcal{S}$.*

Proof. Since
$$\mathcal{H}_n(rs) = r\mathcal{H}_n(s) + \mathcal{H}_n(r)s + n\mathcal{H}_n(s)\mathcal{H}_n(r)$$
for each $r, s \in \mathcal{S}$, then
$$\mathcal{H}_n(r^2) = \mathcal{H}_n(r)r + r\mathcal{H}_n(r) + n(\mathcal{H}_n(r))^2$$
for each $r \in \mathcal{S}$. Now, if $(\mathcal{H}_n(r))^2 = 0$ for each $r \in \mathcal{S}$, then $\mathcal{H}_n(r^2) = r\mathcal{H}_n(r) + \mathcal{H}_n(r)r$ for each $r \in \mathcal{S}$. Thus, $\mathcal{H}_n$ is a Jordan derivation. According to the hypothesis, $\mathcal{H}_n$ is a derivation by Lemma 4. Consequently, by Lemma 7, $\mathcal{H}_n(\mathcal{S}) = (0)$, which is a contradiction.

Lemma 10. *Let $\mathcal{U} \neq (0)$ be a left ideal of a prime ring $\mathcal{S}$. If $\xi \in \mathcal{S}$, $[\xi, [\mathcal{U}, \mathcal{U}]] = (0)$, then $\mathcal{U}[\mathcal{U}, \mathcal{U}] = (0)$ or $[\xi, \mathcal{U}] = (0)$.*

Proof. In general $[\xi, [r, rs]] = 0$ for $r, s \in \mathcal{U}$, i.e., $[\xi, r][r, s] = 0$. Now, replace s with ts for $t \in \mathcal{U}$ and get $[\xi, r]t[r, s] = 0$. So, for each $r \in \mathcal{U}$ either $\mathcal{U}[r, \mathcal{U}] = (0)$ or $[r, \xi] = 0$. Thus, $\mathcal{U}$ represents the union of the two additive subgroups $\{r : [\xi, r] = 0\}$ and $\{r : \mathcal{U}[r, \mathcal{U}] = (0)\}$. Our conclusion is that $\mathcal{U}[\mathcal{U}, \mathcal{U}] = (0)$ or $[\xi, \mathcal{U}] = (0)$.

Theorem 4. *Suppose that $\mathcal{S}$ is a prime ring with char $\mathcal{S} \neq 2n$, $\mathcal{U} \neq (0)$ is an additive subgroup of $\mathcal{S}$ such that $u^2 \in \mathcal{U}$ for any $u \in \mathcal{U}$, and $\mathcal{H}_n \neq 0$ is an n-anti-homoderivation of $\mathcal{S}$. If $\mathcal{H}_n$ is centralizing on $\mathcal{U}$, then $\mathcal{H}_n$ is commuting on $\mathcal{U}$.*

Proof. We have $u + u^2 \in \mathcal{U}$, for each $u \in \mathcal{U}$. So, $[u + u^2, \mathcal{H}_n(u + u^2)] \in \Upsilon(\mathcal{S})$. Using the fact that $[u, \mathcal{H}_n(u)] \in \Upsilon(\mathcal{S})$ we have, $[u, \mathcal{H}_n(u^2)] + [u^2, \mathcal{H}_n(u)] \in \Upsilon(\mathcal{S})$ for each $u \in \mathcal{U}$. Therefore, $[u, \mathcal{H}_n(u)]\{4u + 2n\mathcal{H}_n(u)\} \in \Upsilon(\mathcal{S})$ *for each* $u \in \mathcal{U}$. According to Lemma 1, either $4u + 2n\mathcal{H}_n(u) \in \Upsilon(\mathcal{S})$ or $[u, \mathcal{H}_n(u)] = 0$. Now, if $4u + 2n\mathcal{H}_n(u) \in \Upsilon(\mathcal{S})$, then $0 = [u, 4u + 2n\mathcal{H}_n(u)] = 2n[u, \mathcal{H}_n(u)]$, since the characteristic of $\mathcal{S}$ is not $2n$. Therefore, $[u, \mathcal{H}_n(u)] = 0$ for any $u \in \mathcal{U}$. We arrived at our requirement.

4. Results

The following theorem, which is essential to our current study, is where we begin this section.

Theorem 5. *Let $\mathcal{S}$ be a semiprime ring with a left ideal $\mathcal{U} \neq (0)$, and $\mathcal{H}_n$ an n-anti-homoderivation of $\mathcal{S}$. If $\mathcal{H}_n$ is centralizing on $\mathcal{U}$, then it is commuting on $\mathcal{U}$.*

Proof. Based on our prelude,

$$[\mathcal{H}_n(\upsilon), \upsilon] \in \Upsilon(\mathcal{S}) \;\; for\, all\, \upsilon \in \mathcal{U}. \tag{4}$$

Therefore,

$$[\mathcal{H}_n(\upsilon^2), \upsilon^2] \in \Upsilon(\mathcal{S}),$$

that is,

$$[\mathcal{H}_n(\upsilon)\upsilon, \upsilon^2] + [\upsilon\mathcal{H}_n(\upsilon), \upsilon^2] + n[\mathcal{H}_n(\upsilon)\mathcal{H}_n(\upsilon), \upsilon^2] \in \Upsilon(\mathcal{S}).$$

By (4), we arrive at

$$4\upsilon^2[\mathcal{H}_n(\upsilon), \upsilon] + 2n\mathcal{H}_n(\upsilon)\upsilon[\mathcal{H}_n(\upsilon), \upsilon] + 2n\upsilon\mathcal{H}_n(\upsilon)[\mathcal{H}_n(\upsilon), \upsilon] \in \Upsilon(\mathcal{S}). \tag{5}$$

Commuting (5) with $\mathcal{H}_n(\upsilon)$, we get

$$4(2\upsilon + n\mathcal{H}_n(\upsilon))[\mathcal{H}_n(\upsilon), \upsilon]^2 = 0. \tag{6}$$

But, using (6), we get

$$\begin{aligned} 8[\mathcal{H}_n(\upsilon), \upsilon]^3 &= 8(\mathcal{H}_n(\upsilon)\upsilon - \upsilon\mathcal{H}_n(\upsilon))[\mathcal{H}_n(\upsilon), \upsilon]^2 \\ &= 8\mathcal{H}_n(\upsilon)\upsilon[\mathcal{H}_n(\upsilon), \upsilon]^2 - 8\upsilon\mathcal{H}_n(\upsilon)[\mathcal{H}_n(\upsilon), \upsilon]^2 \\ &= 8\mathcal{H}_n(\upsilon)\upsilon[\mathcal{H}_n(\upsilon), \upsilon]^2 - 8\upsilon[\mathcal{H}_n(\upsilon), \upsilon]^2\mathcal{H}_n(\upsilon) \\ &= 8\mathcal{H}_n(\upsilon)\upsilon[\mathcal{H}_n(\upsilon), \upsilon]^2 + 4n\mathcal{H}_n(\upsilon)^2[\mathcal{H}_n(\upsilon), \upsilon]^2 \\ &= \mathcal{H}_n(\upsilon)\{4(2\upsilon + n\mathcal{H}_n(\upsilon))\}[\mathcal{H}_n(\upsilon), \upsilon]^2 = 0. \end{aligned} \tag{7}$$

Therefore,

$$(2[\mathcal{H}_n(\upsilon), \upsilon])^3 = 0, \; for\, all\, \upsilon \in \mathcal{U}.$$

Since the semiprime $\mathcal{S}$ has no non-zero central nilpotent element, hence

$$2[\mathcal{H}_n(\upsilon), \upsilon] = 0, \; for\, all\, \upsilon \in \mathcal{U}. \tag{8}$$

Moreover,

$$[\mathcal{H}_n(\upsilon), \upsilon^2] = 2[\mathcal{H}_n(\upsilon), \upsilon]\upsilon = 0, \; for\, all\, \upsilon \in \mathcal{U}. \tag{9}$$

Linearizing υ in (8), we get

$$2\{[\mathcal{H}_n(\upsilon), u] + [\mathcal{H}_n(u), \upsilon]\} = 0, \; for\, any\, \upsilon, u \in \mathcal{U}. \tag{10}$$

Also, since $[\mathcal{H}_n(\upsilon), \upsilon] \in \Upsilon(\mathcal{S})$, then

$$[\mathcal{H}_n(\upsilon), u] + [\mathcal{H}_n(u), \upsilon] \in \Upsilon(\mathcal{S}), \; for\, all\, \upsilon, u \in \mathcal{U}. \tag{11}$$

Now, using (8), (10) and (11), we get

$$[\mathcal{H}_n(\upsilon), \upsilon u + u\upsilon] + [\mathcal{H}_n(u), \upsilon^2] = 0, \; for\, all\, \upsilon, u \in \mathcal{U}. \tag{12}$$

Changing u with υu in (12), we get

$$\begin{aligned}[\mathcal{H}_n(\upsilon), \upsilon u\upsilon + \upsilon^2 u] &+ [\mathcal{H}_n(\upsilon)u, \upsilon^2] + [\upsilon\mathcal{H}_n(u), \upsilon^2] \\ &+ n[\mathcal{H}_n(u)\mathcal{H}_n(\upsilon), \upsilon^2] = 0, \; for\, all\, \upsilon, u \in \mathcal{U}.\end{aligned}$$

Using (9) and (12), we get

$$\begin{aligned}&(\upsilon u + u\upsilon)[\mathcal{H}_n(\upsilon), \upsilon] + n[\mathcal{H}_n(u), \upsilon^2]\mathcal{H}_n(\upsilon) + \mathcal{H}_n(\upsilon)[u, \upsilon^2] = 0 \\ &for\, any\, \upsilon, u \in \mathcal{U}.\end{aligned} \tag{13}$$

In (13) replacing $(\upsilon u + u\upsilon)$ by $[\upsilon, u] + 2u\upsilon$ and using (8), we obtain

$$[\upsilon, u][\mathcal{H}_n(\upsilon), \upsilon] + \mathcal{H}_n(\upsilon)[u, \upsilon^2] + n[\mathcal{H}_n(u), \upsilon^2]\mathcal{H}_n(\upsilon) = 0. \tag{14}$$

Case 1: When n is even, it can be expressed as $n = 2z$ for some $z \in \mathbb{Z}$. Also, by(10), we derive the relation $2[\mathcal{H}_n(u), \upsilon] = -2[\mathcal{H}_n(\upsilon), u]$. Consequently, substitute these into the equation. (14) simplifies to

$$\begin{aligned}[\upsilon, u][\mathcal{H}_n(\upsilon), \upsilon] + \mathcal{H}_n(\upsilon)[u, \upsilon^2] = 2z[\mathcal{H}_n(\upsilon), u]\upsilon\mathcal{H}_n(\upsilon) \\ +2z\upsilon[\mathcal{H}_n(\upsilon), u]\mathcal{H}_n(\upsilon).\end{aligned} \tag{15}$$

In (15), substituting $\mathcal{H}_n(\upsilon)\upsilon$ instead of u we get

$$\begin{aligned}&[\upsilon, \mathcal{H}_n(\upsilon)]\upsilon[\mathcal{H}_n(\upsilon), \upsilon] + \mathcal{H}_n(\upsilon)[\mathcal{H}_n(\upsilon), \upsilon^2]\upsilon \\ =\; &2z\mathcal{H}_n(\upsilon)[\mathcal{H}_n(\upsilon), \upsilon]\upsilon\mathcal{H}_n(\upsilon) + 2z\upsilon\mathcal{H}_n(\upsilon)[\mathcal{H}_n(\upsilon), \upsilon]\mathcal{H}_n(\upsilon).\end{aligned} \tag{16}$$

Applying (8) and (9) to (16) and by (4), we now get

$$\upsilon[\mathcal{H}_n(\upsilon), \upsilon]^2 = 0, \; for\, all\, \upsilon \in \mathcal{U}. \tag{17}$$

But, using (4) and (17), we get

$$\begin{aligned}[\mathcal{H}_n(\upsilon), \upsilon]^3 &= [\mathcal{H}_n(\upsilon), \upsilon]^2\mathcal{H}_n(\upsilon)\upsilon - [\mathcal{H}_n(\upsilon), \upsilon]^2\upsilon\mathcal{H}_n(\upsilon) \\ &= \mathcal{H}_n(\upsilon)\upsilon[\mathcal{H}_n(\upsilon), \upsilon]^2 - \upsilon[\mathcal{H}_n(\upsilon), \upsilon]^2\mathcal{H}_n(\upsilon) = 0.\end{aligned}$$

Hence, $[\mathcal{H}_n(\upsilon), \upsilon] = 0$, for each $\upsilon \in \mathcal{U}$. In other words, $\mathcal{H}_n$ is commuting on $\mathcal{U}$, for each even integer n.

Case 2: If n is odd integer, then there exist $z \in \mathbb{Z}$ such that $n = 2z + 1$ and (14) becomes

$$\begin{aligned}&[\upsilon, u][\mathcal{H}_n(\upsilon), \upsilon] + \mathcal{H}_n(\upsilon)[u, \upsilon^2] + 2z\upsilon[\mathcal{H}_n(u), \upsilon]\mathcal{H}_n(\upsilon) \\ +\; &2z[\mathcal{H}_n(u), \upsilon]\upsilon\mathcal{H}_n(\upsilon) + [\mathcal{H}_n(u), \upsilon^2]\mathcal{H}_n(\upsilon) = 0.\end{aligned} \tag{18}$$

So, from (10), we have $2[\mathcal{H}_n(u), \upsilon] = -2[\mathcal{H}_n(\upsilon), u]$, hence (18) becomes

$$[\upsilon, u][\mathcal{H}_n(\upsilon), \upsilon] + \mathcal{H}_n(\upsilon)[u, \upsilon^2] - 2z\upsilon[\mathcal{H}_n(\upsilon), u]\mathcal{H}_n(\upsilon)$$

$$- \quad 2z[\mathcal{H}_n(\upsilon), u]\upsilon\mathcal{H}_n(\upsilon) + [\mathcal{H}_n(u), \upsilon^2]\mathcal{H}_n(\upsilon) = 0. \tag{19}$$

In (19), substituting $\mathcal{H}_n(\upsilon)\upsilon$ instead of u, using (8), and (9) in (19) and $[\mathcal{H}_n(\upsilon), \upsilon] \in \Upsilon(\mathcal{S})$ we get

$$[\mathcal{H}_n(\mathcal{H}_n(\upsilon)\upsilon), \upsilon^2]\mathcal{H}_n(\upsilon) = \upsilon[\mathcal{H}_n(\upsilon), \upsilon]^2, \ for\, all\, \upsilon \in \mathcal{U}. \tag{20}$$

Returning to (12), substituting $\mathcal{H}_n(\upsilon)\upsilon$ for u, we can gain

$$[\mathcal{H}_n(\upsilon), \upsilon\mathcal{H}_n(\upsilon)\upsilon + \mathcal{H}_n(\upsilon)\upsilon^2] = [\upsilon^2, \mathcal{H}_n(\mathcal{H}_n(\upsilon)\upsilon)], \ for\, all\, \upsilon \in \mathcal{U}.$$

That is

$$\begin{aligned} &[\mathcal{H}_n(\upsilon), \upsilon]\mathcal{H}_n(\upsilon)\upsilon + \upsilon\mathcal{H}_n(\upsilon)[\mathcal{H}_n(\upsilon), \upsilon] + [\mathcal{H}_n(\upsilon), \mathcal{H}_n(\upsilon)\upsilon^2] \\ &= [\upsilon^2, \mathcal{H}_n(\mathcal{H}_n(\upsilon)\upsilon)]. \end{aligned} \tag{21}$$

Using (9) in (21), the third term is zero and using $[\mathcal{H}_n(\upsilon), \upsilon] \in \Upsilon(\mathcal{S})$, we can get

$$\{\mathcal{H}_n(\upsilon)\upsilon + \upsilon\mathcal{H}_n(\upsilon)\}[\mathcal{H}_n(\upsilon), \upsilon] = -[\mathcal{H}_n(\mathcal{H}_n(\upsilon)\upsilon), \upsilon^2]\, for\, all\, \upsilon \in \mathcal{U}. \tag{22}$$

But, using (8), we get

$$\begin{aligned} \mathcal{H}_n(\upsilon)\upsilon + \upsilon\mathcal{H}_n(\upsilon)[\mathcal{H}_n(\upsilon), \upsilon] \quad &= \{2\upsilon\mathcal{H}_n(\upsilon) + [\mathcal{H}_n(\upsilon), \upsilon]\}[\mathcal{H}_n(\upsilon), \upsilon] \\ &= 2\upsilon\mathcal{H}_n(\upsilon)[\mathcal{H}_n(\upsilon), \upsilon] + [\mathcal{H}_n(\upsilon), \upsilon]^2 \\ &= [\mathcal{H}_n(\upsilon), \upsilon]^2. \end{aligned} \tag{23}$$

Comparing between (22) and (23), we get

$$[\mathcal{H}_n(\upsilon), \upsilon]^2 = [\upsilon^2, \mathcal{H}_n(\mathcal{H}_n(\upsilon)\upsilon)] \tag{24}$$

Now, using (24) in (20), we get

$$(\upsilon + \mathcal{H}_n(\upsilon))[\mathcal{H}_n(\upsilon), \upsilon]^2 = 0. \tag{25}$$

Now, using (25) and the same technique used in getting (7), we get

$$[\mathcal{H}_n(\upsilon), \upsilon]^3 = 0.$$

For this reason, $[\mathcal{H}_n(\upsilon), \upsilon]$ is central and nilpotent, and $\mathcal{S}$ is semiprime. Hence,

$$[\mathcal{H}_n(\upsilon), \upsilon] = 0, \ for\, all\, \upsilon \in \mathcal{U},$$

In other words, $\mathcal{H}_n$ is commuting on $\mathcal{U}$ for all odd integer n. Hence, the requirement has been achieved.

The following is how the Bell and Martindale conclusion [7, Lemma 4] arises as a particular case of the preceding theorem for $n = 0$.

Corollary 2. *Let $\mathcal{U} \neq (0)$ be a left ideal of a semiprime ring $\mathcal{S}$ and $\mathcal{H}_0$ a derivation of $\mathcal{S}$. If $\mathcal{H}_0$ is centralizing on $\mathcal{U}$, then $\mathcal{H}_0$ is commuting on $\mathcal{U}$.*

In what follows, we present a theorem analogous to Posner's result stated in [1, Theorem 2], providing a broader formulation that applies to the case of an n-anti-homoderivation on an ideal.

Theorem 6. *Consider $\mathcal{S}$ is a prime ring, and $\mathcal{U} \neq (0)$ is an ideal of $\mathcal{S}$. If $\mathcal{S}$ admits a non-zero n-anti-homoderivation $\mathcal{H}_n$ that is ZPV and centralizing on $\mathcal{U}$, then $\mathcal{S}$ is commutative.*

Proof. Using Theorem 5, $\mathcal{H}_n$ is commuting on $\mathcal{U}$. So, $[\mathcal{H}_n(v), v] = 0$ for each $v \in \mathcal{U}$. Substituting v with $v + u$ yields

$$[\mathcal{H}_n(v), u] + [\mathcal{H}_n(u), v] = 0 \;\; for\, each\, v, u \in \mathcal{U}. \tag{26}$$

After substituting u with vu in (26) and using (26), we obtain

$$\mathcal{H}_n(v)[u, v] + n[u, \mathcal{H}_n(v)]\mathcal{H}_n(v) = 0 \;\; for\, any\, v, u \in \mathcal{U}. \tag{27}$$

Changing u by tu in (27), we obtain

$$\begin{aligned} nt[u, \mathcal{H}_n(v)]\mathcal{H}_n(v) + n[t, \mathcal{H}_n(v)]u\mathcal{H}_n(v) \\ +\mathcal{H}_n(v)[t, v]u + \mathcal{H}_n(v)t[u, v] = 0 \; \forall v, t, u \in \mathcal{U}. \end{aligned} \tag{28}$$

Using (27) and (28), we get

$$\begin{aligned} &- t\mathcal{H}_n(v)[u, v] + n[t, \mathcal{H}_n(v)]u\mathcal{H}_n(v) \\ &+ \mathcal{H}_n(v)[t, v]u + \mathcal{H}_n(v)t[u, v] = 0 \qquad \text{for any } v, u, t \in \mathcal{U}. \end{aligned}$$

So,

$$[\mathcal{H}_n(v), t][u, v] + n[t, \mathcal{H}_n(v)]u\mathcal{H}_n(v) + \mathcal{H}_n(v)[t, v]u = 0 \;\; for\, all\, v, u, t \in \mathcal{U}. \tag{29}$$

Substituting from (27) in (29), we get

$$[\mathcal{H}_n(v), t][u, v] + n[t, \mathcal{H}_n(v)]u\mathcal{H}_n(v) - n[t, \mathcal{H}_n(v)]\mathcal{H}_n(v)u = 0 \;\; for\, all\, v, u, t \in \mathcal{U}.$$

So,

$$[\mathcal{H}_n(v), t][u, v] + n[t, \mathcal{H}_n(v)][u, \mathcal{H}_n(v)] = 0 \;\; for\, any\, t, v, u \in \mathcal{U}.$$

This leads to

$$[\mathcal{H}_n(v), t]\{[u, v] - n[u, \mathcal{H}_n(v)]\} = 0 \;\; \forall v, u, t \in \mathcal{U}. \tag{30}$$

Substituting from (26) in (30), we get

$$[\mathcal{H}_n(v), t]\{[u, v] + n[\mathcal{H}_n(u), v]\} = 0 \;\; for\, any\, v, t, u \in \mathcal{U}.$$

So,

$$[\mathcal{H}_n(v), t][u + n\mathcal{H}_n(u), v] = 0 \;\; for\, any\, t, v, u \in \mathcal{U}.$$

Replacing u by $u - n\mathcal{H}_n(u) + n^2\mathcal{H}_n^2(u) + \cdots + (-1)^{k(u)-1}n^{k(u)-1}\mathcal{H}_n^{k(u)-1}(u)$, and using the fact that $\mathcal{H}_n$ is a ZPV on $\mathcal{U}$, we get

$$[\mathcal{H}_n(v), t][u, v] = 0 \ \ \forall v, u, t \in \mathcal{U}. \tag{31}$$

Changing t to xt, where $x \in \mathcal{S}$, in (31) and using (31), we obtain

$$[\mathcal{H}_n(v), x]t[u, v] = 0 \ \ for\, all\, v, u, t \in \mathcal{U}, x \in \mathcal{S}.$$

Therefore, by Lemma 6 and Lemma 3, for any $v \in \mathcal{U}$, we have either $v \in \Upsilon(\mathcal{U}) \subseteq \Upsilon(\mathcal{S})$ or $\mathcal{H}_n(v) \in \Upsilon(\mathcal{S})$. For a fixed v, suppose $\mathcal{H}_n(v) \in \Upsilon(\mathcal{S})$, then by (26)

$$[v, \mathcal{H}_n(u)] = 0 \ \ \forall u \in \mathcal{U}. \tag{32}$$

Changing u to uv, in (32) and utilizing (32), we obtain

$$[u, v]\mathcal{H}_n(v) = 0 \ \ for\, each\, u \in \mathcal{U}. \tag{33}$$

Exchanging yu for u, where $y \in \mathcal{S}$, in (33) and using (33), we arrive at

$$[y, v]u\mathcal{H}_n(v) = 0 \ \ for\, each\, u \in \mathcal{U}, y \in \mathcal{S}.$$

Therefore, either $v \in \Upsilon(\mathcal{S})$ or $\mathcal{H}_n(v) = 0$. As a result, $\mathcal{U}$ is the collections of the following two additive subgroups: $\mathscr{L} = \{v \in \mathcal{U} : \mathcal{H}_n(v) = 0\}$ and $\mathscr{M} = \{v \in \mathcal{U} : v \in \Upsilon(\mathcal{S})\}$. This means that, $\mathcal{U} = \mathscr{L}$ or $\mathcal{U} = \mathscr{M}$. Examine the initial scenario where $\mathcal{U} = \mathscr{L}$. This implies that $\mathcal{H}_n = 0$ on $\mathcal{U}$. Therefore, by Lemma 8, $\mathcal{H}_n = 0$, a contradiction. Hence, $\mathcal{U} = \mathscr{M}$, implying that $\mathcal{S}$ is a commutative ring.

Corollary 3. *A prime ring $\mathcal{S}$ is commutative if it has either a non-zero anti-homoderivation $\mathcal{H}_1$ or a non-zero derivation $\mathcal{H}_0$ that is ZPV and centralizing on a non-zero ideal of $\mathcal{S}$.*

In the following, we establish a result analogous to Posner's theorem from [1, Theorem 2] providing a more comprehensive formulation that applies to the case of an n-anti-homoderivation on one-sided ideals.

Theorem 7. *If the prime ring $\mathcal{S}$ of characteristic not equal to 2 has a non-zero anti-homoderivation $\mathcal{H}_n$ that is ZPV and centralizing on a left ideal $\mathcal{U} \neq (0)$ of $\mathcal{S}$ and $\mathcal{U}$ contains no non-zero nilpotent elements, then the prime ring is commutative.*

Proof. According to our hypothesis and Theorem 5, we possess

$$[\mathcal{H}_n(r), r] = 0\, for\, each\, r \in \mathcal{U}.$$

After substituting r with $r + s$, we gain

$$[\mathcal{H}_n(r), s] + [\mathcal{H}_n(s), r] = 0\, for\, all\, r, s \in \mathcal{U}. \tag{34}$$

By substituting r for rs and applying (34), we acquire

$$n\mathcal{H}_n(s)[\mathcal{H}_n(r), s] + [r, s]\mathcal{H}_n(s) = 0 \tag{35}$$

Since $\mathcal{H}_n$ is a ZPV on $\mathcal{U}$, we can replace r by $r + \mathcal{H}_n(r) + \mathcal{H}_n^2(r) + ... + \mathcal{H}_n^{k(r)-1}(r)$ in (34), where $k(r) > 1$ is the smallest integer satisfies $\mathcal{H}_n^{k(r)}(r) = 0$, we obtain

$$[\mathcal{H}_n^{k(r)-1}(r), \mathcal{H}_n(s)] = 0\,\forall s, r \in \mathcal{U}. \tag{36}$$

By putting $w\mathcal{H}_n(s)$ instead of s in (36), we gain

$$[\mathcal{H}_n^{k(r)-1}(r), w]\mathcal{H}_n^2(s) = 0\, for\, any\, r, s, w \in \mathcal{U}. \tag{37}$$

Replacing w by tw, where $t \in \mathcal{S}$, in (37), we have

$$[\mathcal{H}_n^{k(r)-1}(r), t]w\mathcal{H}_n^2(s) = 0\, for\, each\, r, s, w \in \mathcal{U},\, t \in \mathcal{S}.$$

Therefore, either $\mathcal{U}\mathcal{H}_n^2(s) = (0)$ for each $s \in \mathcal{U}$ or $\mathcal{H}_n^{k(r)-1}(r) \in \Upsilon(\mathcal{S})$ for each $r \in \mathcal{U}$.

Assume first that $\mathcal{U}\mathcal{H}_n^2(s) = (0)$ for each $s \in \mathcal{U}$. Then, $(\mathcal{H}_n^2(s))^2 = 0$ for all $s \in \mathcal{U}$. Since $\mathcal{U}$ possesses no non-zero nilpotent elements, then $\mathcal{H}_n^2(s) = 0$ for each $s \in \mathcal{U}$. By substituting s by s^2 and applying $char\,\mathcal{S} \neq 2$, we acquire $(\mathcal{H}_n(s))^2 = 0\, for\, all\, s \in \mathcal{U}$. Again, since $\mathcal{U}$ has no non-zero nilpotent elements, then $\mathcal{H}_n(s) = 0\, for\, all\, s \in \mathcal{U}$. Replacing s by ts, $t \in \mathcal{S}$, gives $\mathcal{H}_n(t)s = 0\, for\, each\, s \in \mathcal{U}, t \in \mathcal{S}$. Therefore, $\mathcal{H}_n = 0$ on $\mathcal{S}$, a contradiction. Thus,

$$\mathcal{H}_n^{k(r)-1}(r) \in \Upsilon(\mathcal{S})\, for\, each\, r \in \mathcal{U}. \tag{38}$$

Substituting s by $\mathcal{H}_n^{k(s)-2}(s)$ in (35), yields

$$[r, \mathcal{H}_n^{k(s)-2}(s)]\mathcal{H}_n^{k(s)-1}(s) = 0\, for\, all\, r, s \in \mathcal{U}. \tag{39}$$

Changing r with tu, where $u \in \mathcal{U}, t \in \mathcal{S}$, in (39) and using (34), we obtain

$$[t, \mathcal{H}_n^{k(s)-2}(s)]u\mathcal{H}_n^{k(s)-1}(s) = 0$$

for every $u, s \in \mathcal{U}$, $t \in \mathcal{S}$. By replacing s with r, we obtain the relation

$$[t, \mathcal{H}_n^{k(r)-2}(r)]\, u\, \mathcal{H}_n^{k(r)-1}(r) = 0$$

which holds for all $u, r \in \mathcal{U}$ and $t \in \mathcal{S}$. Therefore, for every $r \in \mathcal{U}$ either $\mathcal{U}\mathcal{H}_n^{k(r)-1}(r) = (0)$ or $\mathcal{H}_n^{k(r)-2}(r) \in \Upsilon(\mathcal{S})$. But, if $\mathcal{U}\mathcal{H}_n^{k(r)-1}(r) = (0)$, then $\mathcal{H}_n^{k(r)-1}(r) = 0$, it's a contradiction. So,

$$\mathcal{H}_n^{k(r)-2}(r) \in \Upsilon(\mathcal{S})\, for\, all\, r \in \mathcal{U}. \tag{40}$$

By continuing with the same technique from (38) to (40), we arrive at $\mathcal{U} \subseteq \Upsilon(\mathcal{S})$ and thus, using Lemma 5, $\mathcal{S}$ is commutative.

This yields a version of Herstein's result [8, Theorem 2] adapted to the setting of n-anti-homoderivation.

Theorem 8. *Let $\mathcal{S}$ be a prime ring, $\mathcal{H}_n \neq 0$ an n-anti-homoderivation of $\mathcal{S}$ such that $[\mathcal{H}_n(r), \mathcal{H}_n(s)] = 0$ for every $r, s \in \mathcal{S}$. If char $\mathcal{S} \neq 2$, $\mathcal{S}$ is a commutative.*

Proof. Let $\mathcal{A}$ be the subring of $\mathcal{S}$ that is produced by all $\mathcal{H}_n(r)$, $r \in \mathcal{S}$. If $a \in \mathcal{A}$ and $r \in \mathcal{S}$, we have

$$\mathcal{A} \ni \mathcal{H}_n(ar) = n\mathcal{H}_n(r)\mathcal{H}_n(a) + a\mathcal{H}_n(r) + \mathcal{H}_n(a)r,$$

hence centralizes $\mathcal{A}$. So, if $b \in \mathcal{A}$,

$$0 = b\mathcal{H}_n(ar) - \mathcal{H}_n(ar)b = \mathcal{H}_n(a)[b, r].$$

Then, we have $\mathcal{H}_n(a) = 0$ or $b \in \Upsilon(\mathcal{S})$. If $\mathcal{A} \not\subseteq \Upsilon(\mathcal{S})$, we must have $b \in \mathcal{A}$ such that $b \notin \Upsilon(\mathcal{S})$. Then $\mathcal{H}_n(\mathcal{A}) = (0)$. But $\mathcal{H}_n(\mathcal{S}) \subseteq \mathcal{A}$, hence $\mathcal{H}_n^2(\mathcal{S}) \subseteq \mathcal{H}_n(\mathcal{A}) = (0)$, that is, $\mathcal{H}_n^2(r) = 0$ for every $r \in \mathcal{S}$. As in Theorem 1, $\mathcal{H}_n = 0$. This contradicts $\mathcal{H}_n \neq 0$. Therefore, $\mathcal{A} \subseteq \Upsilon(\mathcal{S})$, i.e., $\mathcal{H}_n(r) \in \Upsilon(\mathcal{S})$ for any $r \in \mathcal{S}$. Replacing r by rs, then $\mathcal{H}_n(rs) \in \Upsilon(\mathcal{S})$ for each $r, s \in \mathcal{S}$. So, $[\mathcal{H}_n(rs), r] = 0$ for any $r, s \in \mathcal{S}$ and hence $\mathcal{H}_n(r)[s, r] = 0$ for any $s, r \in \mathcal{S}$. For each $r \in \mathcal{S}$, either $\mathcal{H}_n(r) = 0$ or $r \in \Upsilon(\mathcal{S})$. Since $\mathcal{H}_n \neq 0$, pick r_0 such that $\mathcal{H}_n(r_0) \neq 0$ then $r_0 \in \Upsilon(\mathcal{S})$. If $\mathcal{H}_n(r) = 0$, $\mathcal{H}_n(r_0 + r) = \mathcal{H}_n(r_0) \neq 0$, hence $r_0 + r \in \Upsilon(\mathcal{S})$. This leaves us with $r \in \Upsilon(\mathcal{S})$. Thus $\mathcal{S}$ is a commutative.

As a continuation of previous studies, we aim to present a counterpart to Daif's result, originally stated in [12, Theorem 2.2], in the framework of n-anti-homoderivations.

Theorem 9. *Let $\mathcal{S}$ be a 2-torsion free semiprime ring and $(0) \neq \mathcal{U}$ an ideal of $\mathcal{S}$. If $\mathcal{S}$ has a non-zero n-anti-homoderivation $\mathcal{H}_n$ which is ZPV on $\mathcal{U}$ and satisfies $[\mathcal{H}_n(s), \mathcal{H}_n(r)] = 0$ for any $s, r \in \mathcal{U}$, then, there is a central non-zero ideal in the ring $\mathcal{S}$.*

Proof. By assumption, we have $[\mathcal{H}_n(s), \mathcal{H}_n(r)] = 0$ for any $s, r \in \mathcal{U}$. Substituting s by st, yields,

$$[\mathcal{H}_n(r), s]\mathcal{H}_n(t) + \mathcal{H}_n(s)[\mathcal{H}_n(r), t] = 0 \, for \, any \, s, r, t \in \mathcal{U}. \tag{41}$$

Substituting t by $t\mathcal{H}_n(u)$, $u \in \mathcal{U}$, to obtain

$$[\mathcal{H}_n(r), s]t\mathcal{H}_n^2(u) + n[\mathcal{H}_n(r), s]\mathcal{H}_n^2(u)\mathcal{H}_n(t) = 0 \, for \, any \, s, r, t, u \in \mathcal{U}. \tag{42}$$

In (41), replacing t by $\mathcal{H}_n(u)$, we have $[\mathcal{H}_n(r), s]\mathcal{H}_n^2(u) = 0$ for any $u, s, r \in \mathcal{U}$. Using this in (42), gives $[\mathcal{H}_n(r), s]t\mathcal{H}_n^2(u) = 0$ for any $s, u, r, t \in \mathcal{U}$. Therefore, $[\mathcal{H}_n(r), s]\mathcal{U}\mathcal{S}\mathcal{H}_n^2(u) = (0)$ for any $r, u, s \in \mathcal{U}$.

Since $\mathcal{S}$ is semiprime, we can find a collection $\{P_i : i \in \Lambda\}$ of ideals that are primes in $\mathcal{S}$ so that $\cap_i P_i = (0)$. Hence for each P_i, we have either

(1) $[\mathcal{H}_n(r), s]\mathcal{U} \subseteq P_i \, for \, any \, r, s \in \mathcal{U}$; or

(2) $\mathcal{H}_n^2(\mathcal{U}) \subseteq P_i$.

But (1) implies that $[\mathcal{H}_n(r), s] \in P_i$ or $\mathcal{U} \subseteq P_i$. If $\mathcal{U} \subseteq P_i$, then $[\mathcal{H}_n(r), s] \in P_i$. So, (1) implies $[\mathcal{H}_n(r), s] \in P_i$.

Now, assuming $\mathcal{H}_n^2(\mathcal{U}) \subseteq P_i$, we have $\mathcal{H}_n^2(rs) \in P_i$ for any $r, s \in \mathcal{U}$. Therefore, $2\mathcal{H}_n(r)\mathcal{H}_n(s) \in P_i$ for any $s, r \in \mathcal{U}$. Substituting ts for s to obtain $2\mathcal{H}_n(r)t\mathcal{H}_n(s) \in P_i$ for any $s, r, t \in \mathcal{U}$. Hence, either $\mathcal{H}_n(\mathcal{U}) \subseteq P_i$ or $2\mathcal{H}_n(r)s \in P_i$, $2s\mathcal{H}_n(r) \in P_i$ for any $s, r \in \mathcal{U}$. Thus, $2[\mathcal{H}_n(r), s] \in P_i$ for any $s, r \in \mathcal{U}$. So case (2) implies $2[\mathcal{H}_n(r), s] \in P_i$ for any $s, r \in \mathcal{U}$. Thus, $2[\mathcal{H}_n(r), s] \in \cap Pi = (0)$ for any $s, r \in \mathcal{U}$, i.e, $2[\mathcal{H}_n(r), s] = 0$ for any $s, r \in \mathcal{U}$. But, $\mathcal{S}$ is 2-torsion free, hence $[\mathcal{H}_n(r), s] = 0$ for any $s, r \in \mathcal{U}$. Therefore, by Lemma 3, $\mathcal{H}_n(\mathcal{U}) \subseteq \Upsilon(\mathcal{U}) \subseteq \Upsilon(\mathcal{S})$, i.e., $\mathcal{H}_n(\mathcal{U}) \subseteq \Upsilon(\mathcal{S})$.

Let $r \in \mathcal{U}$, then $\mathcal{H}_n(r), \mathcal{H}_n(r^2) \in \Upsilon(\mathcal{S})$. So, $2r\mathcal{H}_n(r) \in \Upsilon(\mathcal{S})$, and hence, by 2-torsion freeness, $[r, x]\mathcal{H}_n(r) = 0$ for any $r \in \mathcal{U}, x \in \mathcal{S}$. Replacing x with xy where $y \in \mathcal{S}$, we obtain $[r, x]y\mathcal{H}_n(r) = 0$ for any $r \in \mathcal{U}, x, y \in \mathcal{S}$, i.e., $[r, \mathcal{S}]\mathcal{S}\mathcal{H}_n(r) = (0)$ for all $r \in \mathcal{U}$. Therefore, for each $r \in \mathcal{U}$ either $[r, \mathcal{S}] \subseteq P_i$ or $\mathcal{H}_n(r) \subseteq P_i$. Hence, there are two subsets of $\mathcal{U}$: $\mathcal{K} = \{r \in \mathcal{U} : [r, \mathcal{S}] \subseteq P_i\}$ and $\mathcal{M} = \{r \in \mathcal{U} : \mathcal{H}_n(r) \subseteq P_i\}$ such that both are additive subgroups and $\mathcal{U} = \mathcal{M} \cup \mathcal{K}$. Therefore, $\mathcal{U} = \mathcal{K}$ or $\mathcal{U} = \mathcal{M}$. Thus, $[\mathcal{U}, \mathcal{S}] \subseteq P_i$ or $\mathcal{H}_n(\mathcal{U}) \subseteq P_i$. So, in any case we have $[\mathcal{U}, \mathcal{S}]\mathcal{H}_n(\mathcal{U}) \subseteq P_i$. Thus, $[\mathcal{U}, \mathcal{S}]\mathcal{H}_n(\mathcal{U}) = (0)$. So, $[\mathcal{U}, \mathcal{S}]\mathcal{S}\mathcal{H}_n(\mathcal{U}) = (0)$. which implies that $[\mathcal{S}\mathcal{U}\mathcal{H}_n(\mathcal{U})\mathcal{S}, \mathcal{S}]\mathcal{S}[\mathcal{S}\mathcal{U}\mathcal{H}_n(\mathcal{U})\mathcal{S}, \mathcal{S}] = (0)$. Hence, $[\mathcal{S}\mathcal{U}\mathcal{H}_n(\mathcal{U})\mathcal{S}, \mathcal{S}] = (0)$, i.e., $\mathcal{S}\mathcal{U}\mathcal{H}_n(\mathcal{U})\mathcal{S} \subseteq \Upsilon(\mathcal{S})$. Then, there is a non-zero ideal $(\mathcal{S}\mathcal{U}\mathcal{H}_n(\mathcal{U})\mathcal{S})$ contained in the center of $\mathcal{S}$.

Corollary 4. *If a prime ring $\mathcal{S}$ with char $\mathcal{S} \neq 2$ has an n-anti-homoderivation $\mathcal{H}_n \neq 0$ satisfying $[\mathcal{H}_n(\mathcal{U}), \mathcal{H}_n(\mathcal{U})] = (0)$, where $\mathcal{U} \neq (0)$ is an ideal of $\mathcal{S}$, then $\mathcal{S}$ is commutative*

Building on related work, we aim to present a counterpart of the result by Daif and Bell, originally stated in [26, Theorem 3], in the context of n-anti-homoderivations.

Theorem 10. *Let $\mathcal{U}$ be a non-zero ideal in an $(n-1)$-torsion free semiprime ring $\mathcal{S}$, where $n \in \mathbb{Z}^+ - \{1\}$. If $\mathcal{S}$ admits an n-anti-homoderivation $\mathcal{H}_n$ satisfying $\upsilon s - \mathcal{H}_n(\upsilon s) = s\upsilon - \mathcal{H}_n(s\upsilon)$ for any $\upsilon, s \in \mathcal{U}$, then $\mathcal{U}$ is a central ideal in $\mathcal{S}$.*

Proof. Our hypothesis

$$\upsilon s - \mathcal{H}_n(\upsilon s) = s\upsilon - \mathcal{H}_n(s\upsilon) \, for\, any\, \upsilon, s \in \mathcal{U}. \tag{43}$$

Equation (43) could be rephrased as

$$\mathcal{H}_n([\upsilon, s]) = [\upsilon, s] \, for\, any\, \upsilon, s \in \mathcal{U}. \tag{44}$$

Now, from (43), for any $\upsilon, s, t \in \mathcal{U}$, it follows that

$$[\upsilon, s]t - \mathcal{H}_n([\upsilon, s]t) = t[\upsilon, s] - \mathcal{H}_n(t[\upsilon, s]) \, for\, any\, \upsilon, s, t \in \mathcal{U}.$$

Again, making use of (44) we gain

$$(n-1)\mathcal{H}_n(t)[\upsilon, s] = (n-1)[\upsilon, s]\mathcal{H}_n(t) \, for\, any\, \upsilon, s, t \in \mathcal{U}.$$

Since $\mathcal{S}$ is $(n-1)$-torsion free, $[\upsilon, s]\mathcal{H}_n(t) = \mathcal{H}_n(t)[\upsilon, s]$ for any $s, \upsilon, t \in \mathcal{U}$. Then, by Lemma 2, $\mathcal{H}_n(\mathcal{U})$ centralizes $\mathcal{U}$ and it follows from (44) that $[\upsilon, s] \in \Upsilon(\mathcal{U})$ for any $\upsilon, s \in \mathcal{U}$. Now for any $t \in \mathcal{U}$, we have $[\upsilon, s]t = t[\upsilon, s]$ for any $s, \upsilon \in \mathcal{U}$. Again by Lemma 2, t is located in the center of $\mathcal{U}$. Thus, $\mathcal{U} = \Upsilon(\mathcal{U})$. Since $\mathcal{S}$ is semiprime, by Lemma 3, then $\mathcal{U} \subseteq \Upsilon(\mathcal{S})$.

Corollary 5. *If $n \in \mathbb{Z}^+ - \{1\}$, and $\mathcal{S}$ is a prime ring with a characteristic distinct from $(n-1)$, and $(0) \neq \mathcal{U}$ an ideal of $\mathcal{S}$. If $\mathcal{S}$ has an n-anti-homoderivation $\mathcal{H}_n$ with $\mathcal{H}_n([s,r]) = [s,r]$ for any $s, r \in \mathcal{U}$, then $\mathcal{S}$ is commutative.*

As a part of our investigation, we consider a counterpart to Daif's result from [12, Corollary], formulated in the context of n-anti-homoderivations.

Theorem 11. *Let $\mathcal{S}$ be a prime ring with $char\,\mathcal{S} \neq 2$ and $\mathcal{U} \neq (0)$ a left ideal of $\mathcal{S}$, which has no non-zero nilpotent elements. If $\mathcal{S}$ admits a non-zero n-anti-homoderivation $\mathcal{H}_n$ which is ZPV on $\mathcal{U}$ and satisfies $\mathcal{H}_n(r)\mathcal{H}_n(s) + \mathcal{H}_n(sr) = \mathcal{H}_n(s)\mathcal{H}_n(r) + \mathcal{H}_n(rs)$ for any $r, s \in \mathcal{U}$, then $\mathcal{S}$ is commutative.*

Proof. Consider our hypothesis,

$$\mathcal{H}_n([r,s]) = [\mathcal{H}_n(r), \mathcal{H}_n(s)]\; for\, any\, r, s \in \mathcal{U}. \tag{45}$$

Using ZPV property on $\mathcal{U}$, we can replace s by $s + \mathcal{H}_n(s) + \mathcal{H}_n^2(s) + \cdots + \mathcal{H}_n^{k(s)-1}(s)$, to get

$$\begin{aligned} 0 = \mathcal{H}_n([r, \mathcal{H}_n^{k(s)-1}(s)]) &= \mathcal{H}_n(r\mathcal{H}_n^{k(s)-1}(s) - \mathcal{H}_n^{k(s)-1}(s)r) \\ &= \mathcal{H}_n(r)\mathcal{H}_n^{k(s)-1}(s) - \mathcal{H}_n^{k(s)-1}(s)\mathcal{H}_n(r) \\ &= [\mathcal{H}_n(r), \mathcal{H}_n^{k(s)-1}(s)]\;\; for\, any\, r, s \in \mathcal{U}. \end{aligned} \tag{46}$$

Substituting $w\mathcal{H}_n(r)$ for r, and using (46), yields

$$[w, \mathcal{H}_n^{k(s)-1}(s)]\mathcal{H}_n^2(r) = 0\;\; for\, any\, r, s, w \in \mathcal{U}. \tag{47}$$

By substituting tw for w, where $t \in \mathcal{S}$, in (47), we obtain

$$[t, \mathcal{H}_n^{k(s)-1}(s)]w\mathcal{H}_n^2(r) = 0\;\; for\, any\, r, s, w \in \mathcal{U},$$

i.e.,

$$[\mathcal{S}, \mathcal{H}_n^{k(s)-1}(s)]\mathcal{S}\mathcal{U}\mathcal{H}_n^2(r) = (0)\;\; for\, any\, s \in \mathcal{U},$$

Therefore, either $\mathcal{U}\mathcal{H}_n^2(r) = (0)$ for any $r \in \mathcal{U}$ or $\mathcal{H}_n^{k(s)-1}(s) \in \Upsilon(\mathcal{S})$ for any $s \in \mathcal{U}$.

Assume that $\mathcal{U}\mathcal{H}_n^2(r) = (0)$, for any $r \in \mathcal{U}$. Then, $(\mathcal{H}_n^2(r))^2 = 0$ for any $r \in \mathcal{U}$. Since $\mathcal{U}$ has no non-zero nilpotent elements, then $\mathcal{H}_n^2(r) = 0\, for\, any\, r \in \mathcal{U}$. Putting r^2 instead of r, using $char\,\mathcal{S} \neq 2$, we get $(\mathcal{H}_n(r))^2 = 0\, for\, any\, r \in \mathcal{U}$. Again, since $\mathcal{U}$ has no non-zero nilpotent elements, then $\mathcal{H}_n(r) = 0\, for\, any\, r \in \mathcal{U}$. Putting tr instead of r, where $t \in \mathcal{S}$, we arrive at $\mathcal{H}_n(t)r = 0\, for\, any\, r \in \mathcal{U}, t \in \mathcal{S}$. Therefore, $\mathcal{H}_n = 0$ on $\mathcal{S}$, a contradiction. Thus, $\mathcal{H}_n^{k(s)-1}(s) \in \Upsilon(\mathcal{S})\, for\, any\, s \in \mathcal{U}$.

Now, we substitute r with $\mathcal{H}_n^{k(r)-2}(r)$ in (45), yields

$$\begin{aligned} \mathcal{H}_n([\mathcal{H}_n^{k(r)-2}(r), s]) &= \mathcal{H}_n(\mathcal{H}_n^{k(r)-2}(r)s - s\mathcal{H}_n^{k(r)-2}(r)) \\ &= \mathcal{H}_n^{k(r)-2}(r)\mathcal{H}_n(s) - \mathcal{H}_n(s)\mathcal{H}_n^{k(r)-2}(r) \end{aligned}$$

$$= \quad [\mathcal{H}_n^{k(r)-2}(r), \mathcal{H}_n(s)] = 0 \; for\, all\, r, s \in \mathcal{U}. \tag{48}$$

Substituting $z\mathcal{H}(s)$, $z \in \mathcal{U}$, instead of s and using (48), we get

$$[\mathcal{H}_n^{k(r)-2}(r), z]\mathcal{H}_n^2(s) = 0 \, for\, all\, r, s, z \in \mathcal{U}. \tag{49}$$

In (49), substituting z with tz, where $t \in \mathcal{S}$, we gain $[\mathcal{H}_n^{k(r)-2}(r), t]z\mathcal{H}_n^2(s) = 0$ for any $r, s, z \in \mathcal{U}$, $t \in \mathcal{S}$. Hence, either $\mathcal{U}\mathcal{H}_n^2(s) = (0)$ for any $s \in \mathcal{U}$ or $\mathcal{H}_n^{k(r)-2}(r) \in \Upsilon(\mathcal{S})$, for any $r \in \mathcal{U}$. But, if $\mathcal{U}\mathcal{H}_n^2(s) = (0)$, then $\mathcal{H}_n^2(s) = 0$, which as above leads to $\mathcal{H}_n(s) = 0$ and this is a contradiction. So, $\mathcal{H}_n^{k(r)-2}(r) \in \Upsilon(\mathcal{S}) \, for\, all\, r \in \mathcal{U}$. By continuing with the same technique, we arrive at $\mathcal{U} \subseteq \Upsilon(\mathcal{S})$. According to Lemma 5, the requirement is proven.

By the same way that we used to prove the previous theorem, we can obtain:

Theorem 12. *Let $\mathcal{S}$ be a prime ring with char $\mathcal{S} \neq 2$ and $\mathcal{U} \neq (0)$ a left ideal of $\mathcal{S}$ which has no non-zero nilpotent elements. The ring $\mathcal{S}$ is commutative if $\mathcal{S}$ admits a non-zero n-anti-homoderivation $\mathcal{H}_n$ which is ZPV on $\mathcal{U}$ and satisfies $\mathcal{H}_n(r)\mathcal{H}_n(s) + \mathcal{H}_n(rs) = \mathcal{H}_n(s)\mathcal{H}_n(r) + \mathcal{H}_n(sr)$ for any $s, r \in \mathcal{U}$.*

Theorem 13. *Let $\mathcal{S}$ be a prime ring with char $\mathcal{S} \neq 2$, and $\mathcal{U} \neq (0)$ an ideal of $\mathcal{S}$. The ring $\mathcal{S}$ is commutative if $\mathcal{S}$ has an n-anti-homoderivation $\mathcal{H}_n \neq 0$ that is ZPV and $\mathcal{H}_n(rs) + (n+1)\mathcal{H}_n(r)\mathcal{H}_n(s) + rs = \mathcal{H}_n(sr) + (n+1)\mathcal{H}_n(s)\mathcal{H}_n(r) + sr \;\; for\, any\, s, r \in \mathcal{U}$.*

Proof. By hypothesis,

$$\begin{aligned} &\mathcal{H}_n(rs) + (n+1)\mathcal{H}_n(r)\mathcal{H}_n(s) + rs \\ &= \mathcal{H}_n(sr) + (n+1)\mathcal{H}_n(s)\mathcal{H}_n(r) + sr \quad \text{for all } r, s \in \mathcal{U}. \end{aligned} \tag{50}$$

Equation (50) can be written as

$$[r, s] + \mathcal{H}_n([r, s]) = (n+1)[\mathcal{H}_n(s), \mathcal{H}_n(r)] \;\; for\, each\, s, r \in \mathcal{U}.$$

Consequently,

$$[r, s] + [\mathcal{H}_n(r), \mathcal{H}_n(s)] + [\mathcal{H}_n(r), s] + [r, \mathcal{H}_n(s)] = 0 \;\; for\, all\, s, r \in \mathcal{U}.$$

Thus,

$$[r + \mathcal{H}_n(r), s + \mathcal{H}_n(s)] = 0 \;\; for\, all\, r, s \in \mathcal{U}.$$

Since $\mathcal{H}_n$ is ZPV on $\mathcal{U}$, so $[r, s] = 0$ for each $r, s \in \mathcal{U}$. Thus, $\mathcal{S}$ is commutative.

5. Conclusions

By situating n-anti-homoderivations within the historical trajectory of derivations and their generalizations, this research provides a comprehensive framework for understanding how these mappings influence ring structure. The theorems and lemmas presented not only generalize classical results but also open new avenues for exploring the interplay between additive and multiplicative mappings in ring theory. The work underscores the enduring relevance of derivations and their generalizations in uncovering the algebraic properties of rings while introducing innovative tools for future investigations.

This work suggests several promising research avenues:

(i) Extension to skew n-anti-homoderivations, near-rings, and other algebraic structures.

(ii) Extension of the main theorems from associative prime rings to alternative prime rings of characteristic not 2.

This study focuses on associative rings, but many techniques and identities developed herein may have natural analogues in non-associative structures, particularly alternative and Jordan rings [27, 28].

Open problem Let S be an alternative prime ring with char $S \neq 2$ and $U \neq (0)$ a left ideal of S with no non-zero nilpotent elements. The ring S is commutative if it admits a non-zero n-anti-homoderivation H_n which is ZPV on U and satisfies any one of:

(1) $H_n(r)H_n(s) + H_n(rs) = H_n(s)H_n(r) + H_n(sr)$,

(2) $H_n(r)H_n(s) + H_n(sr) = H_n(s)H_n(r) + H_n(rs)$,

(3) $H_n(rs) + (n+1)H_n(r)H_n(s) + rs = H_n(sr) + (n+1)H_n(s)H_n(r) + sr$,

for all $r, s \in U$.

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