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An Advanced Study of Hermite–Hadamard and Trapezoid-Type Inequalities via Generalized Convexity and Fractional Double Integral Operators

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Abstract. Performing a thought study of fractional inequalities by means of convexity and fractional operators has conspicuous work in the field of analysis. The main object of this paper is to discuss the coordinated convexity, pre-invexity, and also establish fractional double integral operators (FDIO) having generalized Bessel–Maitland function as its kernel. We develop a new generation of Hermite–Hadamard (H–H) and trapezoid-type inequalities through different types of coordinated convexities and pre-invexities with successful implementation of newly designed fractional double integral operators. Moreover, we extract some corollaries, which are generalizations of well-known inequalities for different coordinated convexities that show a strong consolidation of our main results.

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1. Introduction

Due to advanced research of fractional inequalities in different areas of mathematics, the demand for fractional operators has increased. One of the techniques to obtain the

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modified versions of generalized fractional operators by utilized special functions in multi-index generation as its general kernel. Fractional integral and differential operators have significant uses in solving real-world problems and extend classical calculus by shifting from integer orders to fractional ones. Many researchers have worked on generalized fractional integral operators and have discussed its immense applications in the field of inequality [1–5]. The conception of convex functions and its generalization have many significant applications in pure and applied mathematics [6, 7]. In recent years, the theory of convexity has also advanced quickly because of its many applications and its strong link to the idea of inequalities. Convexity and pre-invexity have been extended and generalized in many directions of analysis to improve generations of well-known inequalities. The coordination between convexity and inequalities is very strong and gave more results in aspects of extensions and generations[8–11]. Recently, many scientists have worked on two-dimensional convexities [12–16] in response to advanced analysis and modified Hermite-Hadamard inequalities and their refinements. Resolve many mathematical problems by using the applications of inequalities in a situation where it is impossible to find the exact values.

The Hermite-Hadamard inequality is defined for the convex function [17, 18] $\Phi : J \to \mathbb{R}$, as follows

$$\Phi(\frac{x+y}{2}) \leq \frac{1}{y-x}\int_a^b \Phi(u)du \leq \frac{\Phi(x)+\Phi(y)}{2}, \tag{1}$$

where $x, y \in J \subseteq \mathbb{R}, x < y$.

Modification of fractional operators as well as generalized fractional inequalities related to well-known inequalities and its refinements with different convexities and pre-invexities are discussed [8, 19–22].

In the following section, we discuss the basic concepts related to our work, also describe the new class of generalized double fractional operators having extended Bessel-Maitland function as its kernel. In section 3, we explore the Hermite-Hadamard type inequalities by implementation of newly defined fractional operators with coordinated h-Godunova-Levin convexity. In section 4, we refine the trapezoid type inequalities with generalized double fractional operators and coordinated pre-invexity. In the last section, we discuss the conclusion and future recommendations.

2. Preliminaries

In this section, we discuss the basic definitions which help us to understand our main results. Moreover, we develop fractional double integral operators which have generalized the Bessel-Maitland function as used its kernel.

Definition 1. *[23] The function $\Phi : J \to \mathbb{R}$ is said to be convex if the following inequality holds*

$$\Phi[\rho u + (1-\rho)v] \leq \rho\Phi(u) + (1-\rho)\Phi(v), \tag{2}$$

where $\rho \in [0, 1], \forall u, v \in J$.

Definition 2. *[24] Let $J \subseteq \mathbb{R}$ be an invex set with respect to a real bi-function $\Phi : J \times J \to \mathbb{R}$, if the following relation holds:*

$$v + \lambda\Phi(u, v) \in J, \tag{3}$$

where $u, v \in J, \lambda \in [0, 1]$.

Definition 3. *[24] The function $\phi : J \to \mathbb{R}$ is said to be pre-invex if the following inequality holds:*

$$\phi(v + \lambda\Phi(u, v)) \leq \lambda\phi(u) + (1 - \lambda)\phi(v), \tag{4}$$

where $\lambda \in [0, 1], \forall u, v \in J$ and J are an invex set.

Definition 4. *[25] Let $\Phi : J \subseteq \mathbb{R} \to \mathbb{R}$ be a positive real valued function, which is said to be a Godunova-Levin convex function, $\forall u, v \in J$ and $\rho \in (0, 1)$, if the following inequality holds:*

$$\Phi(\rho u + (1 - \rho)v) \leq \frac{\Phi(u)}{(\rho)} + \frac{\Phi(v)}{(1 - \rho)}. \tag{5}$$

Definition 5. *[26] Let $h : (0, 1) \to \mathbb{R}$ be a non-negative function, then the function $\Phi : J \to \mathbb{R}$ is said to be h- Godunova-Levin, $\forall u, v \in J$ and $\rho \in (0, 1)$, if the inequality holds:*

$$\Phi(\rho u + (1 - \rho)v) \leq \frac{\Phi(u)}{h(\rho)} + \frac{\Phi(v)}{h(1 - \rho)}. \tag{6}$$

Definition 6. *[26] Let $\Phi : J \to \mathbb{R}$ is said to be h-Godunova-Levin pre-invex function with respect to $\varkappa$ for $u, v \in J, \phi \in (0, 1)$, if the following inequality holds:*

$$\Phi(u + \phi\varkappa(v, u)) \leq \frac{\Phi(u)}{h(1 - \phi)} + \frac{\Phi(v)}{h(\phi)}. \tag{7}$$

Definition 7. *[27] The Pochhammer symbol is defined for $\alpha' \in \mathbb{C}$ and $n \in \mathbb{N}$, as follows*

$$(\alpha')_n = \begin{cases} \alpha'(\alpha' + 1)(\alpha' + 2)(\alpha' + 3)...(\alpha' + n - 1), & \text{for } n > 1, \\ 1, & \text{for } n = 1, \ \beta \neq 0. \end{cases} \tag{8}$$

Definition 8. *[27] The gamma function is defined for $\Re(z) > 0$, as follows*

$$\Gamma(z) = \int_0^\infty y^{z-1}e^{-y}dy. \tag{9}$$

Definition 9. *[27] The beta function is defined for $\Re(p) > 0$ and $\Re(h) > 0$, as follows:*

$$\beta(p, h) = \int_0^1 y^{p-1}(1 - y)^{h-1}dy. \tag{10}$$

The beta-gamma relation is as follows

$$\beta(p,h)=\frac{\Gamma(p)\Gamma(h)}{\Gamma(p+h)}. \tag{11}$$

Definition 10. *[28] The extended beta functions are defined for $\Re(m)>0$,$\Re(n)>0$, and $\Re(p)>0$ as follows:*

$$Q_p(m,n)=\int_0^1 z^{m-1}(1-z)^{n-1}\exp\left(\frac{-p}{z(1-z)}\right)dz. \tag{12}$$

Remark 1. *If we replace $p=0$, in the definition* (10)*, then we obtain the classical beta function.*

Definition 11. *[29] The generalized Bessel-Maitland function of eight parameters is defined as follows:*

$$\Upsilon^{\psi,\Phi,\rho,\vartheta}_{\phi,\varkappa,m,\sigma}(y)=\sum_{p=0}^{\infty}\frac{(\Phi)_{\varkappa p}(\vartheta)_{\sigma p}(-y)^p}{\Gamma(\phi p+\psi+1)(\rho)_{mp}}, \tag{13}$$

where $\phi,\psi,\Phi,\rho,\vartheta,\in\mathbb{C},\Re(\phi)>0,\Re(\psi)\geq -1,\Re(\Phi)>0,\Re(\rho)>0,\Re(\vartheta)>0,\varkappa,m,\sigma\geq 0$, and $m,\varkappa>\Re(\phi)+\sigma$.

Definition 12. *[30] The extended generalized Bessel-Maitland function is defined for $\mu,\nu,\kappa,\rho,\gamma,c\in\mathbb{C},\Re(\mu)>0,\Re(\nu)\geq_1,\Re(\kappa)>0,\Re(\rho)>0,\Re(\gamma)>0,\varkappa,m,\sigma\geq 0$ and $m,\varkappa>\Re(\mu)+\sigma$ as follows:*

$$\Upsilon^{\mu,\varkappa,m,\sigma,c}_{\nu,\kappa,\rho,\gamma}(\Omega;p)=\sum_{n=0}^{\infty}\frac{Q_p(\kappa+\varkappa n,c-\kappa)(c)_{\varkappa n}(\gamma_{\sigma n})}{Q(\kappa,c-\kappa)\Gamma(\mu n+\nu+1)(\rho)_{mn}}(-\Omega)^n. \tag{14}$$

Definition 13. *[30] The generalized fractional integral operators, with an extended generalized Bessel-Maitland function as the kernel, are defined for $\mu,\nu,\kappa,\rho,\gamma,c\in\mathbb{C},\Re(\mu)>0,\Re(\nu)\geq -1,\Re(\kappa)>0,\Re(\rho)>0,\Re(\gamma)>0,\varkappa,m,\sigma\geq 0$, and $m,\varkappa>\Re(\mu)+\sigma$, as follows:*

$$(\mathbb{E}^{\mu,\varkappa,m,\sigma,c}_{\nu,\kappa,\rho,\gamma;p^+}f)(x,r)=\int_p^x(x-t)^{\nu}\Upsilon^{\mu,\varkappa,m,\sigma,c}_{\nu,\kappa,\rho,\gamma}(\Omega(x-t)^{\mu};\imath)f(t)dt, \tag{15}$$

and

$$(\mathbb{E}^{\mu,\varkappa,m,\sigma,c}_{\nu,\kappa,\rho,\gamma,;q^-}f)(x,r)=\int_x^q(t-x)^{\nu}\Upsilon^{\mu,\varkappa,m,\sigma,c}_{\nu,\kappa,\rho,\gamma,}(\Omega(t-x)^{\mu};\imath)f(t)dt. \tag{16}$$

In short, the notation can be written as follows

$$(\Lambda^{u^+}_{\nu,\nu'})(\Omega,\Phi)=(\mathbb{E}^{\mu,\varkappa,m,\sigma,c}_{\nu',\kappa,\rho,\gamma;u^+}\Phi)(\nu,\imath), \tag{17}$$

and

$$(\Lambda^{\nu^-}_{u,\nu'})(\Omega,\Phi)=(\mathbb{E}^{\mu,\varkappa,m,\sigma,c}_{\nu',\kappa,\rho,\gamma;\nu^-}\Phi)(u,\imath). \tag{18}$$

In definition 13 described generalized fractional integral operators in aspects of one-dimension (single integral); such operators refined well known inequalities [30]. Most of fractional operators have been generalized due to its kernel but we extend fractional operators in two-dimensional (2-D) region, and discuss its applications in inequalities. Due to this, we define double fractional integral operators (two-dimensional), and successfully implementation to the obtain the new version of inequalities.

Definition 14. *The generalized fractional double-integral operators having an extended generalized Bessel-Maitland function as the kernel are defined, for $\mu, \nu, \kappa, \rho, \gamma, c \in \mathbb{C}, \Re(\mu) > 0, \Re(\nu) \geq -1, \Re(\kappa) > 0, \Re(\rho) > 0, \Re(\gamma) > 0, \varkappa, m, \sigma \geq 0,$ and $m, \varkappa > \Re(\mu) + \sigma$, as follows:*

$$\begin{aligned}&(\mathbb{E}^{\mu,\varkappa,m,\sigma,c}_{\nu,\kappa,\rho,\rho,\gamma;(p_1,p_2)^+} f)((x_1,x_2),r)\\ &= \int_{p_1}^{x_1}\int_{p_2}^{x_2}(x_1-t_1)^{\nu}(x_2-t_2)^{\nu}\Upsilon^{\mu,\varkappa,m,\sigma,c}_{\nu,\kappa,\rho,\gamma}(\Omega[(x_1-t_1)^{\mu}(x_2-t_2)^{\mu}];r)f(t_1,t_2)dt_1dt_2,\end{aligned} \tag{19}$$

and

$$\begin{aligned}&(\mathbb{E}^{\mu,\varkappa,m,\sigma,c}_{\nu,\kappa,\rho,\gamma,;(q_1,q_2)^-} f)((x_1,x_2),r)\\ &= \int_{x_1}^{q_1}\int_{x_2}^{q_2}(t_1-x_1)^{\nu}(t_2-x_2)^{\nu}\Upsilon^{\mu,\varkappa,m,\sigma,c}_{\nu,\kappa,\rho,\gamma,}(\Omega[(t_1-x_1)^{\mu}(t_2-x_2)^{\mu}];r)f(t_1,t_2)dt_1dt_2\end{aligned} \tag{20}$$

In our work, we use the notation for fractional double-integral operators.

$$(\Lambda^{(\mu_1,\mu_2)^+}_{(\nu_1,\nu_2),\nu'})(\Omega,\Phi) = (\mathbb{E}^{\mu,\varkappa,m,\sigma,c}_{\nu',\kappa,\rho,\gamma;(\mu_1,\mu_2)^+}\Phi)((\nu_1,\nu_2),\shortmid), \tag{21}$$

and

$$(\Lambda^{(\nu_1,\nu_2)^-}_{(\mu_1,\mu_2),\nu'})(\Omega,\Phi) = (\mathbb{E}^{\mu,\varkappa,m,\sigma,c}_{\nu',\kappa,\rho,\gamma;(\nu_1,\nu_2)^-}\Phi)((\mu_1,\mu_2),\shortmid). \tag{22}$$

The following remark show the strengthen the idea of newly define fractional operators in two-dimension region (double fractional operators).

Remark 2. *(i) If we replace the substitute values $p_2 = 0$ in equation (19) and $q_2 = 0$ in equation (20), we obtained the single fractional operators which are defined in equations (15), (16) respectively.*

*(ii) If we take the values $p_2 = 0$ in equation (19), $q_2 = 0$ in equation (20), and $\Omega = 0$ $\nu = \nu - 1$ in both equations (19), (20), we have Reimann-Liouville fractional operators defined in [**?**].*

Definition 15. *[15] The coordinated convex (two-dimensional) function $\Phi : J \times J \subset \mathbb{R}^2 \to R$ is defined as follows:*

$$\Phi[\rho\varpi_1 + (1-\rho)v_1, \rho u_2 + (1-\rho)\tau_2] \leq \rho\Phi(\varpi_1, u_2) + (1-\rho)\Phi(v_1,\tau_2), \tag{23}$$

where $\rho \in [0,1], \forall(\varpi_1,u_2),(v_1,\tau_2) \in J \times J$.

Example 2.1. If $\Phi(x,y)=(x+y)^2$, $\rho=0.5$, $\varpi_1=-2$, $v_1=3$, $u_2=1$, $\tau_2=0.5$, then two dimension convex function is evaluated as fallows Now,

$$\begin{aligned}
&\Phi[\rho\varpi_1+(1-\rho)v_1,\rho u_2+(1-\rho)\tau_2]\leq\rho\Phi(\varpi_1,u_2)+(1-\rho)\Phi(v_1,\tau_2)\\
&\Phi[0.5(-2)+(1-0.5)3,0.5(1)+(1-0.5)(0.5)]\leq .5\Phi(-2,1)+(1-0.5)\Phi(3,0.5)\\
&\Phi(0.5,.75)\leq 0.5(1)+0.5(12.25)\\
&1.56\leq 6.625
\end{aligned}$$

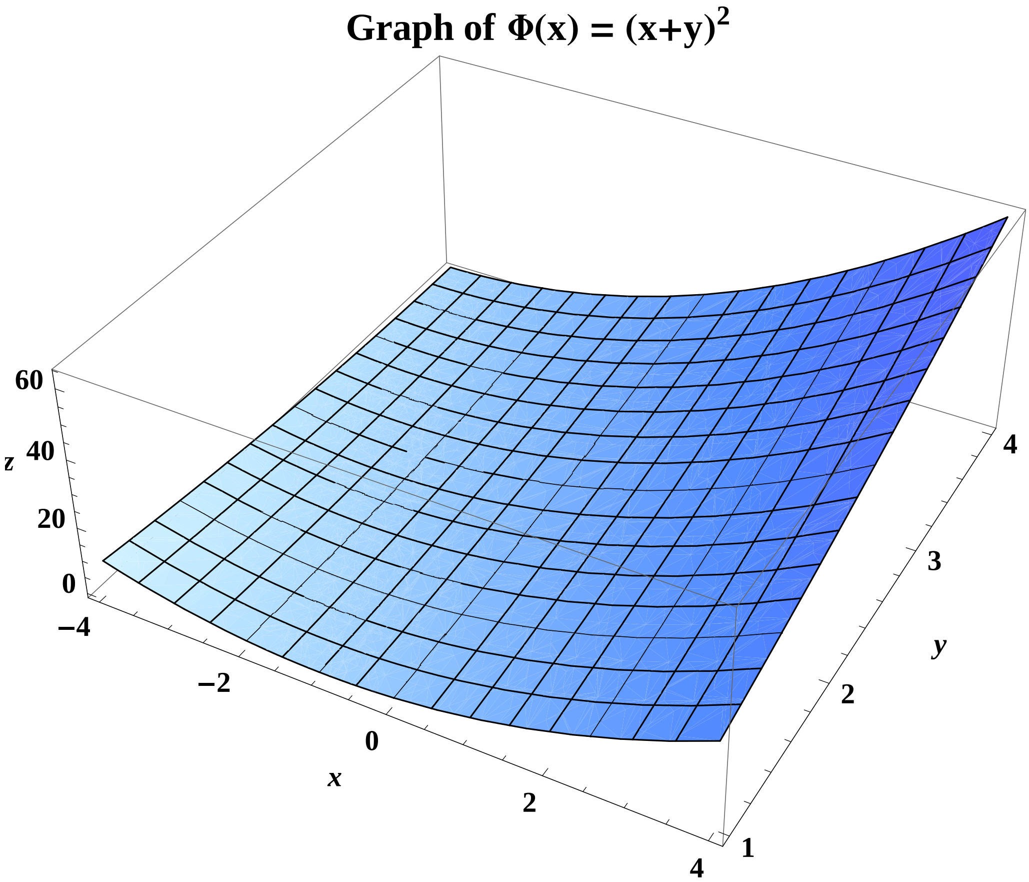

Figure 1: 2-D convex function

Remark 3. *If we replace* $u_2=0,\tau_2=0$ *in equation* (23)*, we have equation* (2)*.*

Definition 16. *[15] An invex set* $J\times J\subseteq\mathbb{R}^2$ *with respect to a real bi-function* $\Phi: J\times J\to R$*, is defined for* $(\varpi_1,u_2),(v_1,\tau_2)\in J\times J,\lambda\in[0,1]$ *as follows:*

$$v_1+\lambda\Phi(\varpi_1,v_1),\tau_2+\lambda\Phi(u_2,\tau_2)\in J. \tag{24}$$

Remark 4. *If we replace* $u_2=0,\tau_2=0$ *in equation* (24)*, then we obtain equation* (3)*.*

Definition 17. *[9] Let $\rho \in [0,1], (\varpi_1, u_2), (v_1, \tau_2) \in J \times J \subset \mathbb{R}^2$, then the coordinated (two-dimensional;2-D) pre-invex function $\Phi : J \times J \subset \mathbb{R}^2 \to R$ is defined as follows:*

$$\Phi[v_1 + \lambda\rho(\varpi_1, v_1), \tau_2 + \lambda\rho(u_2, \tau_2)] \leq (\lambda)\Phi(\varpi_1, u_2) + (1-\lambda)\Phi(v_1, \tau_2). \tag{25}$$

Remark 5. *Putting the values of $u_2 = 0, \tau_2 = 0$ in equation (25), we obtain (4).*

Definition 18. *Let $(\varpi_1, u_2), (v_1, \tau_2) \in J \times J \subset \mathbb{R}^2$ and $\rho \in (0,1)$, then 2-D Godunova-Levin function $\Phi : J \times J \subset \mathbb{R}^2 \to \mathbb{R}$ is defined as follows*

$$\Phi(\rho\varpi_1 + (1-\rho)v_1, \rho u_2 + (1-\rho)\tau_2) \leq \frac{\Phi(\varpi_1, u_2)}{(\rho)} + \frac{\Phi(v_1, \tau_2)}{(1-\rho)}. \tag{26}$$

Remark 6. *Taking the values $u_2 = 0, \tau_2 = 0$ in equation (26), we have equation (5).*

Definition 19. *Let $h : (0,1) \to \mathbb{R}$ be a function and $\Phi : J \times J \subset \mathbb{R}^2 \to \mathbb{R}$ be a non-negative function, then 2-D h-Godunova-Levin convex function is defined as follows:*

$$\Phi(\rho\varpi_1 + (1-\rho)v_1, \rho u_2 + (1-\rho)\tau_2) \leq \frac{\Phi(\varpi_1, u_2)}{h(\rho)} + \frac{\Phi(v_1, \tau_2)}{h(1-\rho)}. \tag{27}$$

where $(\varpi_1, u_2), (v_1, \tau_2) \in J \times J$ and $\rho \in (0,1)$.

Remark 7. *If we put the values of $u_2 = 0, \tau_2 = 0$ in equation (27), then we obtain equation (6).*

Definition 20. *Let $\Phi : J \times J \subset \mathbb{R}^2 \to \mathbb{R}$ be a real valued function, then 2-D Godunova-Levin pre-invex function with respect to $\varkappa$ is defined for all $(\varpi_1, u_2), (v_1, \tau_2) \in J \times J, \phi \in (0,1)$, is defined as follows*

$$\Phi(\varpi_1 + \phi\varkappa(v_1, \varpi_1), u_2 + \phi\varkappa(\tau_2, u_2)) \leq \frac{\Phi(\varpi_1, u_2)}{(1-\phi)} + \frac{\Phi(v_1, \tau_2)}{(\phi)}. \tag{28}$$

Remark 8. *Taking the values $u_2 = 0, \tau_2 = 0$ in equation (28), we obtain equation (7).*

Definition 21. *Let $h : (0,1) \to \mathbb{R}$ be a real valued function and $\Phi : J \times J \subset \mathbb{R}^2 \to R$ be a non-negative function, then the 2-D h-Godunova-Levin pre-invex function with respect to $\varkappa$ is defined for all $(\varpi_1, u_2), (v_1, \tau_2) \in J \times J, \phi \in (0,1)$, as follows:*

$$\Phi(\varpi_1 + \phi\varkappa(v_1, \varpi_1), u_2 + \phi\varkappa(\tau_2, u_2)) \leq \frac{\Phi(\varpi_1, u_2)}{h(1-\phi)} + \frac{\Phi(v_1, \tau_2)}{h(\phi)}. \tag{29}$$

Remark 9. *Taking the values $\tau_2 = 0, u_2 = 0$ in equation (29), we obtain equation (14).*

Definition 22. *[31] Let $\Delta = J_1 \times J_2 \subseteq \mathbb{R}^{\ltimes} \times \mathbb{R}^{\ltimes}$, then the coordinated pre-invex function $\Phi : \Delta \to \mathbb{R}$ is defined in Δ, as follows:*

$$\begin{aligned}\Phi\big(\varpi_1 + \rho\wp_1(\varpi_1, v_1), u_2 + r\wp_2(u_2, \tau_2)\big) = (1-\rho)(1-r)\Phi(\varpi_1, u_2) + \\ (1-\rho)r\Phi(\varpi_1, \tau_2) + \rho(1-r)\Phi(v_1, u_2) + \rho r\Phi(v_1, \tau_2),\end{aligned} \tag{30}$$

where $(\varpi_1, u_2), (\varpi_1, \tau_2), (v_1, u_2), (v_1, v \in \Delta_2)$ and $\rho, r \in [0,1]$

Definition 23. *[31] Let $\Delta = J_1 \times J_2 \subset \mathbb{R}^{\ltimes} \times \mathbb{R}^{\ltimes}$ and $J_1, J_2 \subseteq \mathbb{R}^{\ltimes}$ be two invex sets with respect to $\wp_1 : J_1 \times J_1 \to \mathbb{R}^{\ltimes}$ and $\wp_2 : J_2 \times J_2 \to \mathbb{R}^{\ltimes}$, then the continuous function $\Phi : \Delta \to \mathbb{R}$ is said to be Godunova-Levin type s- pre-invex on the co-ordinates with respect to $\wp_1$ on J_1 and $\wp_2$ in J_2 if for every $\varpi_1, v_1 \in J_1$ and $u_2, \tau_2 \in J_2$ and $\rho, r \in (0,1)$ and $s \in [0,1]$, and defined as follows:*

$$\Phi\big(\varpi_1 + \rho\varkappa_1(v_1, \varpi_1), u_2 + \rho\varkappa_(\tau_2, u_2)\big) \leq \frac{\Phi(\varpi_1, u_2)}{(1-\rho)^s(1-r)^s} + \frac{\Phi(\varpi_1, \tau_2)}{(1-\rho)^s(r)^s} + \frac{\Phi(v_1, \varpi_1)}{(\rho)^s(1-r)^s} + \frac{\Phi(v_1, \tau_2)}{(\rho)^s(r)^s}. \quad (31)$$

Remark 10. *Putting $s = 1$ in the definition* (23)*, we then obtain the following definition.*

Definition 24. *Let $\Delta = J_1 \times J_2 \subset \mathbb{R}^{\ltimes} \times \mathbb{R}^{\ltimes}$ and $J_1, J_2 \subseteq \mathbb{R}^{\ltimes}$ be two invex sets with respect to $\wp_1 : J_1 \times J_1 \to \mathbb{R}^{\ltimes}$ and $\wp_2 : J_2 \times J_2 \to \mathbb{R}^{\ltimes}$, then the continuous function $\Phi : \Delta \to \mathbb{R}$ is said to be a Godunova-Levin coordinated pre-invex function with respect to $\wp_1$on J_1 and $\wp_2$ on J_2 if for every $\varpi_1, v_1 \in J_1$ and $u_2, \tau_2 \in J_2$ and $\rho, r \in (0,1)$, the following inequality holds:*

$$\Phi\big(\varpi_1 + \rho\varkappa_1(v_1, \varpi_1), u_2 + \rho\varkappa_(\tau_2, u_2)\big) \leq \frac{\Phi(\varpi_1, u_2)}{(1-\rho)(1-r)} + \frac{\Phi(\varpi_1, \tau_2)}{(1-\rho)r} + \frac{\Phi(v_1, \varpi_1)}{\rho(1-r)} + \frac{\Phi(v_1, \tau_2)}{\rho r}. \quad (32)$$

Definition 25. *Let $h : (0,1) \to \mathbb{R}$ and $\Delta = J_1 \times J_2 \subset \mathbb{R}^{\ltimes} \times \mathbb{R}^{\ltimes}$ and $J_1 \times J_2 \subseteq \mathbb{R}^{\ltimes}$ be two invex sets with respect to $\wp_1 : J_1 \times J_1 \to \mathbb{R}^{\ltimes}$ and $\wp_2 : J_2 \times J_2 \to \mathbb{R}^{\ltimes}$, then the continuous function $\Phi : \Delta \to \mathbb{R}$ is said to be Godunova-Levin type s- pre-invex on the co-ordinates with respect to $\wp_1$on J_1 and $\wp_2$ on J_2 if for every $\varpi_1, v_1 \in J_1$ and $u_2, \tau_2 \in J_2$ and $\rho, r \in (0,1)$, the following inequality holds*

$$\Phi(\varpi_1 + \rho\varkappa_1(v_1, \varpi_1), u_2 + \rho\varkappa_(\tau_2, u_2)) \leq \frac{\Phi(\varpi_1, u_2)}{h[(1-\rho)(1-r)]} + \frac{\Phi(\varpi_1, \tau_2)}{h[(1-\rho)(r)]} + \frac{\Phi(v_1, \varpi_1)}{h[(\rho)(1-r)]} + \frac{\Phi(v_1, \tau_2)}{h[(\rho)(r)]}. \quad (33)$$

3. Hermite-Hadamard fractional double integral inequalities for coordinated h-Godunova-Levin Convex Function

In this section, we discuss the refinements of Hermite-Hadamard inequalities by implementation of newly defined generalized fractional double-integral (two dimensional region) in definition 14; with 2-D h-Godunova-Levin convex function and also extract some corollaries.

Theorem 1. *Let $\Phi : J \times J \subset \mathbb{R}^2 \to \mathbb{R}$ be a 2-D h-Godunova-Levin convex function, where $0 < \varpi_1 < v_1$, $0 < u_2 < \tau_2$ and $\Phi \in L_1[\varpi_1, v_1], L_2[u_2, \tau_2]$ with $h : (0,1) \to \mathbb{R}$ is a real valued positive function with $h(\rho) \neq 0$ and $\rho \in (0,1)$, then the following inequality holds:*

$$\frac{h(\frac{1}{2})}{2}\left[\Phi(\frac{\varpi_1, u_2}{2} + \frac{v_1, \tau_2}{2})\Lambda^{(v_1,\tau_2)^-}_{(\varpi_1,u_2),\nu'}(\Omega', 1)\right] \leq \left[\frac{1}{2}(\Lambda^{(v_1,\tau_2)^-}_{(\varpi_1,u_2),\nu'}(\Omega', \Phi) + \Lambda^{(\varpi_1,u_2)^+}_{(v_1,\tau_2),\nu'}(\Omega', \Phi))\right]$$
$$\leq \left[(v_1 - \varpi_1)^{\nu'+1}(\tau_2 - u_2)^{\nu'+1}(\frac{\Phi(\varpi_1, u_2) + \Phi(v_1, \tau_2)}{2}) \times \int_0^1 \int_0^1 \left[\frac{1}{h(\rho)} + \frac{1}{h(1-\rho)}\right]\right.$$

$$(\rho^{\nu'} r^{\nu'})\Upsilon^{\mu,\varkappa,m,\sigma,c}_{\nu',\kappa,\rho,\gamma}(\Omega\rho^{\mu}r^{\mu};\shortmid)d\rho dr\Big],$$

where $\Omega' = \left(\frac{\Omega}{\wp_1(v_1,\varpi_1)^{\mu}\wp_2(\tau_2,u_2)^{\mu}}\right)$.

Proof. Consider the 2-D h-Godunova-Levin convex function

$$\Phi(\varkappa x + (1-\varkappa)z), (\varkappa y + (1-\varkappa)w) \leq \frac{\Phi(x,y)}{h(\varkappa)} + \frac{\Phi(z,w)}{h(1-\varkappa)}. \tag{34}$$

Putting the values $x = \rho\varpi_1 + (1-\rho)v_1, y = ru_2 + (1-r)\tau_2, z = (1-\rho)\varpi_1 + \rho v_1, w = (1-r)u_2 + r\tau_2, \varkappa = \frac{1}{2}$ into the equation (34), we have

$$\Phi((\frac{\varpi_1+v_1}{2}),(\frac{u_2+\tau_2}{2})) \leq \frac{1}{h(\frac{1}{2})}\Big[\Phi\Big((\rho\varpi_1)+(1-\rho)v_1),(ru_2+(1-r)\tau_2\Big) + \Phi((1-\rho\varpi_1)$$
$$+(\rho)v_1),((1-r)u_2+r\tau_2)\Big]. \tag{35}$$

Multiplying both sides of the equation (35) by $(\rho^{\nu'} r^{\nu'})\Upsilon^{\mu,\varkappa,m,\sigma,c}_{\nu',\kappa,\rho,\gamma}(\Omega\rho^{\mu}r^{\mu};\shortmid)$ and taking the double-integral of $[0,1]$ with respect to ρ and r, we have

$$\Phi((\frac{\varpi_1+v_1}{2}),(\frac{u_2+\tau_2}{2}))\int_0^1\int_0^1(\rho^{\nu'} r^{\nu'})\Upsilon^{\mu,\varkappa,m,\sigma,c}_{\nu',\kappa,\rho,\gamma}(\Omega\rho^{\mu}r^{\mu};\shortmid)d\rho dr \leq \frac{1}{h(\frac{1}{2})}$$
$$\Big[\int_0^1\int_0^1(\rho^{\nu'} r^{\nu'})\Upsilon^{\mu,\varkappa,m,\sigma,c}_{\nu',\kappa,\rho,\gamma}(\Omega\rho^{\mu}r^{\mu};\shortmid)\Phi\big(\rho\varpi_1+(1-\rho)v_1, ru_2+(1-r)\tau_2\big)d\rho dr +$$
$$\int_0^1\int_0^1(\rho^{\nu'} r^{\nu'})\Upsilon^{\mu,\varkappa,m,\sigma,c}_{\nu',\kappa,\rho,\gamma}(\Omega\rho^{\mu}r^{\mu};\shortmid)\Phi\big((1-\rho)\varpi_1+\rho v_1,(1-r)u_2+r\tau_2\big)d\rho dr\Big]. \tag{36}$$

$$((\frac{\varpi_1+v_1}{2}),(\frac{u_2+\tau_2}{2}))\sum_{n=0}^{\infty}\frac{Q_{\shortmid}(\kappa+\varkappa n, c-\kappa)(c)_{\varkappa n}(\gamma)_{\sigma n}}{Q(\kappa,c-\kappa)\Gamma(\mu n+\nu'+1)(\rho)_{mn}}(-\Omega)^n\int_0^1\int_0^1\rho^{\nu'+\mu n}r^{\nu'+\mu n}d\rho dr$$
$$\leq \sum_{n=0}^{\infty}\frac{Q_{\shortmid}(\kappa+\varkappa n, c-\kappa)(c)_{\varkappa n}(\gamma)_{\sigma n}}{Q(\kappa,c-\kappa)\Gamma(\mu n+\nu'+1)(\rho)_{mn}}(-\Omega)^n\Big[\int_0^1\int_0^1\rho^{\nu'+\mu n}r^{\nu'+\mu n}\Phi(((\rho\varpi_1)$$
$$+(1-\rho)v_1),(r\varpi_1+(1-r)v_1))d\rho dr + \int_0^1\int_0^1\rho^{\nu'+\mu n}r^{\nu'+\mu n}$$
$$\Phi\big((1-\rho)\varpi_1)+\rho v_1,(1-r)u_2+r\tau_2\big)d\rho dr\Big]. \tag{37}$$

Solving the integrals of equation (37), we have

$$\frac{h(\frac{1}{2})}{2}\Big[\Phi(\frac{\varpi_1+v_1}{2}),(\frac{u_2+\tau_2}{2})\Big]\Lambda^{(v_1,\tau_2)^-}_{(\varpi_1,u_2),\nu'}(\Omega',1) \leq \left(\frac{1}{2}\right)\Big[\Lambda^{(v_1,\tau_2)^-}_{(\varpi_1,u_2),\nu'}(\Omega',\Phi) + \Lambda^{(\varpi_1,u_2)^+}_{(v_1,\tau_2),\nu'}(\Omega',\Phi)\Big]. \tag{38}$$

Now for the second part of the inequality, we again consider the two-dimension h-Godunova-Levin convex function; we have

$$\Phi(\rho x+(1-\rho)z,\rho y+(1-\rho)w)\leq \frac{\Phi(x,y)}{h(\rho)}+\frac{\Phi(z,w)}{h(1-\rho)}. \tag{39}$$

and

$$\Phi((1-\rho)x+(\rho)z,\rho y+(1-\rho)w)\leq \frac{\Phi(x,y)}{h(1-\rho)}+\frac{\Phi(z,w)}{h(\rho)}. \tag{40}$$

Adding equation (39) and equation (40), we have

$$\Phi(\rho x+(1-\rho)z,\rho y+(1-\rho)w)+\Phi((1-\rho)x+\rho z,\rho y+(1-\rho)w)\leq \Phi(x,y)+\Phi(z,w) \left[\frac{1}{h(\rho)}+\frac{1}{h(1-\rho)}\right]. \tag{41}$$

Multiplying equation (41) by $(\rho^{\nu'}r^{\nu'})\Upsilon^{\mu,\varkappa,m,\sigma,c}_{\nu',\kappa,\rho,\gamma}(\Omega\rho^{\mu}r^{\mu};\shortmid)$ and then double-integrating over the interval $[0,1]$ with respect to ρ and r, we have

$$\begin{aligned}&\bigg[\int_0^1\int_0^1(\rho^{\nu'}r^{\nu'})\Upsilon^{\mu,\varkappa,m,\sigma,c}_{\nu',\kappa,\rho,\gamma}(\Omega\rho^{\mu}r^{\mu};\shortmid)\Phi\bigg(\rho\varpi_1+(1-\rho)v_1,ru_2+(1-r)\tau_2\bigg)d\rho dr+\\ &\int_0^1\int_0^1(\rho^{\nu'}r^{\nu'})\Upsilon^{\mu,\varkappa,m,\sigma,c}_{\nu',\kappa,\rho,\gamma}(\Omega\rho^{\mu}r^{\mu};\shortmid)\Phi\bigg((1-\rho)\varpi_1+\rho v_1,(1-r)u_2+r\tau_2\bigg)d\rho dr\bigg]\\ &\leq \Phi(\varpi_1,u_2)+\Phi(v_1,\tau_2)\int_0^1\int_0^1\left[\frac{1}{h(\rho)}+\frac{1}{h(1-\rho)}\right](\rho^{\nu'}r^{\nu'})\Upsilon^{\mu,\varkappa,m,\sigma,c}_{\nu',\kappa,\rho,\gamma}(\Omega\rho^{\mu}r^{\mu};\shortmid)d\rho dr.\end{aligned} \tag{42}$$

Solving the integrals of the equation (42), we obtain

$$\begin{aligned}&\left(\frac{1}{2}\right)\left[\Lambda^{(v_1,\tau_2)^-}_{(\varpi_1,u_2),\nu'}(\Omega',\Phi)+\Lambda^{(\varpi_1,u_2)^+}_{(v_1,\tau_2),\nu'}(\Omega',\Phi)\right]\leq ((v_1-\varpi_1)^{\nu'+1}(\tau_2-u_2)^{\nu'+1})\frac{\Phi(\varpi_1,u_2)+\Phi(v_1,\tau_2)}{2}\\ &\int_0^1\int_0^1\left[\frac{1}{h(\rho)}+\frac{1}{h(1-\rho)}\right](\rho^{\nu'}r^{\nu'})\Upsilon^{\mu,\varkappa,m,\sigma,c}_{\nu',\kappa,\rho,\gamma}(\Omega\rho^{\mu}r^{\mu};\shortmid)d\rho dr.\end{aligned} \tag{43}$$

By combining (38) and (43), we have the required inequality.

Corollary 1. *Substituting $h(\rho)=\rho^s$ in theorem [1], we have the Hermite-Hadamard fractional double-integral inequality for two-dimension s-Godunova-Levin convex function*

$$\begin{aligned}&\frac{(\frac{1}{2^s})}{2}\left[\Phi(\frac{\varpi_1,u_2}{2}+\frac{v_1,\tau_2}{2})\Lambda^{(v_1,\tau_2)^-}_{(\varpi_1,u_2),\nu'}(\Omega',1)\right]\leq\left[\frac{1}{2}(\Lambda^{(v_1,\tau_2)^-}_{(\varpi_1,u_2),\nu'}(\Omega',\Phi)+\Lambda^{(\varpi_1,u_2)^+}_{(v_1,\tau_2),\nu'}(\Omega',\Phi))\right]\leq\\ &\bigg[(v_1-\varpi_1)^{\nu'+1}(\tau_2-u_2)^{\nu'+1}(\frac{\Phi(\varpi_1,u_2)+\Phi(v_1,\tau_2)}{2})\int_0^1\int_0^1\left(\frac{1}{(\rho)^s}+\frac{1}{(1-\rho)^s}\right)\\ &(\rho^{\nu'}r^{\nu'})\Upsilon^{\mu,\varkappa,m,\sigma,c}_{\nu',\kappa,\rho,\gamma}(\Omega\rho^{\mu}r^{\mu};\shortmid)d\rho dr\bigg].\end{aligned}$$

Corollary 2. *If we replace $h(\rho) = 1$ in theorem [1] we have another refinement of the Hermite-Hadamard type fractional double-integral inequality.*

$$\frac{1}{2}\left[\Phi(\frac{\varpi_1, u_2}{2} + \frac{v_1, \tau_2}{2})\Lambda^{(v_1,\tau_2)^-}_{(\varpi_1,u_2),\nu'}(\Omega', 1)\right] \leq \left[\frac{1}{2}(\Lambda^{(v_1,\tau_2)^-}_{(\varpi_1,u_2),\nu'}(\Omega', \Phi) + \Lambda^{(\varpi_1,u_2)^+}_{(v_1,\tau_2),\nu'}(\Omega', \Phi))\right] \leq \left[(v_1 - \varpi_1)^{\nu'+1}(\tau_2 - u_2)^{\nu'+1}(\Phi(\varpi_1, u_2) + \Phi(v_1, \tau_2))\Upsilon^{(\varpi_1,u_2)^+}_{(v_1,\tau_2),\nu'}(\Omega', 1)\right].$$

Corollary 3. *If we replace $h(\rho) = \frac{1}{\rho}$ in theorem [1], we obtain the Hermite-Hadamard-type double-integral inequality with coordinated convex function.*

$$\left(\frac{1}{4}\right)\left[\Phi(\frac{\varpi_1, u_2}{2} + \frac{v_1, \tau_2}{2})\Lambda^{(v_1,\tau_2)^-}_{(\varpi_1,u_2),\nu'}(\Omega', 1)\right] \leq \left[\frac{1}{2}(\Lambda^{(v_1,\tau_2)^-}_{(\varpi_1,u_2),\nu'}(\Omega', \Phi) + \Lambda^{(\varpi_1,u_2)^+}_{(v_1,\tau_2),\nu'}(\Omega', \Phi))\right] \leq \left[(v_1 - \varpi_1)^{\nu'+1}(\tau_2 - u_2)^{\nu'+1}(\frac{\Phi(\varpi_1, u_2) + \Phi(v_1, \tau_2)}{2}) \int_0^1 \int_0^1 [\frac{\rho^{\nu'-1}}{1-\rho}](\rho^{\nu'} r^{\nu'})\Upsilon^{\mu,\varkappa,m,\sigma,c}_{\nu',\kappa,\rho,\gamma}(\Omega\rho^{\mu} r^{\mu}; \imath)d\rho dr\right].$$

Corollary 4. *If we substitute $h(\rho) = \rho$ in theorem [1], we have the Hermite-Hadamard-type fractional double-integral inequality for the Godunova-Levin function of the same result as the previous corollary*

Corollary 5. *If we substituting $h(\rho) = \frac{1}{\rho^s}$ into theorem [1], we get the Hermite-Hadamard-type fractional double-integral inequality for a two dimensional s-convex function:*

$$2^{s-1}\left[\Phi(\frac{\varpi_1, u_2}{2} + \frac{v_1, \tau_2}{2})\Lambda^{(v_1,\tau_2)^-}_{(\varpi_1,u_2),\nu'}(\Omega', 1)\right] \leq \left[\frac{1}{2}(\Lambda^{(v_1,\tau_2)^-}_{(\varpi_1,u_2),\nu'}(\Omega', \Phi) + \Lambda^{(\varpi_1,u_2)^+}_{(v_1,\tau_2),\nu'}(\Omega', \Phi))\right] \leq \left[(v_1 - \varpi_1)^{\nu'+1}(\tau_2 - u_2)^{\nu'+1}(\frac{\Phi(\varpi_1, u_2) + \Phi(v_1, \tau_2)}{2}) \int_0^1 \int_0^1 [(\rho)^s + (1-\rho)^s](\rho^{\nu'} r^{\nu'}) \Upsilon^{\mu,\varkappa,m,\sigma,c}_{\nu',\kappa,\rho,\gamma}(\Omega\rho^{\mu} r^{\mu}; \imath)d\rho dr\right].$$

Corollary 6. *By taking the values $p_2 = 0$ in equation* (19) *and $q_2 = 0$ in equation* (20)*, we obtained equations* (15), (16)*, such operators have discussed theorem* (1) *in single fractional integral inequalities and its refinements [1].*

4. Trapezoid fractional double integral inequalities by coordinated h-Godunova-Levin pre-invex function

Here, we develop trapezoid type fractional double-integral inequalities related to Hermite-Hadamard inequalities for 2-D h-Godunova-Levin pre-invex function with the help of a new constructed lemma.

Lemma 2. *Let $\Phi : [\varpi_1 + \wp_1(v_1, \varpi_1), u_2 + \wp_2(\tau_2, u_2)] \to \mathbb{R}^2$ with $(\varpi_1, v_1), (u_2, \tau_2) \in \mathbb{R}^2$, $\Phi \in L_1[\varpi_1 + \wp_1(v_1, \varpi_1), u_2 + \wp_2(\tau_2, u_2)]$ be partially differentiable function and $J = [\varpi_1 + \wp_1(v_1, \varpi_1), u_2 + \wp_2(\tau_2, u_2)]$ is taken to be an open invex set with respect to $\wp_1 : J \times J \to$*

$\mathbb{R}, \wp_2 : J \times J \to \mathbb{R}$ with $\wp_1(v_1, \varpi_1) > 0, \wp_2(\tau_2, u_2) > 0$, for $\varpi_1, v_1, u_2, \tau_2 \in J$, then for the extended generalized Bessel-Maitland function defined in [30], we have the following inequality:

$$\frac{\Big[\Phi(\varpi_1, u_2) + \Phi(\varpi_1 + \wp_1(v_1, \varpi_1), u_2 + \wp_2(\tau_2, u_2))\Big]\Lambda^{\mu,\varkappa,m,\sigma,c}_{\nu',\kappa,\rho,\gamma}(\Omega; \imath)}{2} -$$
$$\frac{\int_{\varpi_1}^{v_1}(\rho - \varpi_1)^{\nu'+\mu n-1}\Phi(\rho, \tau_2)d\rho}{2(v_1 - \varpi_1)^{\nu'+\mu n}} + \frac{\int_{v_1}^{\varpi_1}(\rho - v_1)^{\nu'+\mu n-1}\Phi(\rho, u_2)d\rho}{2(\varpi_1 - v_1)^{\nu'+\mu n}} -$$
$$\frac{\Lambda^{(\varpi_1+\wp_1(v_1,\varpi_1),u_2+\wp_2(\tau_2,u_2))^-}_{(\varpi_1,u_2),(\nu',\nu'-1)}(\Omega', \Phi) + \Lambda^{(\varpi_1,u_2)^+}_{(\varpi_1+\wp_1(v_1,\varpi_1),u_2+\wp_2(\tau_2,u_2)),(\nu',\nu'-1)}(\Omega', \Phi)}{2\wp_1(v_1, \varpi_1)^{\nu}\wp_2(\tau_2, u_2)^{\nu}}$$
$$= \frac{\wp_1(v_1, \varpi_1)\wp_2(\tau_2, u_2)}{2}I$$

where

$$I = \int_0^1\int_0^1(\rho^{\nu'}r^{\nu'})\Upsilon^{\mu,\varkappa,m,\sigma,c}_{\nu',\kappa,\rho,\gamma}(\Omega\rho^{\mu}r^{\mu}; \imath)\frac{\partial^2}{\partial\rho\partial r}(\varpi_1 + \rho\wp_1(v_1, \varpi_1), u_2 + r\wp_2(\tau_2, u_2))d\rho dr$$
$$+ \int_0^1\int_0^1((1 - \rho^{\nu'})(1 - r^{\nu'}))\Upsilon^{\mu,\varkappa,m,\sigma,c}_{\nu',\kappa,\rho,\gamma}(\Omega\rho^{\mu}r^{\mu}; \imath)\frac{\partial^2}{\partial\rho\partial r}(\varpi_1 + \rho\wp_1(v_1, \varpi_1), u_2 + r\wp_2(\tau_2, u_2))d\rho dr,$$

where $\Omega' = \left(\frac{\Omega}{\wp_1(v_1,\varpi_1)^{\mu}\wp_2(\tau_2,u_2)^{\mu}}\right)$.

Proof. Consider the integral

$$I = \int_0^1\int_0^1(\rho^{\nu'}r^{\nu'})\Upsilon^{\mu,\varkappa,m,\sigma,c}_{\nu',\kappa,\rho,\gamma}(\Omega\rho^{\mu}r^{\mu}; \imath)\frac{\partial^2}{\partial\rho\partial r}(\varpi_1 + \rho\wp_1(v_1, \varpi_1), u_2 + r\wp_2(\tau_2, u_2))d\rho dr$$
$$+ \int_0^1\int_0^1((1 - \rho^{\nu'})(1 - r^{\nu'}))\Upsilon^{\mu,\varkappa,m,\sigma,c}_{\nu',\kappa,\rho,\gamma}(\Omega\rho^{\mu}r^{\mu}; \imath)\frac{\partial^2}{\partial\rho\partial r}(\varpi_1 + \rho\wp_1(v_1, \varpi_1), u_2 + r\wp_2(\tau_2, u_2))d\rho dr.$$

Let

$$I = I_1 + I_2$$

Now, we take the integral I_1

$$I_1 = \sum_{n=0}^{\infty}\frac{Q_{\imath}(\kappa + \varkappa n, c - \kappa)(c)_{\varkappa n}(\gamma)_{\sigma n}}{Q(\kappa, c - \kappa)\Gamma(\mu n + \nu` + 1)(\rho)_{mn}}(-\Omega)^n\int_0^1\int_0^1(\rho^{\nu'}r^{\nu'})$$
$$\frac{\partial^2}{\partial\rho\partial r}(\varpi_1 + \rho\wp_1(v_1, \varpi_1), u_2 + r\wp_2(\tau_2, u_2))d\rho dr = \sum_{n=0}^{\infty}\frac{Q_{\imath}(\kappa + \varkappa n, c - \kappa)(c)_{\varkappa n}(\gamma)_{\sigma n}}{Q(\kappa, c - \kappa)\Gamma(\mu n + \nu' + 1)(\rho)_{mn}}(-\Omega)^n$$
$$\int_0^1(\rho^{\nu'})\Big[\int_0^1(r^{\nu'})\frac{\partial^2}{\partial\rho\partial r}(\varpi_1 + \rho\wp_1(v_1, \varpi_1), u_2 + r\wp_2(\tau_2, u_2))dr\Big]d\rho. \tag{44}$$

Taking the last integral on the right hand side of equation (44) and integrating by parts, we have

$$A=\int_0^1 (r^{\nu'})\frac{\partial^2}{\partial\rho\partial r}\Bigg(\varpi_1+\rho\wp_1(v_1,\varpi_1),u_2+r\wp_2(\tau_2,u_2)\Bigg)dr\frac{\partial\Phi}{\partial\rho}\Bigg(\frac{\varpi_1+\rho\wp_1(v_1,\varpi_1),u_2+\wp_2(\tau_2,u_2)}{\wp_2(\tau_2,u_2)}\Bigg)$$
$$-(\nu'+\mu n)\int_0^1 r^{\nu'+\mu n-1}\frac{\partial\Phi}{\partial\rho}\Bigg(\frac{\varpi_1+\rho\wp_1(v_1,\varpi_1),u_2+r\wp_2(\tau_2,u_2)}{\wp_2(\tau_2,u_2)}\Bigg)dr. \tag{45}$$

After simplifying equation (45) and then adding it to equation (44), we obtain

$$I_1=\sum_{n=0}^{\infty}\frac{Q_{\imath}(\kappa+\varkappa n,c-\kappa)(c)_{\varkappa n}(\gamma)_{\sigma n}}{Q(\kappa,c-\kappa)\Gamma(\mu n+\nu'+1)(\rho)_{mn}}(-\Omega)^n\int_0^1(\rho^{\nu'})\frac{\partial\Phi}{\partial\rho}\Bigg(\frac{\varpi_1+\rho\wp_1(v_1,\varpi_1),u_2+\wp_2(\tau_2,u_2)}{\wp_2(\tau_2,u_2)}\Bigg)d\rho$$
$$-(\nu'+\mu n)\int_0^1\int_0^1(\rho^{\nu'})r^{\nu'+\mu n-1}\frac{\partial\Phi}{\partial\rho}\Bigg(\frac{\varpi_1+\rho\wp_1(v_1,\varpi_1),u_2+r\wp_2(\tau_2,u_2)}{\wp_2(\tau_2,u_2)}\Bigg)drd\rho. \tag{46}$$

Taking integration by parts of the equation (46), we then have

$$I_1=\tfrac{1}{\wp_1(v_1,\varpi_1)\wp_2(\tau_2,u_2)}\Bigg[\Phi\Bigg(\varpi_1+\wp_1(v_1,\varpi_1),u_2+\wp_2(\tau_2,u_2)\Bigg)-$$
$$\tfrac{(\nu'+\mu n)\int_{\varpi_1}^{v_1}(\rho-\varpi_1)^{\nu'+\mu n-1}\Phi(\rho,\tau_2)d\rho}{(v_1-\varpi_1)^{\nu'+\mu n}}$$
$$-\tfrac{(\nu'+\mu n)\Upsilon^{(\varpi_1,u_2)^+}_{\left(\varpi_1+\wp_1(v_1,\varpi_1),u_2+\wp_2(\tau_2,u_2)\right),(\nu',\nu'-1)}(\Omega';\Phi)}{\wp_2(\tau_2,u_2)^{\nu'}\wp_1(v_1,\varpi_1)^{\nu'}}\Bigg]. \tag{47}$$

Now, we consider the integral I_2

$$I_2=\int_0^1\int_0^1(1-\rho^{\nu'})(1-r^{\nu'})\Upsilon^{\mu,\varkappa,m,\sigma,c}_{\nu',\kappa,\rho,\gamma}(\Omega\rho^{\mu}r^{\mu};\imath)\frac{\partial^2}{\partial\rho\partial r}\Bigg(\varpi_1+\rho\wp_1(v_1,\varpi_1),u_2+r\wp_2(\tau_2,u_2)\Bigg)d\rho dr. \tag{48}$$

Continuing in the same manner for solving I_2, we have

$$I_2=\frac{1}{\wp_1(v_1,\varpi_1)\wp_2(\tau_2,u_2)}\Bigg[\Phi(\varpi_1,u_2)-\frac{\nu'+\mu n}{(v_1-\varpi_1)^{\nu'+\mu n}}\int_{\varpi_1}^{v_1}(\rho-\varpi_1)^{\nu'+\mu n-1}\Phi(\rho,\tau_2)d\rho-$$
$$\frac{\nu'+\mu n}{\wp_2(\tau_2,u_2)^{\nu'}\wp_1(v_1,\varpi_1)^{\nu'}}\Upsilon^{(\varpi_1+\wp_1(v_1,\varpi_1),u_2+\wp_2(\tau_2,u_2))^-}_{(\varpi_1,u_2),(\nu',\nu'-1)}(\Omega';\Phi)\Bigg] \tag{49}$$

Adding equation (49) and equation (47), we have

$$I=\frac{\Phi(\varpi_1,u_2)+\Phi\Bigg(\varpi_1+\wp_1(v_1,\varpi_1),u_2+\wp_2(\tau_2,u_2)\Bigg)\Upsilon^{\mu,\varkappa,m,\sigma,c}_{\nu',\kappa,\rho,\gamma}(\Omega;\imath)}{\wp_1(v_1,\varpi_1)\wp_2(\tau_2,u_2)}-$$
$$\frac{\int_{\varpi_1}^{v_1}(\rho-\varpi_1)^{\nu'+\mu n-1}\Phi(\rho,\tau_2)d\rho}{(v_1-\varpi_1)^{\nu'+\mu n}\wp_1(v_1,\varpi_1)\wp_2(\tau_2,u_2)}+\frac{\int_{v_1}^{\varpi_1}(\rho-v_1)^{\nu'+\mu n-1}\Phi(\rho,u_2)d\rho}{(\varpi_1-v_1)^{\nu'+\mu n}\wp_1(v_1,\varpi_1)\wp_2(\tau_2,u_2)}$$

$$-\frac{1}{\wp_1(v_1,\varpi_1)^{\nu+1}\wp_2(\tau_2,u_2)^{\nu+1}}\left[\Lambda^{(\varpi_1+\wp(v_1,\varpi_1),u_2+\wp(\tau_2,u_2))^-}_{(\varpi_1,u_2),(\nu',\nu'-1)}(\Omega',\Phi)+\Lambda^{(\varpi_1,u_2)^+}_{(\varpi_1+\wp_1(v_1,\varpi_1),u_2+\wp_2(\tau_2,u_2)),(\nu',\nu'-1)}(\Omega',\Phi)\right] \tag{50}$$

After multiplying $\frac{\wp_1(v_1,\varpi_1)\wp_2(\tau_2,u_2)}{2}$ in the equation (50), we get the required result.

Theorem 3. *Let $\Phi : J = [\varpi_1 + \wp_1(v_1,\varpi_1), u_2 + \wp_2(\tau_2,u_2)] \in (0,\infty)$ with $J \in \mathbb{R}^2$ be a function with partial differentiability in J. Also, consider $|\frac{\partial^2\Phi}{\partial\rho\partial r}|$ 2-D h-Godunova-Levin (coordinated) pre-invex function on J, then for the generalized fractional double-integral operators in the definition* (14) *having an extended generalized Bessel-Maitland function [30] as its kernel, we have the following inequality:*

$$\begin{aligned}
&\frac{\Big[\Phi(\varpi_1,u_2)+\Phi\big(\varpi_1+\wp_1(v_1,\varpi_1),u_2+\wp_2(\tau_2,u_2)\big)\Big]\Upsilon^{\mu,\varkappa,m,\sigma,c}_{\nu',\kappa,\rho,\gamma}(\Omega;\shortmid)}{2}-\frac{\int_{\varpi_1}^{v_1}(\rho-\varpi_1)^{\nu'+\mu n-1}\Phi(\rho,\tau_2)d\rho}{2(v_1-\varpi_1)^{\nu'+\mu n}}\\
&\quad+\frac{\int_{v_1}^{\varpi_1}(\rho-v_1)^{\nu'+\mu n-1}\Phi(\rho,u_2)d\rho}{2(\varpi_1-v_1)^{\nu'+\mu n}}-\frac{1}{2\wp_1(v_1,\varpi_1)^{\nu'}\wp_2(\tau_2,u_2)^{\nu'}}\Bigg[\Lambda^{\big(\varpi_1+\wp_1(v_1,\varpi_1),u_2+\wp_2(\tau_2,u_2)\big)^-}_{(\varpi_1,u_2),(\nu',\nu'-1)}(\Omega',\Phi)\\
&\quad+\Lambda^{(\varpi_1,u_2)^+}_{\big(\varpi_1+\wp_1(v_1,\varpi_1),u_2+\wp_2(\tau_2,u_2)\big),(\nu',\nu'-1)}(\Omega',\Phi)\Bigg]\\
&\quad\le\frac{\wp_1(v_1,\varpi_1)\wp_2(\tau_2,u_2)}{2}\left(\left|\frac{\partial^2}{\partial\rho\partial r}(\varpi_1,\tau_2)\right|+\left|\frac{\partial^2}{\partial\rho\partial r}(\varpi_1,v_1)\right|+\left|\frac{\partial^2}{\partial\rho\partial r}(u_2,\tau_2)\right|+\left|\frac{\partial^2}{\partial\rho\partial r}(u_2,v_1)\right|\right)\\
&\int_0^1\int_0^1\sum_{n=0}^{\infty}\left|\frac{Q_{\shortmid}(\kappa+\varkappa n,c-\kappa)(c)_{\varkappa n}(\gamma)_{\sigma n}}{Q(\kappa,c-\kappa)\Gamma(\mu n+\nu'+1)(\rho)_{mn}}(-\Omega)^n\right|\left|\frac{(\rho^{\nu'+\mu n}r^{\nu'+\mu n})-((1-\rho)^{\nu'+\mu n}(1-r)^{\nu'+\mu n})}{h(\rho r)}\right|d\rho dr
\end{aligned}$$

Proof. Using the lemma (2) and taking the absolute value, we have

$$\begin{aligned}
&\Bigg|\frac{\Big[\Phi(\varpi_1,u_2)+\Phi(\varpi_1+\wp_1(v_1,\varpi_1),u_2+\wp_2(\tau_2,u_2))\Big]\Upsilon^{\mu,\varkappa,m,\sigma,c}_{\nu',\kappa,\rho,\gamma}(\Omega;\shortmid)}{2}-\\
&\frac{\int_{\varpi_1}^{v_1}(\rho-\varpi_1)^{\nu'+\mu n-1}\Phi(\rho,\tau_2)d\rho}{2(v_1-\varpi_1)^{\nu'+\mu n}}+\frac{\int_{v_1}^{\varpi_1}(\rho-v_1)^{\nu'+\mu n-1}\Phi(\rho,u_2)d\rho}{2(\varpi_1-v_1)^{\nu'+\mu n}}-\\
&\frac{\Lambda^{(\varpi_1+\wp_1(v_1,\varpi_1),u_2+\wp_2(\tau_2,u_2))^-}_{(\varpi_1,u_2),(\nu',\nu'-1)}(\Omega',\Phi)+\Lambda^{(\varpi_1,u_2)^+}_{(\varpi_1+\wp_1(v_1,\varpi_1),u_2+\wp_2(\tau_2,u_2)),(\nu',\nu'-1)}(\Omega',\Phi)}{2\wp_1(v_1,\varpi_1)^{\nu}\wp_2(\tau_2,u_2)^{\nu}}\Bigg|\\
&=\left|\frac{\wp_1(v_1,\varpi_1)\wp_2(\tau_2,u_2)}{2}I\right|
\end{aligned}$$

Replacing the value of I, which is in lemma (2), and after applying the modulus property, we have

$$\leq \frac{\wp_1(v_1,\varpi_1)\wp_2(\tau_2,u_2)}{2}\sum_{n=0}^{\infty}\left|\frac{Q_{\shortmid}(\kappa+\varkappa n,c-\kappa)(c)_{\varkappa n}(\gamma)_{\sigma n}}{Q(\kappa,c-\kappa)\Gamma(\mu n+\nu'+1)(\rho)_{mn}}(-\Omega)^n\right|$$
$$\int_0^1\int_0^1\left|\rho^{\nu'+\mu n}r^{\nu'+\mu n}-(1-\rho)^{\nu'+\mu n}(1-r)^{\nu'+\mu n}\right|$$
$$\left|\frac{\partial^2}{\partial\rho\partial r}(\varpi_1+\rho\wp_1(v_1,\varpi_1),u_2+r\wp_2(\tau_2,u_2))\right|d\rho dr$$

$$\leq \frac{\wp_1(v_1,\varpi_1)\wp_2(\tau_2,u_2)}{2}\sum_{n=0}^{\infty}\left|\frac{Q_{\shortmid}(\kappa+\varkappa n,c-\kappa)(c)_{\varkappa n}(\gamma)_{\sigma n}(-\Omega)^n}{Q(\kappa,c-\kappa)\Gamma(\mu n+\nu'+1)(\rho)_{mn}}\right.$$
$$\left|\int_0^1\int_0^1\left|(\rho^{\nu'+\mu n}r^{\nu'+\mu n})-((1-\rho)^{\nu'+\mu n}(1-r)^{\nu'+\mu n})\right|\right.$$
$$\left|\frac{\frac{\partial^2}{\partial\rho\partial r}(\varpi_1,\tau_2)}{h[(1-\rho)(1-r)]}+\frac{\frac{\partial^2}{\partial\rho\partial r}(\varpi_1,v_1)}{h[(1-\rho)(r)]}+\frac{\frac{\partial^2}{\partial\rho\partial r}(u_2,\tau_2)}{h[(\rho)(1-r)]}+\frac{\frac{\partial^2}{\partial\rho\partial r}(u_2,v_1)}{h[(\rho)(r)]}\right|d\rho dr$$

$$\leq \frac{\wp_1(v_1,\varpi_1)\wp_2(\tau_2,u_2)}{2}\sum_{n=0}^{\infty}\left|\frac{Q_{\shortmid}(\kappa+\varkappa n,c-\kappa)(c)_{\varkappa n}(\gamma)_{\sigma n}}{Q(\kappa,c-\kappa)\Gamma(\mu n+\nu'+1)(\rho)_{mn}}(-\Omega)^n\right|\times$$
$$\left[\left|\frac{\partial^2}{\partial\rho\partial r}(\varpi_1,\tau_2)\right|\int_0^1\int_0^1\left|(\rho^{\nu'+\mu n}r^{\nu'+\mu n})-((1-\rho)^{\nu'+\mu n}(1-r)^{\nu'+\mu n})\right|\left|\frac{1}{h[(1-\rho)(1-r)]}\right|d\rho dr\right.$$
$$+\left|\frac{\partial^2}{\partial\rho\partial r}(\varpi_1,v_1)\right|\int_0^1\int_0^1\left|(\rho^{\nu'+\mu n}r^{\nu'+\mu n})-((1-\rho)^{\nu'+\mu n}(1-r)^{\nu'+\mu n})\right|\left|\frac{1}{h[(1-\rho)(r)]}\right|d\rho dr$$
$$+\left|\frac{\partial^2}{\partial\rho\partial r}(u_2,\tau_2)\right|\int_0^1\int_0^1\left|(\rho^{\nu'+\mu n}r^{\nu'+\mu n})-((1-\rho)^{\nu'+\mu n}(1-r)^{\nu'+\mu n})\right|\left|\frac{1}{h[(\rho)(1-r)]}\right|d\rho dr$$
$$\left.+\left|\frac{\partial^2}{\partial\rho\partial r}(u_2,v_1)\right|\int_0^1\int_0^1\left|(\rho^{\nu'+\mu n}r^{\nu'+\mu n})-((1-\rho)^{\nu'+\mu n}(1-r)^{\nu'+\mu n})\right|\left|\frac{1}{h[(\rho)(r)]}\right|d\rho dr\right]$$

$$=\frac{\wp_1(v_1,\varpi_1)\wp_2(\tau_2,u_2)}{2}\left(\left|\frac{\partial^2}{\partial\rho\partial r}(\varpi_1,\tau_2)\right|+\left|\frac{\partial^2}{\partial\rho\partial r}(\varpi_1,v_1)\right|+\left|\frac{\partial^2}{\partial\rho\partial r}(u_2,\tau_2)\right|+\left|\frac{\partial^2}{\partial\rho\partial r}(u_2,v_1)\right|\right)$$
$$\int_0^1\int_0^1\sum_{n=0}^{\infty}\left|\frac{Q_p(\kappa+\varkappa n,c-\kappa)(c)_{\varkappa n}(\gamma)_{\sigma n}}{Q(\kappa,c-\kappa)\Gamma(\mu n+\nu'+1)(\rho)_{mn}}(-\Omega)^n\right|$$
$$\left|\frac{(\rho^{\nu'+\mu n}r^{\nu'+\mu n})-((1-\rho)^{\nu'+\mu n}(1-r)^{\nu'+\mu n})}{h[(\rho)(r)]}\right|d\rho dr.$$

Corollary 7. *If we replace the values* $\wp_1(v_1,\varpi_1)=v_1-\varpi_1$ *and* $\wp_2(\tau_2,u_2)=\tau_2-u_2$ *in theorem [3], we have*

$$\Bigg|\frac{\Phi(\varpi_1,u_2)+\Phi(v_1,\tau_2)\Upsilon^{\mu,\varkappa,m,\sigma,c}_{\nu',\kappa,\rho,\gamma}(\Omega;\shortmid)}{2}-\frac{\int_{\varpi_1}^{v_1}(\rho-\varpi_1)^{\nu'+\mu n-1}\Phi(\rho,\tau_2)d\rho}{2(v_1-\varpi_1)^{\nu'+\mu n}}+$$
$$\frac{\int_{v_1}^{\varpi_1}(\rho-v_1)^{\nu'+\mu n-1}\Phi(\rho,u_2)d\rho}{2(\varpi_1-v_1)^{\nu'+\mu n}}-$$
$$\frac{\Lambda^{\big(\varpi_1+\wp(v_1,\varpi_1),u_2+\wp(\tau_2,u_2)\big)^-}_{(\varpi_1,u_2),(\nu',\nu'-1)}(\Omega',\Phi)+\Lambda^{(\varpi_1,u_2)^+}_{\big(\varpi_1+\wp_1(v_1,\varpi_1),u_2+\wp_2(\tau_2,u_2)\big),(\nu',\nu'-1)}(\Omega',\Phi)}{2(v_1-\varpi_1)^{\nu'}(\tau_2-u_2)^{\nu'}}\Bigg|$$

$$\leq\frac{(v_1-\varpi_1)(\tau_2-u_2)}{2}\left(\left|\frac{\partial^2}{\partial\rho\partial r}(\varpi_1,\tau_2)\right|+\left|\frac{\partial^2}{\partial\rho\partial r}(\varpi_1,v_1)\right|+\left|\frac{\partial^2}{\partial\rho\partial r}(u_2,\tau_2)\right|+\left|\frac{\partial^2}{\partial\rho\partial r}(u_2,v_1)\right|\right)$$
$$\int_0^1\int_0^1\sum_{n=0}^{\infty}\left|\frac{Q_{\shortmid}(\kappa+\varkappa n,c-\kappa)(c)_{\varkappa n}(\gamma)_{\sigma n}}{Q(\kappa,c-\kappa)\Gamma(\mu n+\nu'+1)(\rho)_{mn}}(-\Omega)^n\right|$$
$$\left|\frac{(\rho^{\nu'+\mu n}r^{\nu'+\mu n})-((1-\rho)^{\nu'+\mu n}(1-r)^{\nu'+\mu n})}{h[(\rho)(r)]}\right|d\rho dr.$$

Theorem 4. *Let* $\Phi:J=[\varpi_1+\wp_1(v_1,\varpi_1),u_2+\wp_2(\tau_2,u_2)]\to(0,\infty)$ *be a function,* $J\in\mathbb{R}^2$ *and a partial differentiable function on* J*. Also, let* $|\frac{\partial^2\Phi}{\partial\rho\partial r}|^q$ *be a 2-D h-Godunova-Levin coordinated preinvex function on* J *with* $p>1$ *and* $q=\frac{p}{(p-1)}$*, then for the generalized fractional double-integral operators in definition* (14) *having extended generalized Bessel-Maitland function [30] as its kernel, we have the following inequality:*

$$\Bigg|\frac{\Big[\Phi(\varpi_1,u_2)+\Phi\big(\varpi_1+\wp_1(v_1,\varpi_1),u_2+\wp_2(\tau_2,u_2)\big)\Big]\Upsilon^{\mu,\varkappa,m,\sigma,c}_{\nu',\kappa,\rho,\gamma}(\Omega;\shortmid)}{2}-$$
$$\frac{\int_{\varpi_1}^{v_1}(\rho-\varpi_1)^{\nu'+\mu n-1}\Phi(\rho,\tau_2)d\rho}{2(v_1-\varpi_1)^{\nu'+\mu n}}+\frac{\int_{v_1}^{\varpi_1}(\rho-v_1)^{\nu'+\mu n-1}\Phi(\rho,u_2)d\rho}{2(\varpi_1-v_1)^{\nu'+\mu n}}-$$
$$\frac{\Lambda^{\big(\varpi_1+\wp_1(v_1,\varpi_1),u_2+\wp_2(\tau_2,u_2)\big)^-}_{(\varpi_1,u_2),(\nu',\nu'-1)}(\Omega',\Phi)+\Lambda^{(\varpi_1,u_2)^+}_{\big(\varpi_1+\wp_1(v_1,\varpi_1),u_2+\wp_2(\tau_2,u_2)\big),(\nu',\nu'-1)}(\Omega',\Phi)}{2\wp_1(v_1,\varpi_1)^{\nu}\wp_2(\tau_2,u_2)^{\nu}}\Bigg|$$

$$\leq\frac{\wp_1(v_1,\varpi_1)\wp_2(\tau_2,u_2)}{2}\left(\left|\frac{\partial^2\Phi}{\partial\rho\partial r}(\varpi_1,\tau_2)\right|^q+\left|\frac{\partial^2\Phi}{\partial\rho\partial r}(\varpi_1,v_1)\right|^q+\left|\frac{\partial^2\Phi}{\partial\rho\partial r}(u_2,\tau_2)\right|^q+\left|\frac{\partial^2\Phi}{\partial\rho\partial r}(u_2,v_1)\right|^q\right)^{\frac{1}{q}}$$
$$\frac{\wp_1(v_1,\varpi_1)\wp_2(\tau_2,u_2)}{2}\Bigg(\int_0^1\int_0^1\Big|(\rho^{\nu'+\mu n}r^{\nu'+\mu n})\Upsilon^{\mu,\varkappa,m,\sigma,c}_{\nu',\kappa,\rho,\gamma}(\Omega\rho^{\mu}r^{\mu};\shortmid)-(1-\rho)^{\nu'+\mu n}(1-r)^{\nu'+\mu n}$$
$$\Upsilon^{\mu,\varkappa,m,\sigma,c}_{\nu',\kappa,\rho,\gamma}(\Omega(1-\rho)^{\mu}(1-r)^{\mu};\shortmid)\Big|^p d\rho dr\Bigg)^{\frac{1}{p}}\left(\int_0^1\int_0^1\frac{1}{h(\rho r)}d\rho dr\right)^{\frac{1}{q}}.$$

Proof. Using the lemma (2) and taking the absolute value, we have

$$\Bigg|\frac{\Big[\Phi(\varpi_1,u_2)+\Phi(\varpi_1+\wp_1(v_1,\varpi_1),u_2+\wp_2(\tau_2,u_2))\Big]\Upsilon^{\mu,\varkappa,m,\sigma,c}_{\nu',\kappa,\rho,\gamma}(\Omega;\rceil)}{2}-$$

$$\frac{\int_{\varpi_1}^{v_1}(\rho-\varpi_1)^{\nu'+\mu n-1}\Phi(\rho,\tau_2)d\rho}{2(v_1-\varpi_1)^{\nu'+\mu n}}+\frac{\int_{v_1}^{\varpi_1}(\rho-v_1)^{\nu'+\mu n-1}\Phi(\rho,u_2)d\rho}{2(\varpi_1-v_1)^{\nu'+\mu n}}-$$

$$\frac{\Lambda^{(\varpi_1+\wp_1(v_1,\varpi_1),u_2+\wp_2(\tau_2,u_2))^-}_{(\varpi_1,u_2),(\nu',\nu'-1)}(\Omega',\Phi)+\Lambda^{(\varpi_1,u_2)^+}_{(\varpi_1+\wp_1(v_1,\varpi_1),u_2+\wp_2(\tau_2,u_2)),(\nu',\nu'-1)}(\Omega',\Phi)}{2\wp_1(v_1,\varpi_1)^{\nu}\wp_2(\tau_2,u_2)^{\nu}}\Bigg|$$

$$=\left|\frac{\wp_1(v_1,\varpi_1)\wp_2(\tau_2,u_2)}{2}I\right|$$

$$\leq\frac{\wp_1(v_1,\varpi_1)\wp_2(\tau_2,u_2)}{2}\int_0^1\int_0^1\Big|(\rho^{\nu'+\mu n}r^{\nu'+\mu n})\Upsilon^{\mu,\varkappa,m,\sigma,c}_{\nu',\kappa,\rho,\gamma}(\Omega\rho^{\mu}r^{\mu};\rceil)-(1-\rho)^{\nu'+\mu n}(1-r)^{\nu'+\mu n}$$

$$\Upsilon^{\mu,\varkappa,m,\sigma,c}_{\nu',\kappa,\rho,\gamma}(\Omega(1-\rho)^{\mu}(1-r)^{\mu};\rceil)\Big|\Big|\frac{\partial^2\Phi}{\partial\rho\partial r}\big(\varpi_1+\rho\wp_1(v_1,\varpi_1),u_2+r\wp_2(\tau_2,u_2)\big)\Big|d\rho dr \quad (51)$$

Applying the integral inequality Hölder's in equation (51), we have the following.

$$\leq\frac{\wp_1(v_1,\varpi_1)\wp_2(\tau_2,u_2)}{2}\Bigg[\int_0^1\int_0^1\Big|(\rho^{\nu'+\mu n}r^{\nu'+\mu n})\Upsilon^{\mu,\varkappa,m,\sigma,c}_{\nu',\kappa,\rho,\gamma}(\Omega\rho^{\mu}r^{\mu};\rceil)-((1-\rho)^{\nu'+\mu n}$$

$$(1-r)^{\nu'+\mu n}\Upsilon^{\mu,\varkappa,m,\sigma,c}_{\nu',\kappa,\rho,\gamma}(\Omega(1-\rho)^{\mu}(1-r)^{\mu};\rceil))\Big|^{p}d\rho dr\Bigg]^{\frac{1}{p}}$$

$$\Bigg[\int_0^1\int_0^1\Big|\frac{\partial^2\Phi}{\partial\rho\partial r}\big(\varpi_1+\rho\wp_1(v_1,\varpi_1),u_2+r\wp_2(\tau_2,u_2)\big)\Big|^{q}\Bigg]^{\frac{1}{q}}d\rho dr, \quad (52)$$

where $\frac{1}{p}+\frac{1}{q}=1$

Now, using the definition of two dimension h- Godunova-Levin (coordinated) pre-invex, we have

$$\int_0^1\int_0^1\Big|\frac{\partial^2}{\partial\rho\partial r}\big(\varpi_1+\rho\wp_1(v_1,\varpi_1),u_2+r\wp_2(\tau_2,u_2)\big)\Big|^{q}d\rho dr$$

$$\leq\int_0^1\int_0^1\left(\frac{\left|\frac{\partial^2\Phi}{\partial\rho\partial r}(\varpi_1,\tau_2)\right|^q}{h[(1-\rho)(1-r)]}+\frac{\left|\frac{\partial^2\Phi}{\partial\rho\partial r}(\varpi_1,v_1)\right|^q}{h[(1-\rho)(r)]}+\frac{\left|\frac{\partial^2\Phi}{\partial\rho\partial r}(u_2,\tau_2)\right|^q}{h[(\rho)(1-r)]}+\frac{\left|\frac{\partial^2\Phi}{\partial\rho\partial r}(u_2,v_1)\right|^q}{h[(\rho)(r)]}\right)d\rho dr$$

$$\leq\left(\left|\frac{\partial^2\Phi}{\partial\rho\partial r}(\varpi_1,\tau_2)\right|^q+\left|\frac{\partial^2\Phi}{\partial\rho\partial r}(\varpi_1,v_1)\right|^q+\left|\frac{\partial^2\Phi}{\partial\rho\partial r}(u_2,\tau_2)\right|^q+\left|\frac{\partial^2\Phi}{\partial\rho\partial r}(u_2,v_1)\right|^q\right)$$

$$\int_0^1\int_0^1\frac{1}{h[(\rho)(r)]}d\rho dr.$$

5. Conclusion

In this research work, we discussed the coordinated [two-dimensional; (2-D)] convexity, pre-invexity and also described the new generation of fractional double integral operators (FDIO) having extended version of the Bessel-Maitland function as its kernel. The refinements of Hermite-Hadamard and trapezoid type inequalities are discussed with the coordinated convexity by implementation of newly developed fractional double integral operators. Also, we discussed the corollaries, which show the strengthened the idea of our main results. We concluded that the fractional operators can be modified not only in aspects of their kernel but also generalized by its dimensions of fractional integrals (double-integrals). Similarly, convexity and pre-invexity can also be generalized by their dimensions. The extensions of generalization of well-known inequalities and such type refinements could be possible by successfully implementation of FDIO with coordinated convexity and pre-invexities. These significant results of (2-D)-convexity and the new version of fractional operators make a great contribution to exploring new families of inequalities.

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Availability of data and material

Data sharing is not applicable to this paper as no datasets were generated or analyzed during the current study.

Competing interests

The authors declare that there is no conflict of interest regarding the publication of this paper.

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