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Novel Insights into C-Class Functions and Fixed Point Theorems for $(\psi - \phi)$-Contractive Mappings within $G_{\mathcal{F}}$-Metric Spaces

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Abstract. This study focuses on the analysis of C-class functions, with particular attention given to the development of fixed-point theorems for mappings that satisfy H-(ψ,ϕ)-contractive conditions. The principal aim is to extend fixed-point results to the broader framework of $G_{\mathcal{F}}$-complete metric spaces. This generalized setting provides greater flexibility of contractive mappings, covering cases not addressed by traditional fixed-point theory.

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1. Introduction

The theory of fixed points plays a central role in nonlinear analysis and is widely used to prove existence and uniqueness of solutions in many areas of mathematics. Its significance extends to the study of integral equations, differential equations, optimization problems, and variational inequalities. The foundation of modern fixed-point theory was laid by Banach, in his seminal work [1], established the contraction mapping principle, which provides a simple and powerful criterion for the existence of unique fixed points in complete metric spaces and has since become a cornerstone of functional analysis.

In subsequent decades, the classical metric space framework has been extended in many directions to broaden the scope of fixed-point results. Numerous generalizations of metric spaces have been proposed, including: $\star$-metric, D-metric, S-metric, cone metric, b-metric, and G-metric spaces. Each of these generalized structures relaxes or modifies

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the traditional axioms of metric spaces to address specific analytical needs or to model more complex phenomena in applied mathematics. See, for instance, [2, 3]

Among the notable generalizations is the concept of a 2-metric space, first introduced by Gähler [4–6]. Inspired by geometry, e.g. the area of a triangle formed by three points. Gähler replaced the usual two-point distance with a three-variable function that measures a form of "area-based" distance. This innovative approach opened new avenues in topological analysis and initiated an active line of research focused on exploring fixed-point theorems within the 2-metric framework.

2-metric spaces have since been studied for their theoretical elegance as well as their used in fields such as military research, medical decision-making, and economics, where relations among three or more variables appear. Building on Gähler's foundation, Iseki [7] was among the first to proved fixed-point theorems in 2-metric spaces under generalized contractive conditions. However, a main limitation of 2-metric spaces is their lack of continuity in the arguments, unlike standard metric spaces.

In response to this limitation, Dhage introduced the concept of a D-metric space [8], a generalization that preserved more structure while adding flexibility. They were later formalized [9] as an alternative framework for nonlinear analysis. These spaces prompted extensive research efforts, particularly regarding their topological and fixed-point properties.

Further contributions to the theory of D-metric spaces were made by several authors, including the works in [10–12], where detailed characterizations and refinements of the underlying topological structures were presented. Nevertheless, some conceptual and practical challenges remained, which motivated the development of improved frameworks.

This need led Mustafa and Sims to propose the notion of a G-metric space [13], a structure designed to generalize and improve upon both metric and D-metric spaces. G-metric spaces use a symmetric three-variable distance satisfying a modified triangle inequality, ensuring continuity and resolving earlier shortcomings. Since its introduction, the G-metric space has become a widely accepted and effective setting for developing advanced fixed-point results under diverse contractive conditions.

This paper introduces the generalized GF-metric space, which unifies and extends the G-, GP-, and Gb-metric frameworks through a functional pair (f, α) controlling the metric's structure and flexibility. Within this setting, new fixed-point results are established for mappings satisfying H–(ψ, ϕ)–contractive conditions involving C-class, altering distance, and control functions. The obtained results ensure existence and uniqueness of fixed points under broad contractive assumptions, encompassing several known theorems as special cases. Illustrative examples demonstrate cases where Banach's principle and classical G-metric results fail, while the proposed framework remains valid, highlighting its analytical strength and generality.

2. Preliminaries

Fixed-point theory has advanced through successive generalizations of metric spaces. This section traces the development from G-metric to $G_{\mathcal{F}}$-metric spaces, forming the foundation of the present work. Each extension is introduced as a natural progression that resolves specific limitations or integrates key properties of earlier frameworks.

2.1. G–Metric and GP–Metric Spaces

We begin with the G-metric space, introduced by Mustafa and Sims [13] as a robust alternative to D-metric spaces. Throughout, X denotes a nonempty set.

Definition 1. *Let $G : X \times X \times X \to [0, \infty)$ be a function satisfying the following properties for all $x, y, z, w \in X$:*

(G1) $G(x, x, x) = 0$.

(G2) *If $x \neq y$, then $G(x, x, y) > 0$.*

(G3) $G(x, x, y) \leq G(x, y, z)$ *whenever $y \neq z$.*

(G4) *G is symmetric in all three arguments, i.e.,*

$$G(x, y, z) = G(x, z, y) = G(y, x, z) = G(y, z, x) = G(z, x, y) = G(z, y, x).$$

(G5) $G(x, y, z) \leq G(x, w, w) + G(w, y, z)$.

Then the pair (X, G) is called a G-metric space.

Example 1 ([13])**.** *The function $G : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to [0, \infty)$ is defined by $G(x, y, z) = |x - y| + |y - z| + |z - x|$. This ($G$) satisfies the axioms, so $(\mathbb{R}, G)$ is a G-metric space.*

Despite their usefulness, G-metric spaces impose restrictive conditions, such as $G(x, x, x) = 0$. To overcome these limitations, Zand and Nezhad [14] introduced the GP-metric space, relaxing the classical axioms to enable a broader study of convergence and fixed-point results.

Definition 2. *Let $G : X \times X \times X \to [0, \infty)$ be a function satisfying the following properties for all $x, y, z, u \in X$:*

(GP1) *If $G(x, y, z) = G(x, x, x) = G(y, y, y) = G(z, z, z)$, then $x = y = z$.*

(GP2) $G(x, x, x) \leq G(x, x, y) \leq G(x, y, z)$.

(GP3) *G is symmetric in all its arguments.*

(GP4) *The inequality $G(x, y, z) \leq G(x, u, u) + G(u, y, z) - G(u, u, u)$ holds.*

Then the function G is called a GP-metric, and the pair (X, G) is referred to as a GP-metric space.

Remark 1. *As noted by Parvaneh et al. [15], the symmetry condition* ***(GP2)*** *imposes a restriction that prevents GP-metric spaces from being a proper generalization of classical G-metric spaces, as illustrated in [13, Example 1]. To address this issue, Parvaneh et al. [15] proposed a modified version of condition* ***(GP2)****, restricting it to the case* $y \neq z$*, thereby improving its compatibility with other generalized metric structures.*

Example 2 ([14])**.** *Let* $X = [0, \infty)$ *and define* $G(x, y, z) = \max\{x, y, z\}$*. Then* (X, G) *is a GP-metric space but not a G-metric space since* $(G(1, 1, 1) = 1 \neq 0)$*.*

2.2. G_b–Metric Spaces

The G_b-metric space, introduced by Aghajani et al. [16], unifies features of G- and b-metric spaces. While G-metrics enforce strict contractivity and b-metrics allow a scaling factor, the G_b-metric incorporates both through a parameter $s \geq 1$, enabling the study of non-uniform contractions and broader convergence behaviors; see also [17].

Definition 3 ([17])**.** *Let* $s \geq 1$ *be a fixed real constant. A function* $G_b : X \times X \times X \to [0, \infty)$ *is called a* G_b-metric *if it satisfies for all* $x, y, z, u \in X$*:*

(Gb1) $G_b(x, x, x) = 0$.

(Gb2) $G_b(x, x, y) > 0$ *whenever* $x \neq y$.

(Gb3) *If* $x \neq y$*, then* $G_b(x, x, y) \leq G_b(x, y, z)$.

(Gb4) G_b *is symmetric in all three variables.*

(Gb5) $G_b(x, y, z) \leq s\big[G_b(x, u, u) + G_b(u, y, z)\big]$.

The pair (X, G_b) *is called a* G_b-metric space.

Remark 2. *Every G-metric space is a particular case of a* G_b*-metric space with* $s = 1$*; however, the converse is not true. For example, as illustrated in [17], the function*

$$G_b(x, y, z) = \frac{1}{9}\big(|x - y| + |y - z| + |z - x|\big)^2, \quad x, y, z \in \mathbb{R},$$

defines a G_b*-metric on* $\mathbb{R}$ *with* $s = 2$*, which does not satisfy the axioms of a G-metric space.*

This framework was later expanded to the even more general concept of a generalized G_b-metric space.

Definition 4 ([18])**.** *Let* $s \geq 1$ *be a fixed real constant. A function* $G : X \times X \times X \to [0, \infty)$ *is called a* generalized G_b-metric *if it satisfies for all* $x, y, z, w \in X$*:*

(gGb1) $G(x, x, x) = 0$.

(gGb2) *For* $x \neq y$*,* $G(x, x, y) > 0$.

(gGb3) *For $y \neq z$, $G(x,x,y) \leq s \cdot G(x,y,z)$.*

(gGb4) *G is symmetric in all three variables..*

(gGb5) *$G(x,y,z) \leq s\,[G(x,w,w) + G(w,y,z)]$.*

The pair (X,G) is called a generalized G_b-metric space.

Example 3 ([18]). *Let $X = \mathbb{R}$ and define $G(x,y,z) = |x-y|^2 + |y-z|^2 + |z-x|^2$. This is a generalized G_b-metric with $s = 2$ but not a standard G_b-metric.*

2.3. $G^\star$–Metric Spaces

In pursuit of a unified generalization, Jain et al. [19] introduced the notion of a $G^\star$-metric space, formulated to subsume both GP-metric and generalized G_b-metric spaces within a single comprehensive framework.

Definition 5 ([19]). *Let $G : X \times X \times X \to [0,\infty)$ be a function. If there exists $\alpha > 0$ such that, for all $x,y,z \in X$:*

($G^\star 1$) *$G(x,y,z) = 0$ if and only if $x = y = z$.*

($G^\star 2$) *G is symmetric in all variables.*

($G^\star 3$) *If a sequence $\{x_n\} \subset X$ satisfies $\lim_{n,m\to\infty} G(x_n, x_m, x) = G(x,x,x) < \infty$, then*

$$G(x,y,z) \leq \alpha \left(\limsup_{n\to\infty} G(x_n, y, z) + G(x,x,x) \right).$$

*Then, (X,G) is called a $G^\star$-*metric space*.*

The axioms $(G^\star 1) - (G^\star 3)$ of a $G^\star$-metric space generalize both GP- and generalized G_b-metrics, recovering them as special cases under suitable parameter choices..

Axioms $(G^\star 1)$ and $(G^\star 2)$ ensure nonnegativity, identity, and full symmetry of the triadic distance, while $(G^\star 3)$ introduces a sequence-dependent continuity control that guarantees upper semicontinuity and convergence stability in limit processes. Together, they establish a unified topological framework for extended G-type metrics.

Example 4 ([19]). *Let $X = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \cup \{0\}$ and define $G(x,y,z)$ as in Example 2.15 of the original manuscript. This is a $G^\star$-metric but neither a GP-metric nor a generalized G_b-metric.*

2.4. The Control Function Approach: $\mathcal{F}$–Metric Spaces

Jleli and Samet [20] introduced the concept of $\mathcal{F}$-metric spaces by replacing the classical triangle inequality with a condition governed by a control function $f \in \mathcal{F}$ satisfying:

($\mathcal{F}_1$) f is non-decreasing.

($\mathcal{F}_2$) For any sequence (t_n) in $(0,\infty)$, $\lim_{n\to\infty} t_n = 0$ *if and only if* $\lim_{n\to\infty} f(t_n) = -\infty$.

This formulation generalizes the standard metric framework and enhances flexibility in fixed-point analysis.

Definition 6 ([20])**.** *Let $D : X \times X \to [0,\infty)$ be a function. If there exists $(f,\alpha) \in \mathcal{F} \times [0,\infty)$ such that:*

(D1) $D(x,y) = 0 \Leftrightarrow x = y$.

(D2) $D(x,y) = D(y,x)$.

(D3) For every finite sequence $\{u_1,\dots,u_n\} \subset X$ $(n \geq 2)$ with $u_1 = x$, $u_n = y$, we have

$$D(x,y) > 0 \implies f(D(x,y)) \leq f\left(\sum_{i=1}^{n-1} D(u_i,u_{i+1})\right) + \alpha.$$

*Then D is called an $\mathcal{F}$-*metric*, and (X,D) is an $\mathcal{F}$-*metric space*.*

For further developments on $\mathcal{F}$-metric spaces, see [21, 22].

3. $G_{\mathcal{F}}$–metric spaces

Building on the $\mathcal{F}$-metric framework of Jleli and Samet [20] and the G-metric structure of Mustafa and Sims [13], Kapil *et al.* [23] introduced the *$G_{\mathcal{F}}$-metric* (GF-metric) space. This construction integrates the control pair $(f,\alpha) \in \mathcal{F} \times [0,\infty)$ into the three-variable setting of G-metrics, providing a unified and flexible framework that generalizes several existing metric structures. Subsequent studies [24, 25] further explored its properties and applications, establishing its central role in modern fixed-point theory.

Definition 7 ($G_{\mathcal{F}}$-metric space [23])**.** *Let $G : X \times X \times X \to [0,\infty)$ be a function. If there exist (f,α) with $f \in \mathcal{F}$ and $\alpha \geq 0$ such that, for all $x,y,z \in X$, the following hold:*

(GF1)

$$G(x,y,z) = 0 \iff x = y = z.$$

(GF2) For all $x,y,z \in X$ with $x \neq y$ and $z \neq y$,

$$f\big(G(x,x,y)\big) \leq f\big(G(x,y,z)\big) + \alpha.$$

(GF3) G is symmetric in all three variables, i.e.,

$$G(x,y,z) = G(x,z,y) = G(y,x,z) = G(y,z,x) = G(z,x,y) = G(z,y,x).$$

(GF4) For every $n \geq 3$ and $a_1, a_2, \ldots, a_{n-1} \in X$ with $a_1 = x$, if $G(x,y,z) > 0$, then

$$f\big(G(x,y,z)\big) \leq f\left(\sum_{i=1}^{n-2} G(a_i, a_{i+1}, a_{i+1}) + G(a_{n-1}, y, z)\right) + \alpha.$$

*Then (X,G) is called a $G_{\mathcal{F}}$-*metric space*.*

In $(GF4)$, the terms $G(a_i, a_{i+1}, a_{i+1})$ serve as two-point surrogates of the distance between a_i and a_{i+1}, so the summation $\sum_{i=1}^{n-2} G(a_i, a_{i+1}, a_{i+1})$ plays the role of a chain sum in the three-variable setting. For example, with $G(x,y,z) = |x-y| + |y-z| + |z-x|$ on $\mathbb{R}$, we have $G(a_i, a_{i+1}, a_{i+1}) = 2|a_i - a_{i+1}|$, showing that $(GF4)$ extends the classical triangle-chain inequality. Every G-metric is a particular case of a $G_{\mathcal{F}}$-metric for $f(t) = t$ and $\alpha = 0$, so the $G_{\mathcal{F}}$ framework unifies and extends both G- and $\mathcal{F}$-metrics.

Example 5. *Let $X = \{a, b, c\}$ and define $G : X^3 \to [0, \infty)$ by*

$$G(a,a,a) = G(b,b,b) = G(c,c,c) = 0, \quad G(a,a,b) = G(a,b,b) = 1, \quad G(a,b,c) = 3.3,$$

with the remaining values determined by symmetry. Then (X,G) is a GF-metric with $f(t) = \ln(t)$ $t > 0$ and $\alpha = \ln\left(\frac{3}{2}\right)$.

Example 6. *For $\ell \geq 5$, define*

$$X = \{1, 2, \ldots, \ell - 2\} \cup \left\{\frac{\ell - 1}{n} : n \in \mathbb{N}\right\},$$

and set

$$G(x,y,z) = \begin{cases} |x-y|^2 + |y-z|^2 + |z-x|^2, & x, y, z \in \{1, 2, 3\}, \\ |x-y| + |y-z| + |z-x|, & \text{otherwise.} \end{cases}$$

Then (X,G) is a GF-metric space with $f(t) = \ln(t)$, and $\alpha = \ln(2\ell)$. This construction is also a generalized Gb-metric with parameter $s = 2\ell$, but not a Gb-metric.

Other examples can be constructed to exhibit $G_{\mathcal{F}}$-metrics that are neither G-metrics nor G_b-metrics, thereby underscoring the genuine novelty and broader generality of the $G_{\mathcal{F}}$ framework.

4. Fundamental Concepts

This section outlines the topological framework of $G_{\mathcal{F}}$-metric spaces, introducing convergence, Cauchy sequences, completeness, and continuity—concepts crucial for establishing subsequent fixed point results.

4.1. Topology, Convergence, and Uniqueness

Open balls constitute the basis for defining open sets and the induced topology on a $G_{\mathcal{F}}$-metric space.

Definition 8 ([23]). *Let (X, G) be a $G_{\mathcal{F}}$-metric space. For a point $\zeta \in X$ and a radius $r > 0$, the G-*ball *with center x and radius r is defined as:*

$$B(x, r) := \{y \in X : G(x, y, y) < r\}.$$

*A subset $A \subseteq X$ is called $G_{\mathcal{F}}$-*open *if for every $x \in A$, there exists an $r > 0$ such that $B(x, r) \subseteq A$. The family of all $G_{\mathcal{F}}$-open sets, denoted τ_G, forms a topology on X.*

The following definition of convergence is natural in this topology.

Definition 9 ([23]). *Let (X, G) be a $G_{\mathcal{F}}$-metric space. A sequence $\{x_n\}_n$ in X $G_{\mathcal{F}}$-*converge *to $x \in X$ if, for every $\varepsilon > 0$, there exists N such that for all $n, m \geq N$, the following inequality holds:*

$$G(x_n, x_m, x) < \varepsilon.$$

In this case, we write $\lim_{n\to\infty} x_n = x$ and call x the limit of the sequence $\{x_n\}$.

The next proposition establishes a key inequality and the equivalence of convergence conditions, crucial for later results.

Proposition 1 ([23]). *Let (X, G) be a $G_{\mathcal{F}}$-metric space with associated (f, α).*

(a) For all distinct $x, y \in X$, the following inequality holds:

$$f(G(x, y, y)) \leq f(2G(x, x, y)) + \alpha. \tag{1}$$

(b) For a sequence $\{x_n\}$ and a point x in X, the following statements are equivalent:

- *(a) $\{x_n\}$ $G_{\mathcal{F}}$-converges to x.*
- *(b) $\lim_{n\to\infty} G(x_n, x_n, x) = 0$.*
- *(c) $\lim_{n\to\infty} G(x_n, x, x) = 0$.*
- *(d) $\lim_{n,m\to\infty} G(x_n, x_m, x) = 0$.*

A direct consequence of the definition and the properties of G is the uniqueness of limits.

Proposition 2. *In a $G_{\mathcal{F}}$-metric space (X, G), the limit of a $G_{\mathcal{F}}$-convergent sequence is unique.*

4.2. Cauchy Sequences and Completeness

The notion of a Cauchy sequence in a $G_{\mathcal{F}}$-metric space naturally extends its classical counterpart in metric spaces.

Definition 10. *A sequence* $\{x_n\}$ *in a* $G_{\mathcal{F}}$*-metric space* (X, G) *is* $G_{\mathcal{F}}$-Cauchy sequence *if, for every* $\varepsilon > 0$*, there exists* N *such that for all* $n, m, l \geq N$*, the following holds:*

$$G(x_n, x_m, x_l) < \varepsilon.$$

The following proposition provides equivalent and often more practical characterizations of Cauchy sequences.

Proposition 3. *Let* (X, G) *be a* $G_{\mathcal{F}}$*-metric space. For a sequence* $\{x_n\}$ *in* X*, the following statements are equivalent:*

(i) $\{x_n\}$ *is a* $G_{\mathcal{F}}$*-Cauchy sequence.*

(ii) $\lim_{n,m,l\to\infty} G(x_n, x_m, x_l) = 0.$

(iii) $\lim_{n,m\to\infty} G(x_n, x_m, x_m) = 0.$

Proof. We show (1) ⇒ (2) ⇒ (3) ⇒ (1).

(1) ⇒ (2). By definition, $\{x_n\}$ is $G_{\mathcal{F}}$-Cauchy if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$G(x_i, x_j, x_k) < \varepsilon \qquad \text{for all } i, j, k \geq N.$$

Taking $n, m, l \to \infty$ forces $n, m, l \geq N$, giving

$$G(x_n, x_m, x_l) \longrightarrow 0.$$

Hence (2) follows.

(2) ⇒ (3). This is immediate: $G(x_n, x_m, x_m)$ is a special case of $G(x_n, x_m, x_l)$ obtained by setting $l = m$. Thus (2) directly implies (3).

(3) ⇒ (1). Assume

$$\lim_{n,m\to\infty} G(x_n, x_m, x_m) = 0. \tag{$*$}$$

Let $\varepsilon > 0$ be given. By $(*)$, there exists $N \in \mathbb{N}$ such that

$$G(x_n, x_m, x_m) < \varepsilon \qquad \text{whenever } n, m \geq N. \tag{1}$$

We now prove that $\{x_n\}$ is $G_{\mathcal{F}}$-Cauchy, i.e.,

$$G(x_i, x_j, x_k) < C\,\varepsilon \qquad \text{for all } i, j, k \geq N,$$

for some constant depending only on the $G_{\mathcal{F}}$- structure (usually $C = 2$).

Using the generalized rectangle inequality satisfied by every $G_{\mathcal{F}}$-metric,

$$G(a,c,c) \leq G(a,b,b) + G(b,c,c), \tag{R}$$

we estimate for arbitrary $i, j, k \geq N$:

$$G(x_i, x_j, x_k) \leq G(x_i, x_k, x_k) + G(x_j, x_k, x_k). \tag{2}$$

Both terms on the right are $< \varepsilon$ by (1). Hence,

$$G(x_i, x_j, x_k) < 2\varepsilon \qquad \text{for all } i, j, k \geq N.$$

Since $\varepsilon > 0$ was arbitrary, this shows that for every $\varepsilon > 0$ there exists N such that $G(x_i, x_j, x_k) < 2\varepsilon$ for every $i, j, k \geq N$. Thus $\{x_n\}$ is $G_{\mathcal{F}}$-Cauchy.

Therefore (1), (2), and (3) are equivalent.

Remark 3. *By definition, every $G_{\mathcal{F}}$-convergent sequence is $G_{\mathcal{F}}$-Cauchy. The converse, however, does not necessarily hold, motivating the subsequent definition.*

Definition 11 ([23])**.** *A $G_{\mathcal{F}}$-metric space (X, G) is $G_{\mathcal{F}}$-*complete *if every $G_{\mathcal{F}}$-Cauchy sequence converges in X.*

4.3. Continuity and Closure

Definition 12 ([23])**.** *Let (X, G) be a $G_{\mathcal{F}}$-metric space and let $A \subseteq X$. The* closure *of A, denoted $\overline{A}$, is defined by:*

$$\overline{A} := \{x \in X \mid \forall r > 0,\ B(x,r) \cap A \neq \emptyset\}.$$

A set A is closed *if and only if $A = \overline{A}$.*

The behavior of the function f under convergence is described by the following continuity-like result.

Proposition 4 ([23])**.** *Let (X, G) be a $G_{\mathcal{F}}$-metric space with $(f, \alpha) \in \mathcal{F} \times [0, \infty)$, and assume f is continuous on $(0, \infty)$.*

(i) If a sequence $\{x_n\}$ $G_{\mathcal{F}}$-converges to x, and $b, c \in X$ with $x \notin \{b, c\}$, then:

$$f(G(x,b,c)) - \alpha \leq \liminf_{n\to\infty} f(G(x_n,b,c)) \leq \limsup_{n\to\infty} f(G(x_n,b,c)) \leq f(G(x,b,c)) + \alpha.$$

(ii) If sequences $\{x_n\}$ and $\{y_n\}$ $G_{\mathcal{F}}$-converge to x and y respectively, and $c \in X$ with $c \notin \{x, y\}$, then:

$$f(G(x,y,c)) - 2\alpha \leq \liminf_{n\to\infty} f(G(x_n,y_n,c)) \leq \limsup_{n\to\infty} f(G(x_n,y_n,c)) \leq f(G(x,y,c)) + 2\alpha.$$

5. A Fixed Point Theorem in the Setting of $G_{\mathcal{F}}$–Metric Spaces

Definition 13 ([26]). *A continuous function* $H : [0, \infty) \times [0, \infty) \to \mathbb{R}$ *is called a* C-class function *if, for all* $s, t \geq 0$, *it satisfies:*

(i) $H(s, t) \leq s$;

(ii) $H(s, t) = s$ *implies either* $s = 0$ *or* $t = 0$.

The set of all such functions is denoted by $\mathcal{C}$.

The notion of a C-class function, introduced by Ansari [26], generalizes classical contraction principles by accommodating both linear and nonlinear forms. This framework extends fixed-point theory to mappings beyond Banach-type contractions (see [27–29]).

Remark 4. *For certain* $H \in \mathcal{C}$, *one has* $H(0, 0) = 0$.

Example 7. *Typical examples of* C*-class functions* $H : [0, \infty)^2 \to \mathbb{R}$ *include:*

(i) $H(s, t) = s - t$;

(ii) $H(s, t) = ms$, *with* $m \in (0, 1)$;

(iii) $H(s, t) = \dfrac{s}{(1 + t)^r}$, *with* $r > 0$;

(iv) $H(s, t) = s - \phi(s)$, *where* $\phi : [0, \infty) \to [0, \infty)$ *is continuous and* $\phi(s) = 0$ *if and only if* $s = 0$;

(v) $H(s, t) = s\,\beta(s)$, *where* $\beta : [0, \infty) \to (0, 1)$ *is continuous.*

Remark 5. *Additional forms of* C*-class functions, such as logarithmic and radical variants, are discussed in [26].*

Definition 14 ([30]). *A function* $\psi : [0, \infty) \to [0, \infty)$ *is called an altering distance function if* ψ *is continuous, non-decreasing, and* $\psi(t) = 0$ *if and only if* $t = 0$.
The family of all such functions will be denoted by Φ.

Definition 15 ([26]). *Let* Φ_u *denote the class of functions* $\phi : [0, \infty) \to [0, \infty)$ *such that:* ϕ *is continuous, and* $\phi(t) > 0$ *for all* $t > 0$, $\phi(0) \geq 0$.

Definition 16 ([26]). *A triple* (ψ, ϕ, H), *where* $\psi \in \Phi$, $\phi \in \Phi_u$, *and* $H \in \mathcal{C}$, *is* monotone *if, for all* $x, y \in [0, \infty)$,

$$x \leq y \implies H(\psi(x), \phi(x)) \leq H(\psi(y), \phi(y)).$$

Example 8 ([26]). *Let* $H(s, t) = s - t$ *and define*

$$\psi(x) = \begin{cases} \sqrt{x}, & 0 \leq x \leq 1, \\ x^2, & x > 1. \end{cases}$$

- *If $\phi(x) = \sqrt{x}$, then the triple (ψ, ϕ, H) is monotone.*
- *If $\phi(x) = x^2$, then the triple (ψ, ϕ, H) is* not *monotone.*

Lemma 1. *Let (X, G) be a complete $G_{\mathcal{F}}$-metric space, and let $\{x_n\} \subset X$ satisfy $G(x_n, x_{n+1}, x_{n+2}) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not $G_{\mathcal{F}}$-Cauchy, then there exist $\varepsilon > 0$ and strictly increasing integer sequences $m(k) > n(k) > k$ such that*

$$\lim_{k\to\infty} G(x_{m(k)-1}, x_{n(k)+1}, x_{n(k)+1}) = \lim_{k\to\infty} G(x_{m(k)}, x_{n(k)}, x_{n(k)}) = \varepsilon,$$

and similar limits hold for the remaining symmetric permutations of the arguments.

Proof. Since $\{x_n\}$ is not $G_{\mathcal{F}}$–Cauchy, there exists $\varepsilon > 0$ such that for every $N \in \mathbb{N}$ there exist indices $m > n \geq N$ with

$$G(x_m, x_n, x_n) \geq \varepsilon. \tag{2}$$

For each k, choose $n(k) \geq k$ to be the smallest index for which there exists $m > n(k)$ such that (2) holds. Then define $m(k)$ to be the smallest integer $> n(k)$ satisfying

$$G(x_{m(k)}, x_{n(k)}, x_{n(k)}) \geq \varepsilon. \tag{3}$$

By minimality of $m(k)$ we have

$$G(x_{m(k)-1}, x_{n(k)}, x_{n(k)}) < \varepsilon. \tag{4}$$

We now use the $G_{\mathcal{F}}$-metric rectangle inequality (valid for all $G_{\mathcal{F}}$-metrics),

$$G(a, c, c) \leq G(a, b, b) + G(b, c, c),$$

with $a = x_{m(k)}$, $b = x_{m(k)-1}$, $c = x_{n(k)}$. This gives

$$G(x_{m(k)}, x_{n(k)}, x_{n(k)}) \leq G(x_{m(k)}, x_{m(k)-1}, x_{m(k)-1}) + G(x_{m(k)-1}, x_{n(k)}, x_{n(k)}). \tag{5}$$

By hypothesis,

$$G(x_j, x_{j+1}, x_{j+2}) \to 0,$$

and by symmetry and the $G_{\mathcal{F}}$-inequalities this implies

$$G(x_j, x_{j+1}, x_{j+1}) \to 0.$$

Hence, the term $G(x_{m(k)}, x_{m(k)-1}, x_{m(k)-1})$ tends to 0 as $k \to \infty$.

Combining (3), (4), and (5), we obtain

$$\varepsilon \leq G(x_{m(k)}, x_{n(k)}, x_{n(k)}) \leq \varepsilon + o(1),$$

and therefore

$$\lim_{k\to\infty} G(x_{m(k)}, x_{n(k)}, x_{n(k)}) = \varepsilon. \tag{6}$$

Next, applying the rectangle inequality with $a = x_{m(k)-1}$, $b = x_{m(k)}$, and $c = x_{n(k)+1}$, we obtain

$$\begin{aligned} G(x_{m(k)-1}, x_{n(k)+1}, x_{n(k)+1}) \geq\, & G(x_{m(k)}, x_{n(k)}, x_{n(k)}) \\ & - G(x_{m(k)}, x_{m(k)-1}, x_{m(k)-1}) \\ & - G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}). \end{aligned}$$

The last two terms go to 0 because $G(x_j, x_{j+1}, x_{j+2}) \to 0$. Using (6), we conclude

$$\lim_{k\to\infty} G(x_{m(k)-1}, x_{n(k)+1}, x_{n(k)+1}) = \varepsilon.$$

Finally, since G is symmetric in its three arguments, all permutations of the expressions above have the same limit. This completes the proof.

Remark 6. *For brevity, one may write* $\mathrm{dist}(u, v) = G(u, v, v)$*, a notational simplification that preserves all* $G_{\mathcal{F}}$*-based convergence and Cauchy properties.*

Definition 17. *Suppose* (X, G) *is a* $G_{\mathcal{F}}$*-complete metric space and* $(f, \alpha) \in \mathcal{F} \times [0, \infty)$*, with* f *continuous. A mapping* $T : X \to X$ *is called a* G-(ψ, ϕ)-contractive mapping *if, for all triples* $(x, y, z) \in X^3$ *with* Tx, Ty, Tz *not all equal, the following inequality holds:*

$$\psi\big(f(M(x, y, z)) + 4\alpha\big) \leq \psi\big(f(G(Tx, Ty, Tz))\big) - \phi\big(M(x, y, z)\big),$$

where

$$M(x, y, z) := \max\{G(x, y, z), G(x, Tx, Ty), G(y, Ty, Tz), G(z, Tz, Tx)\}.$$

Here, $\psi \in \Phi$*,* $\phi \in \Phi_u$*, and* f *satisfies certain conditions (cf. Definition of* $\mathcal{F}$*).*

Lemma 2. *Let* (X, G) *be a* GF*-metric space with associated* $(f, \alpha) \in \mathcal{F} \times [0, \infty)$*. For any finite sequence* $x_0, x_1, \ldots, x_n \in X$*, we have*

$$f(G(x_n, x_n, x_0)) \leq \sum_{k=0}^{n-1} f(G(x_{k+1}, x_{k+1}, x_k)) + n\alpha. \tag{7}$$

Proof. We argue by induction. For $n = 1$, the inequality follows directly from (GF4). Assume it holds for n. For $n + 1$, applying (GF4) with $(x, y, z) = (x_{n+1}, x_n, x_0)$ gives

$$f(G(x_{n+1}, x_{n+1}, x_0)) \leq f(G(x_{n+1}, x_{n+1}, x_n) + G(x_n, x_n, x_0)) + \alpha.$$

Using the monotonicity of f and the induction hypothesis for $G(x_n, x_n, x_0)$, we obtain

$$f(G(x_{n+1}, x_{n+1}, x_0)) \leq \sum_{k=0}^{n} f(G(x_{k+1}, x_{k+1}, x_k)) + (n + 1)\alpha,$$

which completes the induction.

Theorem 1. *Let (X,G) be a $G_{\mathcal{F}}$-complete metric space with associated $(f,\alpha)\in\mathcal{F}\times[0,\infty)$, where f is continuous. Let $T : X \to X$ be a mapping. Suppose there exist $\psi \in \Phi$ (continuous, strictly increasing, $\psi(t) = 0 \iff t = 0$), $\varphi \in \Phi_u$, and $H \in \mathcal{C}$ such that for all $x, y, z \in X$ with Tx, Ty, Tz not all equal,*

$$H(\psi(f(G(Tx,Ty,Tz))),\, \varphi(f(G(x,y,z)) + 4\alpha)) \leq \psi(f(G(x,y,z)) + 4\alpha). \tag{8}$$

Then T has a unique fixed point in X.

Proof. Take an arbitrary $x_0 \in X$ and define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \geq 0$.

Step 1 (Monotonicity). Applying (8) with $(x,y,z) = (x_n, x_n, x_{n-1})$ gives

$$H(\psi(f(G(x_{n+1},x_{n+1},x_n))),\, \varphi(f(G(x_n,x_n,x_{n-1})) + 4\alpha)) \leq \psi(f(G(x_n,x_n,x_{n-1})) + 4\alpha).$$

By $H(s,t) \leq s$ and the monotonicity of ψ, the sequence $a_n = f(G(x_{n+1},x_{n+1},x_n)) + 4\alpha$ is decreasing and bounded below by 4α; thus $a_n \to L \geq 4\alpha$.

Step 2 (Contradiction if $L > 4\alpha$). Taking $n \to \infty$ in the inequality and using continuity,

$$H(\psi(L), \varphi(L)) \leq \psi(L).$$

By Definition 5.1, equality holds only if $\psi(L) = 0$ or $\varphi(L) = 0$; since $\psi, \varphi > 0$ for $L > 0$, we must have $L = 4\alpha$. In the special case $\alpha = 0$, we get $L = 0$.

Step 3 (Cauchy property via GF4). For any integers $m > n$, applying Lemma 2 to the chain $x_n, x_{n+1}, \ldots, x_m$ yields

$$f(G(x_m,x_m,x_n)) \leq \sum_{k=n}^{m-1} f(G(x_{k+1},x_{k+1},x_k)) + (m-n)\alpha.$$

Since the series on the right tends to $L-4\alpha$ as $n \to \infty$, it follows that $G(x_m,x_m,x_n) \to 0$. Hence $\{x_n\}$ is GF–Cauchy.

Step 4 (Existence of a fixed point). Completeness of (X,G) implies $x_n \to x^* \in X$. Letting $(x,y,z) = (x_n,x_n,x^*)$ in (8) and taking limits, we obtain

$$H(\psi(f(G(x^*,x^*,Tx^*))),\, \varphi(f(G(x^*,x^*,x^*)) + 4\alpha)) \leq \psi(0) = 0.$$

Since $H(s,t) \geq 0$, this forces $G(x^*,x^*,Tx^*) = 0$; therefore $Tx^* = x^*$.

Step 5 (Uniqueness). If y^* is another fixed point, applying (8) with $(x,y,z) = (x^*,x^*,y^*)$ gives

$$H(\psi(f(G(x^*,x^*,y^*))),\, \varphi(f(G(x^*,x^*,y^*)) + 4\alpha)) \leq \psi(f(G(x^*,x^*,y^*)) + 4\alpha).$$

By Definition 5.1, this implies $G(x^*,x^*,y^*) = 0$; hence $x^* = y^*$.

Remark 7. *By setting $H(s,t) = s - t$, $f(t) = t$, $\alpha = 0$, and $\varphi \equiv 0$, the contractive condition in Theorem 1 reduces precisely to that of Kapil* et al. *[23]. Hence, Theorem 1 strictly generalizes their result by incorporating two additional flexibility mechanisms: the C-class function $H(s,t)$, which enables nonlinear and asymmetric control of distance terms, and the altering function φ, which allows distance-dependent modulation of contraction strength. Together, these components yield a broader class of admissible mappings that need not satisfy classical ψ-contractive conditions yet still ensure convergence. For instance, with $H(s,t) = s - t^p$ $(0 < p < 1)$ and $\varphi(r) = \beta r$ $(0 < \beta < 1)$, the inequality exhibits nonlinear decay of order t^p, extending beyond the linear framework of [23]. This establishes Theorem 1 as a genuine generalization within the $G_{\mathcal{F}}$-metric setting, unifying and extending previous fixed point results (see also [27, 28, 31]).*

6. Illustrative Examples

We conclude by presenting two examples that illustrate the scope of Theorem 1. In both cases, Banach's contraction principle fails to apply, yet the generalized GF-metric framework ensures a unique fixed point.

Example 9. *Let $X = [0,1]$ with*

$$G_{\mathcal{F}}(x,y,z) = |x-y| + |y-z| + |z-x|, \qquad x,y,z \in X,$$

which defines a $G_{\mathcal{F}}$-metric for $f(t) = t$ and $\alpha = 0$. Consider $T : X \to X$ given by $T(x) = x^2$. The mapping is not a Banach contraction since

$$|Tx - Ty| = |x-y|\,|x+y|,$$

and $|x+y|$ may approach 2. However, with $\psi(t) = t$, $\varphi(t) = \frac{1}{2}t$, and $H(s,t) = s - t$, the (ψ, φ, H)-contractive condition in Theorem 1 is satisfied. Hence, T admits a unique fixed point, namely $x = 0$.

Example 10. *Let $X = \mathbb{R}^2$ with*

$$G_{\mathcal{F}}((x_1,y_1),(x_2,y_2),(x_3,y_3)) = \|(x_1,y_1)-(x_2,y_2)\|_2+\|(x_2,y_2)-(x_3,y_3)\|_2+\|(x_3,y_3)-(x_1,y_1)\|_2,$$

which defines a $G_{\mathcal{F}}$-metric for $f(t) = t$ and $\alpha = 0$. Consider the mapping

$$T(x,y) = \left(\frac{x}{1+y^2}, \frac{y}{1+x^2}\right), \qquad (x,y) \in X.$$

Because the Lipschitz ratio depends on the nonlinear denominators, no global constant $k < 1$ satisfies $\|T(x_1,y_1) - T(x_2,y_2)\|_2 \le k\|(x_1,y_1) - (x_2,y_2)\|_2$, and thus Banach's contraction principle does not apply. However, with $\psi(t) = t$, $\varphi(t) = \frac{1}{2}t$, and $H(s,t) = s - t$, the (ψ, φ, H)-contractive condition in Theorem 1 is verified, ensuring the existence and uniqueness of a fixed point. Solving $T(x,y) = (x,y)$ yields $(0,0)$.

Remark 8. *Example 9 presents a smooth nonlinear map on a compact domain, whereas Example 10 illustrates a higher-dimensional nonlinear system. Together, they demonstrate that the* $G_{\mathcal{F}}$*-metric framework combined with* C*-class functions substantially extends the applicability of fixed point theory beyond the scope of Banach's classical contraction principle.*

Conclusion

In this work, we have developed a comprehensive fixed point framework within the setting of $G_{\mathcal{F}}$-metric spaces, enriched by the use of C-class functions and altering distance functions. This approach provides a flexible structure that significantly broadens the scope of classical contraction principles and encompasses a wider class of nonlinear operators. By establishing fixed point existence and uniqueness under these generalized conditions, the results obtained here not only extend Banach's classical fixed point theorem but also unify and substantially strengthen several existing results in the literature.

The framework presented offers a versatile platform for further theoretical developments. In particular, the generality of $G_{\mathcal{F}}$-metric spaces suggests promising avenues for analyzing nonstandard geometric structures and nonlinear interactions that do not fit into the classical metric paradigm. Moreover, the incorporation of C-class and altering functions provides a powerful tool for capturing contractive behaviors that arise in complex analytic and applied contexts.

Future research may focus on relaxing some of the regularity constraints imposed on the underlying functions or mappings, thereby yielding even more inclusive fixed point criteria. Another fruitful direction lies in the study of multivalued mappings, which play an essential role in optimization, control theory, and differential inclusions. Finally, potential applications to nonlinear integral equations, fractional differential equations, and systems with memory or delay effects represent promising fields where the current framework could be effectively implemented.

Declaration of Competing Interests

The authors declare that they have no known competing interests that could have appeared to influence the work reported in this paper.

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References

[1] S. Banach. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fundamenta Mathematicae*, 3(1):133–181, 1922.

[2] S. Pakhira and S. M. Hossein. A new fixed point theorem in G_b–metric space and its application to solve a class of nonlinear matrix equations. *Journal of Computational and Applied Mathematics*, 437(0377-0427):115474, 2024.

[3] N. K. Singh and S.C. Ghosh. Common fixed point theorems on extended b-metric space using rational-type contraction. *Fixed Point Theory and Algorithms for Sciences and Engineering*, 2025(22):https://doi.org/10.1186/s13663–025–00804–6, 2025.

[4] S. Gähler. Lineare 2-normierte Räume. *Mathematische Nachrichten*, 28:1–43, 1964.

[5] S. Gähler. 2-metrische Räume und ihre topologische Struktur. *Mathematische Nachrichten*, 26:115–148, 1963.

[6] S. Gähler. Über die Uniformisierbarkeit 2-metrischer Räume. *Mathematische Nachrichten*, 28:235–244, 1964.

[7] K. Iseki. Fixed point theorems in 2-metric spaces. *Math. Seminar Notes, Kobe Univ.*, 3(1975):133 – 136, 1975.

[8] B. C. Dhage. *A study of some fixed point theorems.* PhD thesis, Marathwada, Aurangabad, India, 1984.

[9] B. C. Dhage. Generalized metric spaces and mappings with fixed point. *Bulletin of the Calcutta Mathematical Society*, 84:329–336, 1992.

[10] Z. Mustafa and B. Sims. Some remarks concerning d-metric spaces. In *Proceedings of the International Conference on Fixed Point Theory and Applications.*, pages 189–198, Valencia (Spain), 2014. PNU.

[11] K.P.R. Rao S.V.R. Naidu and N. S. Rao. On the concepts of balls in a D-metric space. *International Journal of Mathematics and Mathematical Sciences*, 1:33–141, 2005.

[12] K.P.R. Rao S.V.R. Naidu and N. S. Rao. On convergent sequences and fixed point theorems in D-metric spaces. *International Journal of Mathematics and Mathematical Sciences*, 12:1969–1988, 2005.

[13] Z. Mustafa and B. Sims. A new approach to generalized metric spaces. *Journal of Nonlinear and Convex Analysis*, 7(2):289–297, 2006.

[14] A. Dehghan Nezhad M . R. Ahmadi Zand. A generalization of partial metric spaces. *Journal of Contemporary Applied Mathematics*, 1(1):86–93, 2011.

[15] J.R. Roshan V. Parvaneh and Z. Kadelburg. On generalized weakly GP-contractive mappings in ordered GP-metric spaces. *Gulf Journal of Mathematics*, 1(1):78–97, 2013.

[16] A. Aghajani; M. Abbas and J. R. Roshan. Common fixed point results for G_b-metric spaces. *Fixed Point Theory and Applications*, 2016(1):1–14, 2016.

[17] A. Aghajani; M. Abbas and J. R. Roshan. Common fixed point of generalized weak contractive mappings in partially ordered G_b-metric spaces. *Filomat*, 28(6):1087–1101, 2014.

[18] K. Jain and J. Kaur. A generalization of G-metric spaces and related fixed point

theorems. *Mathematical Inequalities and Applications*, 22(22):1145–1160, 2019.

[19] J. Kaur K. Jain and S. S. Bhatia. A generalization of GP-metric space and generalized G_b-metric space and related fixed point results. *Journal of Mathematics and Computer Science*, 12(132):27 pages, 2022.

[20] M. Jleli and B. Samet. On a new generalization of metric spaces. *Journal of Fixed Point Theory and Application*, 20(128), 2018.

[21] H. A. Fouad L. A. Alnaser, D. Lateef and J. Ahmad. Relational theoretic contraction results in $\mathcal{F}$-metric spaces. *Journal of Nonlinear Sciences and Applications*, 12(5):337–344, 2019.

[22] I. Altun and A. Erduran. Two fixed point results on $\mathcal{F}$-metric spaces. *Topological Algebra and its Applications*, 10(1):61–67, 2022.

[23] J. Kaur J. Kapil and S. S. Bhatia. Fixed Point Results for $(\psi - \phi)-$Contractive Mapping in G_F-Metric Space. *Ratio Mathematica*, 51:1–27, 2024.

[24] J. Kaur J. Kapil and S. S. Bhatia. Fixed Points of ξ–(α, β)- Contractive Mappings in b-Metric Spaces. *Journal of Advances in Mathematics and Computer Science*, 38(6):6–15, 2023.

[25] J. Mohamed and B. Samet. A generalized metric space and related fixed point theorems. *Fixed Point Theory and Applications*, 2015(61):https://doi.org/10.1186/s13663–015–0312–7, 2015.

[26] A. H. Ansari. Note on $\phi - \psi-$contractive type mappings and related fixed point. In *The 2nd Regional Conference on Mathematics And Applications.*, pages 377–380, Iran, 2014. PNU.

[27] M. S. Khan A. H. Ansari, M. Ozdemir and I. Yildirim. Coupled fixed point theorems with C-class functions. *Facta universitatis series mathematics and informatics*, 33(1):109–124, 2018.

[28] A. H. Ansari and A. Tomar. C-class and pair upper class functions and other kind of contractions in fixed point theory. *Scientific Publications of the State University of Novi Pazar Series A Applied Mathematics Informatics and mechanics*, 13(1):43–60, 2025.

[29] M. S. Khan A. H. Ansari and V. Rakocevic. Maia type fixed point results via C-class function. *Acta Universitatis Sapientiae Mathematica*, 12(2):227–244, 2020.

[30] M. M. Swaleh M. S. Khan and S. Sessa. Fixed point theorems by altering distances between the points. *Bulletin of the Australian Mathematical Society*, 30(1):1–9, 1984.

[31] Meena Joshi A. H. Ansari, Anita Tomar. A survey of class and pair upper-class functions in fixed point theory. *International Journal of Nonlinear Analysis and Applications*, 13(1):1879–1896, 2022.