EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2026, Vol. 19, Issue 1, Article Number 7371
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

On the Diophantine Equation $L_n - L_m = 11 \cdot 2^a$

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Abstract. Using Baker's method, we completely solve the title equation in positive integers n, m and a for L_n the nth Lucas number.

2020 Mathematics Subject Classifications: 11B39, 11D61

Key Words and Phrases: Diophantine equation, lucas number, Baker's method

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1. Introduction

The Fibonacci sequence (F_n) can be defined recursively as $F_0 = 0, F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. The Lucas sequence (L_n) is defined using the same recurrence relation as the Fibonacci sequence with initial conditions $L_0 = 2, L_1 = 1$. They are two of the most well-studied second order linear recursive sequences. Finding all square terms and, more generally, perfect powers in the Fibonacci and Lucas sequences is a problem which received particular attention. In 2006, Bugeaud, Mignotte and Siksek [1] showed that the only perfect powers in the Fibonacci and Lucas sequences are $F_0 = 0, F_1 = F_2 = 1, F_6 = 8 = 2^3, F_{12} = 144 = 12^2$, and $L_1 = 1, L_3 = 4 = 2^2$, respectively. In the last decade, some exponential Diophantine equations containing the terms of second order linear recurrences sequences have been studied. For example, the Diophantine equation $L_m + L_n = 2^a$ was solved by Bravo and Luca [2]. Two years later, the same authors solved the equation $F_n + F_m = 2^a$. Meanwhile, the equation $F_n + F_m + F_l = 2^a$ has been solved by E. F. Bravo and J. J. Bravo [3]. In [4], Pink and Ziegler dealt with the more general Diophantine equation $u_n + u_m = wp_1^{Z_1} p_2^{Z_2} \cdots p_s^{Z_s}$ and they solved this equation in the case that $w = 1, p_1, \cdots p_{46}$ are all prime numbers, less than 200 and u_n is the Fibonacci sequence or Lucas sequence. In [5], Şiaf and Keskin solved $F_n - F_m = 2^a$, and the more general equation $F_n \pm F_m = y^a$ for positive integers y and a has been studied by Kebli, Kihel, Larone and Luca in [6] and by Kihel and Larone

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DOI: https://doi.org/10.29020/nybg.ejpam.v19i1.7371

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in [7]. And there are another studied case about Diophantine equation by Bitim in [8] , by Şiaf and Keskin in [9] and by Gaha and Mezroui in [10].

In this paper, we consider the equation

$$L_n - L_m = 11 \cdot 2^a, \tag{1}$$

and find all solutions $n, m,$ and a in positive integers. This study can be viewed as a continuation of the previous works on this subject. In Section 2, we introduce necessary preliminary results. In Section 3 we obtain a large bound on n, then in Section 4 we reduce this bound to one that allows a brute force search to finish the proof of Theorem 1. Calculations done for this paper were carried out using SageMath [11] and can be viewed on CoCalc.

Theorem 1. *The only solutions of the equation $L_n - L_m = 11 \cdot 2^a$ in non-negative integers (n, m, a) with $n > m$ are*

$$(n, m, a) \in \{(6, 4, 0), (7, 6, 0), (7, 4, 1), (8, 2, 2)\}.$$

2. Auxiliary Results

Let $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ denote the roots of the polynomial $x^2 - x - 1$. We can write

$$L_n = \alpha^n + \beta^n.$$

Note that $\frac{3}{2} < \alpha < \frac{5}{3}$ and $-\frac{2}{3} < \beta < -\frac{1}{2}$ and that $\alpha^n = \frac{L_n + F_n\sqrt{5}}{2}$. The Lucas numbers can be bounded between expressions in α:

$$\alpha^{n-1} \leq L_n \leq 2\alpha^n \text{ for } n \geq 0. \tag{2}$$

where we can proved the inequality (2) by induction

Let η be an algebraic number of degree d with minimal polynomial

$$a_0 x^d + a_1 x^{d-1} + \cdots + a_d = a_0 \prod_{i=1}^{d} \left(X - \eta^{(i)}\right) \in \mathbb{Z}[x]$$

where the a_i are integers with $a_0 > 0$ and $\eta^{(i)}$ are the conjugates of η. Then

$$\mathrm{h}(\eta) = \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log\left(\max\left\{\left|\eta^{(i)}\right|, 1\right\}\right)\right)$$

is called the logarithmic height of η. If $\eta = a \,/\, b$ is a rational number with $\gcd(a, b) = 1$ and $b \geq 1$, then $\mathrm{h}(\eta) = \log(\max\{|a|, b\})$. The following properties of the logarithmic height are widely used:

$$\mathrm{h}(\eta \pm \gamma) \leq \mathrm{h}(\eta) + h(\gamma) + \log(2)$$

$$\mathrm{h}\left(\eta\gamma^{\pm 1}\right) \leq \mathrm{h}(\eta) + \mathrm{h}(\gamma) \tag{3}$$
$$\mathrm{h}\left(\eta^{s}\right) = |s|\,\mathrm{h}(\eta).$$

where s denotes any integer.

The following theorem of Matveev [12] will provide us with lower bounds for our linear forms in three logarithms.

Theorem 2 (Matveev [12]). *Let $n \in \mathbb{Z}^{+}$. Let $\mathbb{L}$ be a real number field of degree D and let $\eta_1, \ldots, \eta_l$ be non-zero elements of $\mathbb{L}$. Let $b_1, b_2, \ldots, b_l$ be integers and define*

$$B := \max\left\{|b_1|, \ldots, |b_l|\right\}$$

and

$$\Lambda := \eta_1^{b_1} \cdots \eta_l^{b_l} - 1 = \left(\prod_{i=1}^{l} \eta_i^{b_i}\right) - 1.$$

Let $A_1, \ldots, A_l$ be real numbers such that

$$A_j \geq \max\left\{D\mathrm{h}\left(\eta_j\right), \left|\log\left(\eta_j\right)\right|, 0.16\right\}, 1 \leq j \leq l.$$

Assume $\Lambda \neq 0$. Then we have

$$\log|\Lambda| > -1.4 \times 30^{l+3} \times l^{4.5} \times D^2 \times A_1 \cdots A_l (1 + \log D)(1 + \log B).$$

We will employ the following version of the Baker-Davenport reduction method due to Bravo, Gómez and Luca. Here, $\|\cdot\|$ will denote the distance from x to the nearest integer, that is, $\|x\| = \min\{|x - n| : n \in \mathbb{Z}\}$ for any real number x.

Lemma 1 (Bravo, Gómez and Luca [13]). *Let N be a positive integer. Let p/q be a convergent of the continued fraction expansion of the irrational κ such that $q > 6N$, and let A, B, μ be real numbers with $A > 0$ and $B > 1$. Furthermore, let $\varepsilon = ||\mu q|| - N \cdot ||\kappa q||$, where $||\cdot||$ denotes the distance from the nearest integer. If $\varepsilon > 0$, then there is no solution of the inequality*

$$0 < |u\kappa - v + \mu| < AB^{-w}$$

in positive integers u, v and w with

$$u \leq N \quad \textit{and} \quad w \geq \frac{\log\left(Aq/\varepsilon\right)}{\log B}.$$

3. Initial Bounds on $n - m$ and n

Let (n, m, a) be a solution to equation (1). Certainly $n > m$ with $1 \leq m$, $n \leq 100$, and if $n - m = 1$, the equation becomes $L_{n-2} = 11 \cdot 2^a$, which has been shown [14] to have only the solution $n = 7, a = 0$, and so results in the solution $(7, 6, 0)$ to (1). When $n - m = 2$, we get $L_n - L_{n-2} = L_{n-1}$, and so we conclude $n = 6$ and add $(6, 4, 0)$ to our list of solutions. When $n - m = 3$, equation (1) becomes $L_n - L_{n-3} = 2L_{n-2} = 11 \cdot 2^a$

and so $a > 0$. Dividing by 2 gives $L_{n-2} = 11 \cdot 2^{a-1}$, which similarly has only the solution $n = 7, a = 1$, giving us the solution $(7, 4, 1)$. When $n - m = 4$, we have $L_n - L_{n-4} = 5F_{n-2}$, and so it is $0 \pmod 5$ and can never be a solution to (1). For the rest of this section, we will assume that $n - m > 4$ and, since $n, m > 0$, also $n > 5$. By (1) and (2), we obtain the inequality

$$2^a < 11 \cdot 2^a = L_n - L_m < L_n < 2\alpha^n < 2^{n+1}$$

and so we can assume $a \leq n$.

3.1. A Bound on $n - m$

We will use Theorem 2 to obtain a bound on the difference $n - m$. To set up an appropriate linear form in logarithms, we begin by rearranging equation (1) as $\alpha^n - 11 \cdot 2^a = L_m - \beta^n$ and taking absolute values, we get

$$|\alpha^n - 11 \cdot 2^a| = |L_m - \beta^n| \leq L_m + |\beta|^n < 2\alpha^m + \frac{1}{2}$$

for $n > 0$, where we used inequality (2). If we multiply both sides of the above inequality by $1/\alpha^n$, we obtain

$$\left|1 - \alpha^{-n} \cdot 11 \cdot 2^a\right| < \frac{\alpha^{-n}}{2} + 2\alpha^{m-n} < \frac{1}{\alpha^{n-m}} + \frac{2}{\alpha^{n-m}} = \frac{3}{\alpha^{n-m}}, \tag{4}$$

where we used the facts $\alpha^m > 1/2$ and $n > m$.

In the notation of Theorem 2 Let $\gamma_1 := 11, \gamma_2 := \alpha, \gamma_3 := 2$ and $b_1 := 1, b_2 := -n, b_3 := a$. The numbers γ_i for $i = 1, 2, 3$ are positive real numbers and elements of the field $\mathbb{F} = \mathbb{Q}(\sqrt{5})$, so $D = 2$. The number $\Lambda_1 := 11 \cdot 2^a \alpha^{-n} - 1$ is nonzero. Indeed, if $\Lambda_1 = 0$ then we get

$$\alpha^n = \frac{L_n + F_n\sqrt{5}}{2} = 11 \cdot 2^a,$$

and the left side of this equality is only rational if $F_n = 0$, which is never true for $n > 0$. Since $\mathrm{h}(\gamma_1) = \log 11, \mathrm{h}(\gamma_2) = \frac{\log \alpha}{2} = 0.2406\ldots$ and $\mathrm{h}(\gamma_3) = \log 2$, we can take $A_1 := 2.2, A_2 := 0.5, A_3 := 1.4$. Since $a \leq n$ we can take $B := \max\{|a|, |-n|, 1\} = n$. Using inequality (4) and Theorem 2 we get

$$\frac{3}{\alpha^{n-m}} > |\Lambda_1| > \exp\left(-1.4 \cdot 30^{3+3} \cdot 3^{4.5} \cdot 2^2(1 + \log 2)(1 + \log n)(1 + \log n) \cdot 2.2 \cdot 0.5 \cdot 1.4\right),$$

and so

$$(n-m)\log\alpha - \log 3 < 1.4 \cdot 30^{3+3} \cdot 3^{4.5} \cdot 2^2(1+\log 2)(1+\log n)(1+\log n) \cdot 2.2 \cdot 0.5 \cdot 1.4. \tag{5}$$

3.2. The Second Linear Form in Logarithms

We now derive a second linear form in logarithms which, after another application of Theorem 2 and the substitution of the bound on $n-m$ in (5), will result in an absolute upper bound on n. Rearrange equation (1) as $\alpha^n - \alpha^m - 11 \cdot 2^a = -\beta^n + \beta^m$. An application of the triangle inequality yields

$$\left|\alpha^n \left(1-\alpha^{m-n}\right) - 11 \cdot 2^a\right| = |-\beta^n + \beta^m| \leq |\beta^n| + |\beta^m| < 1,$$

where the rightmost inequality follows from the fact that $|\beta^n| + |\beta^m| < 1$ for $n > 5$ and $n - m > 4$. Multiplying both sides of the above inequality by $\alpha^n (1-\alpha^{n-m})$, we obtain

$$\left|1 - \alpha^{-n} \cdot 11 \cdot 2^a \left(1-\alpha^{m-n}\right)^{-1}\right| < \frac{1}{\alpha^n \left(1-\alpha^{m-n}\right)}. \tag{6}$$

Since

$$\alpha^{m-n} = \frac{1}{\alpha^{n-m}} < \frac{1}{\alpha} < \frac{2}{3},$$

we have

$$\frac{1}{1-\alpha^{m-n}} < \frac{3}{2} < 3.$$

From (6), it follows that

$$\left|1 - \alpha^{-n} \cdot 11 \cdot 2^a \left(1-\alpha^{m-n}\right)^{-1}\right| < \frac{3}{\alpha^n}. \tag{7}$$

Since

$$\alpha^n - \alpha^m = \frac{L_n - L_m + (F_n - F_m)\sqrt{5}}{2},$$

is only ever an integer when $F_n = F_m$, which does not hold when $n-m > 4$ as we assume, the value $\Lambda_2 := \alpha^{-n} \cdot 11 \cdot 2^a \left(1-\alpha^{m-n}\right)^{-1} - 1$ is nonzero. Taking $\gamma_1 := \alpha, \gamma_2 := 2, \gamma_3 := 11\left(1-\alpha^{m-n}\right)$ and $b_1 := -n, b_2 := a, b_3 := 1$, we can see that all three γ are positive real numbers in the field $\mathbb{F} = \mathbb{Q}(\sqrt{5})$, so $D = 2$. In the same way, since $h(\gamma_1) = \frac{\log \alpha}{2} = 0.2406\ldots$ and $h(\gamma_2) = \log 2$ by (2), we can take $A_1 := 0.5, A_2 := 1.4$. Using the properties of the height in (2) gives $h(\gamma_3) \leq \log 6 + (n-m)\log\alpha$ and so we take $A_3 := \log 36 + (n-m)\log\alpha$. Finally, since $a \leq n$, it follows that $B := \max\{|a|, |-n|, 1\} = n$. Applying Theorem 2 to equality (7), we obtain

$$\frac{3}{\alpha^n} > |\Lambda_2| > \exp((-C)(1+\log 2)(1+\log n) \cdot 0.5 \cdot 1.4 \cdot (\log 36 + (n-m)(\log\alpha))$$

or, after taking logarithms,

$$n\log\alpha - \log 3 < (-C)(1+\log 2)(1+\log n) \cdot 0.5 \cdot 1.4 \cdot (\log 36 + (n-m)(\log\alpha) \tag{8}$$

where $C = 1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2$. Applying the inequality (5) into the previous inequality, a programming search gave us that $n < 8.4 \cdot 10^{27}$.

4. Reducing the Bounds

We apply the inequality $|x| < 2\,|e^x - 1|$ to Λ_1. Note that this inequality is valid for real values of x satisfying $-1.5 \leq x < 0$ or $x > 0$. The bound of -1.5 is not sharp and could be made precise using the Lambert W function. This gives

$$|n \log \alpha - a \log 2 - \log 11| < 2|\Lambda_1| < \frac{6}{\alpha^{n-m}}.$$

Divide by $\log 2$ to obtain

$$\left| n\frac{\log \alpha}{\log 2} - a - \frac{\log 11}{\log 2} \right| < \frac{6}{(\log 2)\alpha^{n-m}}. \tag{9}$$

Using the fact that the bound $n < 8.4 \times 10^{27}$ has already been established, we apply Lemma 1 with the following values.

$$u = n, \quad v = a, \quad w = n - m, \quad A = \frac{6}{\log 2}, \quad B = \alpha,$$
$$\kappa = \frac{\log \alpha}{\log 2}, \quad \mu = \frac{\log \frac{1}{11}}{\log 2}, \quad N = 8.4 \times 10^{27}.$$

Using SageMath, we find that the 65th convergent is the first with denominator greater than $6N$ and we calculate $q_{65} = 133370345034021137584089207921$. The associated value of ε, is positive

$$\varepsilon = ||\mu q_{65}|| - N \cdot ||\kappa q_{65}|| = 0.288018808887095,$$

so we have the bound

$$w = n - m < \frac{\log (Aq/\varepsilon)}{\log B} < 146.43433.$$

We substitute $n - m \leq 146$ into (8) to get an improved bound on n, finding $n < 3.85 \times 10^{15}$. We can apply Lemma 1 again with the same values for every parameter except the updated value $N = 3.85 \times 10^{15}$. This gives us an improved bound of $w = n - m < 87.86$. Using $n - m \leq 87$ in inequality (12) gives a slightly improved bound on n of $n < 2.34 \times 10^{15}$, but further iteration does not give us any more improvements.

Now we apply Lemma 1 to Λ_2. Using $|x| < 2|e^x - 1|$ again, we get

$$\left| n \log \alpha - a \log 2 - \log \frac{11}{1 - \alpha^{m-n}} \right| < 2|\Lambda_1| < \frac{6}{\alpha^n}.$$

Dividing by $\log 2$ again gives an inequality similar to (9),

$$\left| n\frac{\log \alpha}{\log 2} - a + \frac{\log \frac{1-\alpha^{m-n}}{11}}{\log 2} \right| < \frac{6}{(\log 2)\alpha^n}. \tag{10}$$

We now use Lemma 1 another 83 times, once for each value of $\mu = \frac{\log \frac{1-\alpha^{m-n}}{11}}{\log 2}$ with $5 \leq n - m \leq 87$. The values of κ, A and B are all the same as before, however this time $u = w = n$ and $N = 2.34 \times 10^{15}$. In all cases other than $n - m = 10$, the denominator of the 37th convergent, $q_{37} = 78462338394551841$, is larger than $6N$ and yields a positive value of ε, and Lemma 1 tells us that (10) has no solutions with $n < 2.34 \times 10^{15}$ and $n \geq 93.5$. We conclude that $n \leq 93$ in these cases.

When $n - m = 10$, we have $\frac{\log \frac{1-\alpha^{m-n}}{11}}{\log 2} = -5\frac{\log \alpha}{\log 2}$, and the value of ε is always negative. In this situation, we write inequality (10) as

$$\left| \frac{\log \alpha}{\log 2} - \frac{a}{n-5} \right| < \frac{6}{(n-5)(\log 2)\alpha^n} < \frac{1}{2(n-5)^2}. \tag{11}$$

By a criterion of Legendre, this tells us that $\frac{a}{n-5}$ must be a convergent of $\kappa = \frac{\log \alpha}{\log 2}$. By our most recent bound on n, we know that it must be one of the first 37 convergents of κ. Moreover, supposing it is the ℓth convergent, it will also satisfy

$$\frac{1}{(a_\ell + 2)(n-5)^2} < \left| \kappa - \frac{a}{n-5} \right|,$$

for $a_{\ell+1}$ the ℓth partial quotient. We find that the maximum of the first 37 partial quotients is $a_{18} = 134$, and so

$$\frac{1}{(136)(n-5)} < |(n-5)\kappa - a| < \frac{6}{\log 2\alpha^n}.$$

Comparing the upper and lower bounds for $|(n-5)\kappa - a|$ above, we find that $n < 21$ in this case, and so $n \leq 93$ in all cases.

We could run a computer search from this point, but we can also use this new bound on n to obtain an even better bound on $n - m$. After repeating the reduction procedure on (9) with $N = 93$, we find that $n - m \leq 20$. Repeating another 15 reductions on 10, once for each value of $n - m$ satisfying $5 \leq n - m \leq 20$ aside from $n - m = 10$, we arrive at a final bound of $n \leq 34$. Since we already have the bound $n \leq 20$ in the case $n - m = 10$, we do not repeat the continued fraction argument. After running a brute force search for n and m with $5 \leq n - m \leq 20$, with $1 \leq m < n \leq 34$ and satisfying $L_n - L_m = 11 \cdot 2^a$ for some integer a, we find only the fourth solution $n = 8, m = 2, a = 2$ listed in Theorem 1.

Acknowledgements

This research has been obtained financial support from School of Science, King Mongkut's Institute of Technology Ladkrabang. Nutchanon Chiamchanya is the recipient of the Graduate Study Fellowship of the School of Science, King Mongkut's Institute of Technology Ladkrabang, Thailand. This research was funded by the RA/TA graduate scholarship from the School of Science, King Mongkut's Institute of Technology Ladkrabang, grant number RA/TA-2566-M-009.

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