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Some Contribution to Multivalued Fixed Point Problems on a Closed Ball with Applications

Tahair Rasham[1,*], Badriah Alamri[2], Mohamed Hussein[3], Nida Bashir[4], M. Daher Albalwi[5]

[1] *Department of Mathematics, University of Poonch Rawalakot, Azad Kashmir, Pakistan*
[2] *Department of Mathematics and Statistics, College of Science, University of Jeddah, Jeddah, Saudi Arabia*
[3] *Department of Chemistry, Faculty of Science, King Khalid University, P.O. Box 9004, Abha 61413, Saudi Arabia*
[4] *Department of Mathematics, University of Poonch Rawalakot, Azad Kashmir, Pakistan*
[5] *Yanbu Industrial College, The Royal Commission for Jubail and Yanbu, Yanbu 30436, Saudi Arabia*

Abstract. The objective of this research is to prove novel theorems for two discrete families of multi-nonlinear dominated operators that satisfy the hybrid type locally contractions in the framework of complete strong b-metric-like spaces. Our approach combines two distinct types of mappings: one from a weaker class of non-decreasing mappings, and the other from a class of multi-dominated mappings. Furthermore, some latest findings for graph contraction involving with family of multi-graph dominated structure are introduced. Several illustrative examples are presented to show the validity of the hypothesis underlying our results. Furthermore, two related applications are given to highlight the novelty of our findings. Our findings have prompted adjustments to various recent and classical outcomes in the academic literature, offering further proof of the innovation and significance of our work.

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1. Introduction

Fixed point (FP) theory holds significant importance within the realm of mathematics, making substantial contributions in both pure and applied mathematics such as optimization theory, differential and integral equations, partial differential equations, coding theory,

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Email addresses: tahairrasham@upr.edu.pk (T. Rasham), baalamri@uj.edu.sa (B. Alamri), bilal.ahmed96742@gmail.com (M. D. Albalwi)

approximation theory, dynamical system, fractional calculus and many different fields of sciences. Nadler [1] explored the Banach FP result [2] for multivalued mappings. Lateral, the authors of [3–9] expanded upon this work, extending it to different setting of metric spaces. Bakhtin [10] and Czerwik [11] discussed Banach's FP result in the framework of b-metric spaces (bMS). Afterward, Hussain et al. [12] introduced another generalization of bMS and named as b-metric-like space (bMLS). Lateral, Mehmood et al. [13], and Rasham et al. [14] demonstrated the existence of some FP problems within the context of a complete bMLS.

The authors of [15] established the notion of strong b-metric spaces (sBMS) and proved that sBMS is a better framework than bMS. They investigated that in sBMS an open ball is always open set instead in bMS open ball is not necessarily an open interval. After that, Tassadiq et al. [9] demonstrated FP results for self and set-valued mappings involving the complete sBMS. Afterward a latest variant of Kannan's FP result discussed by Doan et al. [16] in the setting of sBMS.

Wardowski [17] presented a novel multiplication of Banach contraction principle result and named it F-contraction and introduced some new FP theorems. Lateral, Acar et al. [18], Nazam et al. [19], Nicolae [6], Padcharoen et al. [20] and Rasham et al. [21] discussed different extensions of Wardowski's results [17]. Lateral, Mehmood et al. [13] proved FP findings for a single family of multi-maps holding locally F-contraction and their applications. Recetly, Rasham et al. [21] discussed novel FP problems for two families of multi-maps and their applications on integral and functional equations. This article introduces some latest FP results involving closed ball in complete strong-b-metric-like-space (sbMLS), The described contribution of this research can be delineated as follows:

- This article investigates innovative multi FP solutions for discrete families of multivalued dominated operators that satisfy a hybrid type contraction specified on a closed ball intersected with iterative sequence in the context of sbMLS.

- In contrast to the sparse attention given to multi-dominated mappings in existing literature, our study marks the first exploration of FP results pertaining to ordered multi-dominated operators in the context of ordered complete sbMLS.

- A new notion known as a couple of multi-graph dominated operators is introduced on a closed ball within these spaces, along with presenting novel results on graph contraction using a multi-graph structure.

- Lastly, to emphasize the uniqueness of our recent outcomes, we provide applications that illustrate how these discoveries can be used to derive the collective solution of integral and fractional differential equations.

The manuscript is organized in the succeeding sections: Section 2, introduces and elaborates on both foundational and innovative definitions, accompanied by illustrative examples. In Section 3, we present novel FP theorems concerning discrete families of multi-dominated nonlinear operators that satisfy hybrid type contractions on a closed ball and their examples. In Section 4, we establish FP theorems for multi-graph dominated operators equipped with graphical structures. In Section 5, we explore the application of our primary outcome to integral equations. In Section 6, we prove application of our primary findings to explore the common solution of fractional-differential equations. Lastly, in Section 7, we provide a comprehensive summary of our whole study and outline potential directions for future research.

2. Preliminaries

Definition 1. *[22]*

Let $b > 1$ and $\varnothing \neq \Upsilon$. The function $\Lambda_b : \Upsilon \times \Upsilon \to [0, \infty)$ is said to be strong b-metric-like on Υ if the given assumptions hold for each $y, y_1, y_2 \in \Upsilon$;

1 If $\Lambda_b(y, y_1) = 0$, then $y = y_1$;

2 $\Lambda_b(y, y_1) = \Lambda_b(y_1, y)$;

3 $\Lambda_b(y, y_2) \leq \Lambda_b(y, y_1) + b\Lambda_b(y_1, y_2)$. The pair (Υ, Λ_b) is said a strong b-metric-like space, shortly as sbMLS. For $g \in \Upsilon$ and $\varepsilon > 0$, $B(g; \varepsilon) = \{p \in \Upsilon : \Lambda_b(g, p) \leq \varepsilon\}$ be a closed ball in sbMLS.

Definition 2. *[22]*

Let (Υ, Λ_b) be a sbMLS.

1 A sequence $\{s_n\}$ in Υ is said to be convergent to a point $s \in \Upsilon$ such that $\lim_{n\to+\infty} \Lambda_b(s_n, s) \to 0$.

2 A sequence $\{s_n\}$ in Υ is said a Cauchy if for every $\epsilon > 0$ there exists a natural number $\delta(\epsilon) = \delta$ such that $\Lambda_b(s_n, s_m) < \epsilon$ for all $n, m \geq \delta$.

3 A sbMLS (Υ, Λ_b) is considered complete if every Cauchy sequence $\{s_n\}$ in Υ converges to a point $s \in \Upsilon$.

Definition 3. *[9]*

Let (Υ, Λ_b) be a sbMLS and $W \subseteq \Upsilon$. Then an element $h \in W$ is considered to be a best approximation in W when

$$\Lambda_b(v, W) = \inf_{h \in W} \Lambda_b(v, h) \quad \text{for all } v \in \Upsilon.$$

Hence $P(\Upsilon)$ denote the set consisting all closed compact subsets of Υ.

Let Ψ_b represents the class of whole non-increasing functions $\Psi_b : [0, +\infty) \to [0, +\infty)$ implies that $\sum_{k=1}^{+\infty} \Psi_b^k(g) < +\infty$ where $\Psi_b(g) < g$ and Ψ_b^k denotes k-th iterative term of Ψ_b.

Definition 4. *[22]*

The function $H : P(\Upsilon) \times P(\Upsilon) \to \mathbb{R}^+$ *given by*

$$H(L, K) = \max\left\{\sup_{a\in L} \Lambda_b(a, K), \sup_{b\in K} \Lambda_b(L, b)\right\},$$

for all $L, K \in P(\Upsilon)$ *is called Hausdorff strong b-metric-like on* $P(\Upsilon)$

Definition 5. *[21]*

Let $\varnothing \neq \Upsilon$ *and* $\alpha : \Upsilon \times \Upsilon \to R^+$ *is a function. Let* $S, T : \Upsilon \to P(\Upsilon)$ *be a couple of multi-maps. Then,* S *and* T *are said to be* α_**-admissible if for all* $v, w \in \Upsilon$ *and* $\alpha(v, w) \geq 1$ *implies that* $\alpha_*(Sv, Tw) \geq 1$*, where*

$$\alpha_*(Sv, Tw) = \inf\{\alpha(v, w) : v \in Sv, w \in T(v)\} > 1.$$

Definition 6. *[14] Let* $\varnothing \neq \Upsilon$ *and* $\alpha : \Upsilon \times \Upsilon \to R^+$ *be a function. Let* $Z \subseteq \Upsilon$*,* $T : \Upsilon \to P(\Upsilon)$ *be a multi-valued mapping. Then,* T *is considered to be* α_**-dominated on* Z*, whenever* $\alpha_*(w, Tw) \geq 1$ *for all* $w \in Z$*, where* $\alpha_*(w, Tw) = \inf\{\alpha(v, w) : w \in T(v)\}$.

Definition 7. *[17] Consider the metric space* (Υ, d)*. The mapping* $T : \Upsilon \to \Upsilon$ *is called* $\mathcal{F}$*-contraction if there exists* $\tau > 0$ *so that for all* $x, y \in \Upsilon$ *with* $d(Tx, Ty) > 0$ *holds the given inequality defined as:*

$$\tau + \mathcal{F}(d(Tx, Ty)) \leq \mathcal{F}(d(x, y)),$$

where $\mathcal{F} : R^+ \to R$ *is fulfill the following assumptions:*

F1 $\mathcal{F}$ *is a function of strictly increasing;*

F2 For every sequence $\{a_n\}$ *in* $(0, \infty)$ *and* $\lim_{n\to\infty} a_n = 0$ *implies* $\lim_{n\to\infty} \mathcal{F}(a_n) = -\infty$*;*

F3 If there exists $k \in (0, 1)$ *such that* $\lim_{a\to 0^+} a^k \mathcal{F}(a) = 0$.

Example 1. *[21]*

Let Υ *be a non-empty set and the mapping* $\alpha : \Upsilon \times \Upsilon \to [0, \infty)$ *be given by*

$$\alpha(x, z) = \begin{cases} 1 & \text{if } x > z, \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

Define the mappings $S, T : \Upsilon \to P(\Upsilon)$ *by* $Sr = [-4+r, -3+r]$ *and* $Tu = [-2+u, -1+u]$*, respectively. Then* S *and* T *are* α_**-dominated but not* α_**-admissible.*

Lemma 2.9 [22]. Let (Υ, Λ_b) be a sbMLS. Let $(P(\Upsilon), H)$ be a strong Hausdorff b-metric-like space on $P(\Upsilon)$. For each $g \in G$, there exists $h_g \in H$ such that

$$H(G, H) \geq \Lambda_b(g, h_g).$$

Proof. If $H(G, H) \geq \Lambda_b(g, H)$ for all $g \in G$. As H is a compact set there exists $h_g \in H$ satisfies $H(G, H) = \Lambda_b(g, h_g)$. Now we have, $H(G, H) \geq \Lambda_b(g, h_g)$. Now, if

$$H(C, D) = \sup_{h_v\in D} \Lambda_b(C, h_v) \geq \sup_{v\in C} \Lambda_b(v, D) \geq \Lambda_b(g, h_g).$$

Hence, proved.

3. Main Results

Let (Υ, Λ_b) be a sbMLS, $\vartheta_0 \in \Upsilon$ and $\{K_o : o \in N^o\}$, $\{L_e : e \in N^e\}$ be the discrete families of α_*-dominated multi-maps from Υ to $P(\Upsilon)$. Let $\vartheta_1 \in K_1(\vartheta_0)$ be an element such that $\Lambda_b(\vartheta_0, K_1(\vartheta_0)) = \Lambda_b(\vartheta_0, \vartheta_1)$. Let $\vartheta_2 \in L_2(\vartheta_1)$ be an element such that $\Lambda_b(\vartheta_1, L_2(\vartheta_1)) = \Lambda_b(\vartheta_1, \vartheta_2)$, where $1 \in N^o$ and $2 \in N^e$. Let $\vartheta_3 \in K_3(\vartheta_2)$ be such that $\Lambda_b(\vartheta_2, K_3(\vartheta_2)) = \Lambda_b(\vartheta_2, \vartheta_3)$. Let $\vartheta_4 \in L_4(\vartheta_3)$ be an element such that $\Lambda_b(\vartheta_3, L_4(\vartheta_3)) = \Lambda_b(\vartheta_3, \vartheta_4)$, where $3 \in N^o$ and $4 \in N^e$. Continuing in this way, we get a sequence ϑ_n in Υ that fulfills $\vartheta_{2n+1} \in K_i(\vartheta_{2n})$ and $\vartheta_{2n+2} \in L_j(\vartheta_{2n+1})$, $i = 2n+1 \in N^o$ (odd naturals), $j = 2n+1 \in N^e$ (even naturals) for $n = 0, 1, 2, \ldots$. Also $\Lambda_b(\vartheta_{2n}, K_i(\vartheta_{2n})) = \Lambda_b(\vartheta_{2n}, \vartheta_{2n+1})$ and $\Lambda_b(\vartheta_{2n+1}, L_j(\vartheta_{2n+1})) = \Lambda_b(\vartheta_{2n+1}, \vartheta_{2n+2})$. We denote this type of the sequence as $\{L_e K_o(\vartheta_n)\}$. For $\vartheta, \gamma \in \Upsilon$ and $a > 0$ define $\Xi_{(o,e)}(\vartheta, \gamma)$ by

$$\Xi_{(o,e)}(\vartheta, \gamma) = \max \left\{ \Lambda_b(\vartheta, \gamma), \frac{\Lambda_b(\vartheta, K_o(\vartheta)) \cdot \Lambda_b(\gamma, L_e(\gamma))}{a + \Lambda_b(\vartheta, \gamma)}, \Lambda_b(\vartheta, K_o(\vartheta)), \Lambda_b(\gamma, L_e(\gamma)) \right\}.$$

Theorem 1. *Let (Υ, Λ_b) be a sbMLS with constant a coefficient and $b > 1$. Let $r > 0$, $\vartheta_0 \in B(\vartheta_0, r)$, $\alpha : \Upsilon \times \Upsilon \to R^+$ is a function, $\{K_o : o \in N^o\}$ and $\{L_e : e \in N^e\}$ be a pair of discrete families of α_*-dominated multi-maps from Υ to $P(\Upsilon)$ on $B(\vartheta_0, r)$. Assume that for some $\mho_b \in \Psi_b$, there exist $\tau > 0$ and a function $\mathcal{F}$ of strictly increasing satisfying the given assumptions:*

a $\tau + \mathcal{F}(H(K_o(\vartheta), L_e(\gamma))) \leq \mathcal{F}(\mho_b(\Xi_{(o,e)}(\vartheta, \gamma)))$, *(3.1) for each* $\vartheta, \gamma \in B(\vartheta_0, r) \cap \{L_e K_o(\vartheta_n)\}$ *and* $\alpha(\vartheta, \gamma) \geq 1$, $H(K_o(\vartheta), L_e(\gamma)) > 0$*;*

b $\sum_{i=0}^{n} b^i \{\mho_b^i \Lambda_b(\vartheta_0, K_o(\vartheta_0))\} \leq r$, *(3.2) for all* $n \in N \cup \{0\}$*. Then* $\{L_e K_o(\vartheta_n)\}$ *be a sequence in* $B(\vartheta_0, r)$, $\alpha(\vartheta_n, \vartheta_{n+1}) \geq 1$ *for all* $n \in N \cup \{0\}$ *and* $\{L_e K_o(\vartheta_n)\} \to \vartheta^* \in B(\vartheta_0, r)$.

c (3.1) exists for ϑ^* *and either* $\alpha(\vartheta_n, \vartheta^*) \geq 1$ *or* $\alpha(\vartheta^*, \vartheta_n) \geq 1$*, for each* $n \in N \cup \{0\}$*. Then* ϑ^* *is a common multi-FP of* K_o *and* L_e *in* $B(\vartheta_0, r)$ *for all* $o \in N^o$ *and* $e \in N^e$.

Proof. Let the sequence $\{L_e K_o(\vartheta_n)\}$. From (3.2) we obtain

$$\Lambda_b(\vartheta_0, \vartheta_1) \leq \sum_{i=0}^{n} b^i \{\mho_b^i \Lambda_b(\vartheta_0, K_o(\vartheta_0))\} \leq r.$$

It shows that $\vartheta_1 \in B(\vartheta_0, r)$. Let $\vartheta_1, \vartheta_2, \vartheta_3, \ldots, \vartheta_j \in B(\vartheta_0, r)$ for some $j \in N$. Assume that $j = 2i+1$, where $i = 1, 2, 3, \ldots, (j-1)/2$. Since $\{K_o : o \in N^o\}$ and $\{L_e : e \in N^e\}$ is a pair of discrete families of α_*-dominated multi-maps in $B(\vartheta_0, r)$.

Since $\alpha_*(\vartheta_{2i}, K_o(\vartheta_{2i})) \geq 1$ and $\alpha_*(\vartheta_{2i+1}, L_e(\vartheta_{2i+1})) \geq 1$. As $\alpha_*(\vartheta_{2i}, K_o(\vartheta_{2i})) \geq 1$ for all $o \in N^o$ and $e \in N^e$ this implies $\inf\{\alpha(\vartheta_{2i}, h) \geq 1 : h \in K_o(\vartheta_{2i})\} \geq 1$. Also $\vartheta_{2i+1} \in K_c(\vartheta_{2i})$ for some $c \in N^o$ so $\alpha(\vartheta_{2i}, \vartheta_{2i+1}) \geq 1$ and $\vartheta_{2i+2} \in L_g(\vartheta_{2i+1})$ for some $g \in N^e$. Now, by apply Lemma 2.9 and inequality (3.1), we have

$$\tau + \mathcal{F}(\Lambda_b(\vartheta_{2i+1}, \vartheta_{2i+2})) \leq \tau + \mathcal{F}(H(K_c(\vartheta_{2i}), L_g(\vartheta_{2i+1}))),$$

$$
\begin{aligned}
&\leq \mathcal{F}(\mho_b(\Xi_{(c,g)}(\vartheta_{2i},\vartheta_{2i+1}))),\\
&\leq \mathcal{F}\Bigg(\mho_b\Bigg(\max\Bigg\{\Lambda_b(\vartheta_{2i},\vartheta_{2i+1}),\\
&\quad \frac{\Lambda_b(\vartheta_{2i},K_c(\vartheta_{2i}))\cdot\Lambda_b(\vartheta_{2i+1},L_g(\vartheta_{2i+1}))}{a+\Lambda_b(\vartheta_{2i},\vartheta_{2i+1})},\\
&\quad \Lambda_b(\vartheta_{2i},K_c(\vartheta_{2i})),\Lambda_b(\vartheta_{2i+1},L_g(\vartheta_{2i+1}))\Bigg\}\Bigg)\Bigg),\\
&\leq \mathcal{F}\Bigg(\mho_b\Bigg(\max\Bigg\{\Lambda_b(\vartheta_{2i},\vartheta_{2i+1}),\\
&\quad \frac{\Lambda_b(\vartheta_{2i},\vartheta_{2i+1})\cdot\Lambda_b(\vartheta_{2i+1},\vartheta_{2i+2})}{a+\Lambda_b(\vartheta_{2i},\vartheta_{2i+1})},\\
&\quad \Lambda_b(\vartheta_{2i},\vartheta_{2i+1}),\Lambda_b(\vartheta_{2i+1},\vartheta_{2i+2})\Bigg\}\Bigg)\Bigg),\\
&\leq \mathcal{F}\big(\mho_b\big(\max(\Lambda_b(\vartheta_{2i},\vartheta_{2i+1}),\Lambda_b(\vartheta_{2i+1},\vartheta_{2i+2}))\big)\big).
\end{aligned}
$$

If $\max\{\Lambda_b(\vartheta_{2i},\vartheta_{2i+1}),\Lambda_b(\vartheta_{2i+1},\vartheta_{2i+2})\}=\Lambda_b(\vartheta_{2i+1},\vartheta_{2i+2})$, then

$$\tau+\mathcal{F}(\Lambda_b(\vartheta_{2i+1},\vartheta_{2i+2}))\leq \mathcal{F}(\mho_b(\Lambda_b(\vartheta_{2i+1},\vartheta_{2i+2}))).$$

Since $\mathcal{F}$ is a mapping of strictly increasing, we get

$$\Lambda_b(\vartheta_{2i+1},\vartheta_{2i+2})<\mho_b(\Lambda_b(\vartheta_{2i+1},\vartheta_{2i+2})).$$

This contradicts to the fact that $\mho_b(u)<u$ for $u>0$. So

$$\max\{\Lambda_b(\vartheta_{2i},\vartheta_{2i+1}),\Lambda_b(\vartheta_{2i+1},\vartheta_{2i+2})\}=\Lambda_b(\vartheta_{2i},\vartheta_{2i+1}).$$

Hence, we obtain

$$
\begin{aligned}
\mathcal{F}(\Lambda_b(\vartheta_{2i+1},\vartheta_{2i+2}))&<\mathcal{F}(\mho_b(\Lambda_b(\vartheta_{2i},\vartheta_{2i+1}))),\\
\Lambda_b(\vartheta_{2i+1},\vartheta_{2i+2})&<\mho_b(\Lambda_b(\vartheta_{2i},\vartheta_{2i+1})). \qquad (3.3)
\end{aligned}
$$

$\alpha_*(\vartheta_{2i-1},L_w(\vartheta_{2i-1}))\geq 1$ and $\vartheta_{2i}\in L_w(\vartheta_{2i-1})$, so $\alpha_*(\vartheta_{2i-1},\vartheta_{2i})\geq 1$ for some $w\in N^e$. Now by applying Lemma 2.10 and inequality (3.1), we get

$$
\begin{aligned}
\tau+\mathcal{F}(\Lambda_b(\vartheta_{2i},\vartheta_{2i+1}))&\leq \tau+\mathcal{F}(H(L_w(\vartheta_{2i-1}),K_e(\vartheta_{2i}))),\\
&\leq \mathcal{F}(\mho_b(\Xi_{(e,w)}(\vartheta_{2i},\vartheta_{2i-1}))),\\
&\leq \mathcal{F}\Bigg(\mho_b\Bigg(\max\Bigg\{\Lambda_b(\vartheta_{2i},\vartheta_{2i-1}),\\
&\quad \frac{\Lambda_b(\vartheta_{2i},K_e(\vartheta_{2i}))\cdot\Lambda_b(\vartheta_{2i-1},K_e(\vartheta_{2i-1}))}{a+\Lambda_b(\vartheta_{2i},\vartheta_{2i-1})},
\end{aligned}
$$

$$\Lambda_b(\vartheta_{2i}, K_e(\vartheta_{2i})), \Lambda_b(\vartheta_{2i-1}, L_w(\vartheta_{2i-1}))\Bigg\}\Bigg)\Bigg),$$

$$\leq \mathcal{F}\Bigg(\mho_b\Bigg(\max\Bigg\{\Lambda_b(\vartheta_{2i}, \vartheta_{2i-1}),$$

$$\frac{\Lambda_b(\vartheta_{2i}, \vartheta_{2i+1}) \cdot \Lambda_b(\vartheta, \vartheta_{2i})}{a + \Lambda_b(\vartheta_{2i}, \vartheta_{2i-1})},$$

$$\Lambda_b(\vartheta_{2i}, \vartheta_{2i+1}), \Lambda_b(\vartheta_{2i-1}, \vartheta_{2i})\Bigg\}\Bigg)\Bigg),$$

$$\leq \mathcal{F}(\mho_b(\max\{\Lambda_b(\vartheta_{2i}, \vartheta_{2i+1}), \Lambda_b(\vartheta_{2i-1}, \vartheta_{2i})\})).$$

Since $\mathcal{F}$ is a mapping of strictly increasing, we have

$$\Lambda_b(\vartheta_{2i}, \vartheta_{2i+1}) < \mho_b(\max\{\Lambda_b(\vartheta_{2i-1}, \vartheta_{2i}), \Lambda_b(\vartheta_{2i}, \vartheta_{2i+1})\}).$$

If $\max\{\Lambda_b(\vartheta_{2i-1}, \vartheta_{2i}), \Lambda_b(\vartheta_{2i}, \vartheta_{2i+1})\} = \Lambda_b(\vartheta_{2i}, \vartheta_{2i+1})$, then

$$\Lambda_b(\vartheta_{2i}, \vartheta_{2i+1}) < \mho_b(\Lambda_b(\vartheta_{2i}, \vartheta_{2i+1})).$$

This contradicts to $\mho_b(u) < u$ for $u > 0$. So, we get

$$\Lambda_b(\vartheta_{2i}, \vartheta_{2i+1}) < \mho_b(\Lambda_b(\vartheta_{2i-1}, \vartheta_{2i})).$$

As $\mho_b$ be a non-decreasing function,

$$\mho_b(\Lambda_b(\vartheta_{2i}, \vartheta_{2i+1})) < \mho_b(\Lambda_b(\vartheta_{2i-1}, \vartheta_{2i})). \tag{3.4}$$

Applying the inequality stated above in (3.3), we derived that

$$\mho_b(\Lambda_b(\vartheta_{2i}, \vartheta_{2i+1})) < \mho_b^2(\Lambda_b(\vartheta_{2i-1}, \vartheta_{2i})).$$

Following this approach, we obtain

$$\Lambda_b(\vartheta_{2i+1}, \vartheta_{2i+2}) < \mho_b^{2i+1}(\Lambda_b(\vartheta_0, \vartheta_1)). \tag{3.5}$$

Instead, if $j = 2i$ where $i = 1, 2, 3, \ldots, j/2$, by proceeding the same method and applying (3.4), we have

$$\Lambda_b(\vartheta_{2i}, \vartheta_{2i+1}) < \mho_b^{2i}(\Lambda_b(\vartheta_0, \vartheta_1)). \tag{3.6}$$

Now, inequalities (3.5) and (3.6) can be combined expressed as

$$\Lambda_b(\vartheta_j, \vartheta_{j+1}) < \mho_b^j(\Lambda_b(\vartheta_0, \vartheta_1)), \quad \text{for all } j \in N. \tag{3.7}$$

Now, by using triangular property and (3.7), we get

$$\Lambda_b(\vartheta_j, \vartheta_{j+1}) \leq \Lambda_b(\vartheta_0, \vartheta_1) + b\Lambda_b(\vartheta_1, \vartheta_2) + b^2\Lambda_b(\vartheta_2, \vartheta_3) + \cdots + b^j\Lambda_b(\vartheta_j, \vartheta_{j+1}),$$

$$< \Lambda_b(\vartheta_0,\vartheta_1) + b\mho_b(\Lambda_b(\vartheta_0,\vartheta_1)) + b^2\mho_b^2(\Lambda_b(\vartheta_0,\vartheta_1)) + \cdots + b^j\mho_b^j(\Lambda_b(\vartheta_0,\vartheta_1)),$$
$$< \sum_{i=0}^{j} b^i(\mho_b^i(\Lambda_b(\vartheta_0,\vartheta_1))) < r.$$

Thus, $\vartheta_{j+1} \in B(\vartheta_0, r)$. Hence, $\vartheta_n \in B(\vartheta_0, r)$ for each $n \in N$. Hence, $\{L_eK_o(\vartheta_n)\} \to \upsilon \in B(\vartheta_0, r)$. Since $\{K_o : o \in N^o\}$ and $\{L_e : e \in N^e\}$ be a pair of discrete families of α_*-dominated multi-maps on $B(\vartheta_0, r)$. Also $\alpha_*(\vartheta_{2n}, K_o(\vartheta_{2n})) \geq 1$ or $\alpha_*(\vartheta_{2n+1}, L_e(\vartheta_{2n+1})) \geq 1$ indicates that $\alpha(\vartheta_n, \vartheta_{n+1}) \geq 1$. Hence (3.7) can be expressed as,

$$\Lambda_b(\vartheta_n,\vartheta_{n+1}) < \mho_b^n(\Lambda_b(\vartheta_0,\vartheta_1)), \quad \text{for all } n \in N. \tag{3.8}$$

Then, the convergence of the series $\sum_{k=0}^{\infty} b^k\mho_b^k(\mho_b^{s-1}(\Lambda_b(\vartheta_0,\vartheta_1)))$ shows that $\sum_{k=0}^{\infty} b^k\mho_b^k(t) < +\infty$ for each $s \in N$. As $\mho_b(u) < u$, we have

$$b^{n+1}\mho_b^{n+1}(\mho_b^{s-1}((\Lambda_b(\vartheta_0,\vartheta_1)))) < b^n\mho_b^n(\mho_b^{s-1}((\Lambda_b(\vartheta_0,\vartheta_1)))).$$

Fix $\epsilon > 0$ there is $s(\epsilon) \in N$, so that

$$b\mho_b(\mho_b^{s(\epsilon)-1}((\Lambda_b(\vartheta_0,\vartheta_1)))) + b\mho_b^2(\mho_b^{s(\epsilon)-1}((\Lambda_b(\vartheta_0,\vartheta_1)))) + \cdots < \epsilon.$$

For each $n, m \in N$ where $m > n > s(\epsilon)$, we deduce that

$$\begin{aligned}\Lambda_b(\vartheta_n,\vartheta_m) &\leq \Lambda_b(\vartheta_n,\vartheta_{n+1}) + b\Lambda_b(\vartheta_{n+1},\vartheta_{n+2}) + \cdots + b^{m-n}\Lambda_b(\vartheta_{m-1},\vartheta_m),\\ &< \mho_b^n(\Lambda_b(\vartheta_0,\vartheta_1)) + b\mho_b^{n+1}(\Lambda_b(\vartheta_0,\vartheta_1)) + \cdots + b^{m-n}\mho_b^{m-1}(\Lambda_b(\vartheta_0,\vartheta_1)),\\ &< \mho_b(\mho_b^{s(\epsilon)-1}(\Lambda_b(\vartheta_0,\vartheta_1))) + b(\mho_b^2(\mho_b^{s(\epsilon)-1}(\Lambda_b(\vartheta_0,\vartheta_1)))) + \cdots < \epsilon.\end{aligned}$$

Hence, the sequence $\{L_eK_o(\vartheta_n)\}$ Cauchy in $B(\vartheta_0, r)$. Since $(B(\vartheta_0, r), \Lambda_b)$ is a complete subspace of a sbMLS, so there is $\upsilon \in B(\vartheta_0, r)$ such that $\{L_eK_o(\vartheta_n)\} \to \upsilon$ as $n \to +\infty$, then

$$\lim_{n\to+\infty} \Lambda_b(\vartheta_n,\upsilon) = 0. \tag{3.9}$$

Now, by utilizing Lemma 2.9 and (2.1), one can writes

$$\begin{aligned}\Lambda_b(\upsilon,K_o(\upsilon)) &\leq \Lambda_b(\upsilon,\vartheta_{2n+2}) + b\Lambda_b(\vartheta_{2n+2},K_o(\upsilon)).\\ &\leq \Lambda_b(\upsilon,\vartheta_{2n+2}) + bH(L_e(\vartheta_{2n+1}),K_o(\upsilon)).\end{aligned}$$

By assumption $\alpha(\vartheta_n, \upsilon) \geq 1$. Suppose that $\Lambda_b(\upsilon, K_o(\upsilon)) > 0$, then there exists positive integer k so that $\Lambda_b(\vartheta_n, K_o(\upsilon)) > 0$ for each $n \geq k$. For $n \geq k$, we have

$$\begin{aligned}\Lambda_b(\upsilon,K_o(\upsilon)) &< \Lambda_b(\upsilon,\vartheta_{2n+2})\\ &\quad + b\left(\mho_b\left(\max\left\{\Lambda_b(\upsilon,\vartheta_{2n+1}), \frac{\Lambda_b(\vartheta_{2n+1},\vartheta_{2n+2})\cdot\Lambda_b(\upsilon,K_o(\upsilon))}{a+\Lambda_b(\upsilon,K_o(\upsilon))}, \Lambda_b(\upsilon,K_o(\upsilon)), \Lambda_b(\vartheta_{2n+1},\vartheta_{2n+2})\right\}\right)\right)\end{aligned}$$

Taking $n \to +\infty$ and applying (3.9), we get

$$\Lambda_b(\upsilon,K_o(\upsilon)) < b\mho_b(\Lambda_b(\upsilon,K_o(\upsilon))) < \Lambda_b(\upsilon,K_o(\upsilon)).$$

This leads to a contradiction, thus our assumption must be incorrect. Hence, $\Lambda_b(\upsilon, K_o(\upsilon)) = 0$ or $\upsilon \in K_o(\upsilon)$. In a similar way, by using Lemma 2.9 and (3.9), we can demonstrate that $\upsilon \in L_e(\upsilon)$. Hence, K_o and L_e both have a unique FP υ in $B(\vartheta_0, r)$.

If we use $\{K_o : o \in N^o\} = \{L_e : e \in N^e\}$ in Theorem 1, then we obtain the upcoming outcome through analysis.

Corollary 1. *Let (Υ, Λ_b) be a sbMLS. Let $r > 0$, $\vartheta_0 \in B(\vartheta_0, r)$, $\alpha : \Upsilon \times \Upsilon \to R^+$ and $\{K_o : o \in N^o\}$ is the family of multi α_*-dominated operators from Υ to $P(\Upsilon)$ on $B(\vartheta_0, r)$. Suppose that for some $\mho_b \in \Psi_b$, there exist $\tau > 0$ and a function of strictly increasing $\mathcal{F}$ satisfying the assumptions;*

a $\tau + \mathcal{F}(H(K_o(\vartheta), K_o(y))) \leq \mathcal{F}(\mho_b(\Xi_{(o,o)}(\vartheta, y)))$, *(3.10) for each $\vartheta, \gamma \in B(\vartheta_0, r) \cap \{\Upsilon K_o(\vartheta_n)\}$ also $\alpha(\vartheta, \gamma) \geq 1$, $H(K_o(\vartheta), K_o(\gamma)) > 0$;*

b $\sum_{i=0}^{n} b^i \{\mho_b^i(\Lambda_b(\vartheta_0, K_o(\vartheta_0))\} \leq r$, *(3.11) for every $n \in N \cup \{0\}$ and $b > 1$.*

Then, the sequence $\{\Upsilon K_o(\vartheta_n)\}$ exists in $B(\vartheta_0, r)$ and $\alpha(\vartheta_n, \vartheta_{n+1}) \geq 1$ for all $n \in N \cup \{0\}$ moreover $\{\Upsilon K_o(\vartheta_n)\} \to \vartheta^* \in B(\vartheta_0, r)$.

c *(3.10) exists for ϑ^* and also $\alpha(\vartheta_n, \vartheta^*) \geq 1$ or $\alpha(\vartheta^*, \vartheta_n) \geq 1$, for each $n \in N \cup \{0\}$. Then ϑ^* is a unique FP of K_o in $B(\vartheta_0, r)$.*

If we apply self-operators as a replacement for discrete families of set-valued operators in Theorem 1, so we obtain the following outcome through analysis.

Corollary 2. *Let (Υ, Λ_b) be a sbMLS. Let $r > 0$, $\vartheta_0 \in B(\vartheta_0, r)$, $\alpha : \Upsilon \times \Upsilon \to R^+$ and $K, L : \Upsilon \to \Upsilon$ be a couple of self-maps on $B(\vartheta_0, r)$. Assume that for some $\mho_b \in \Psi_b$, there are $\tau > 0$ and a function $\mathcal{F}$ of strictly-increasing satisfying the given assumptions:*

a $\tau + \mathcal{F}(\Lambda_b(K(\vartheta), L(\gamma))) \leq \mathcal{F}(\mho_b(\Xi(\vartheta, \gamma)))$, *(3.12) for each $\vartheta, \gamma \in B(\vartheta_0, r) \cap \{\vartheta_n\}$ and $\alpha(\vartheta, \gamma) \geq 1$, $\Lambda_b(K(\vartheta), L(\gamma)) > 0$;*

b $\sum_{i=0}^{n} b^i \{\mho_b^i(\Lambda_b(\vartheta_0, K(\vartheta_0))\} \leq r$, *(3.13) for every $n \in N \cup \{0\}$ and $b > 1$.*

Then $\{\vartheta_n\}$ belongs to $B(\vartheta_0, r)$ and $\alpha(\vartheta_n, \vartheta_{n+1}) \geq 1$ for all $n \in N \cup \{0\}$ moreover $\{\vartheta_n\} \to \vartheta^* \in B(\vartheta_0, r)$.

c *(3.12) exists for ϑ^* also $\alpha(\vartheta_n, \vartheta^*) \geq 1$ or $\alpha(\vartheta^*, \vartheta_n) \geq 1$, for each $n \in N \cup \{0\}$. Then ϑ^* is a unique FP of both K and L in $B(\vartheta_0, r)$.*

Nieto [23] proved ordered sequences and their convergence behaviors and discussed FP theorems in ordered complete metrics spaces. Recently, Rasham et al. [19] showed FP problems in ordered complete multiplicative metric spaces.

Definition 8. *[21] Let $\varnothing \neq \Upsilon$, $\preceq$ be a partial order in Υ and $H \subseteq \Upsilon$. If $t \preceq H$ for each $u \in H$ and we deduce that $t \preceq u$. Then $\{K_o : o \in N^o\}$ from Υ to $P(\Upsilon)$ is considered to be multi ordered-preserved dominated mapping on H if $t \preceq K_o(t)$ for each $t \in H \subseteq \Upsilon$.*

Theorem 2. *Let $(\Upsilon, \preceq, \Lambda_b)$ be an ordered-complete sbMLS. Let $r > 0$, $\vartheta_0 \in B(\vartheta_0, r)$, $\{K_o : o \in N^o\}$, $\{L_e : e \in N^e\}$ are two discrete families of semi $\preceq$-dominated multi-maps from Υ to $P(\Upsilon)$ on $B(\vartheta_0, r)$. Assume that $\mho_b \in \Psi_b$ there exist $\tau > 0$ and a function $\mathcal{F}$ of strictly-increasing satisfying the given restrictions:*

a $\tau + \mathcal{F}(H(K_o(\vartheta), L_e(\gamma))) \leq \mathcal{F}(\mho_b(\Xi_{(o,e)}(\vartheta, \gamma)))$, *(3.14) whenever $\vartheta, \gamma \in B(\vartheta_0, r) \cap \{L_e K_o(\vartheta_n)\}$, $\vartheta \preceq \gamma$ and $H(K_o(\vartheta), L_e(\gamma)) > 0$;*

b $\sum_{i=0}^{n} b^i \{\mho_b^i(\Lambda_b(\vartheta_0, K_o(\vartheta_0))\} \leq r.$ *(3.15)*

Then, $\{L_e K_o(\vartheta_n)\}$ in $B(\vartheta_0, r)$ for each $n \in N \cup \{0\}$ with $b > 1$, $\{TL(\vartheta_n)\} \to \vartheta^* \in B(\vartheta_0, r)$.

c *(3.14) exists for ϑ^* also $\vartheta_n \preceq \vartheta^*$ or $\vartheta^* \preceq \vartheta_n$ for each $n \in N \cup \{0\}$. Formerly ϑ^* is a unique FP of both K_o and L_e in $B(\vartheta_0, r)$ for all $o \in N^o$ and $e \in N^e$.*

Proof. Let $\alpha : \Upsilon \times \Upsilon \to R^+$ is a mapping defined by $\alpha(\vartheta, \gamma) = 1$ for each $\vartheta \in B(\vartheta_0, r)$, $\vartheta \preceq \gamma$ or $\gamma \preceq \vartheta$ and $\alpha(\vartheta, \gamma) = 0$ for all $\vartheta, \gamma \in \Upsilon$. Subsequently $\{K_o : o \in N^o\}$ and $\{L_e : e \in N^e\}$ are double discrete families of $\preceq$-dominated multi-maps from Υ to $P(\Upsilon)$ on $B(\vartheta_0, r)$, so $\vartheta \preceq K_o(\vartheta)$ and $\vartheta \preceq L_e(\vartheta)$ for all $\vartheta \in B(\vartheta_0, r)$. This indicates that $\vartheta \preceq t$ for all $t \in K_o(\vartheta)$ and $\vartheta \preceq u$ for all $u \in L_e(\vartheta)$. So, $\alpha(\vartheta, t) = 1$ for all $t \in K_o(\vartheta)$ and $\alpha(\vartheta, u) = 1$ for all $u \in L_e(\vartheta)$. This signifies that $\inf\{\alpha(\vartheta, \gamma) : \gamma \in K_o(\vartheta)\} = 1$ and $\inf\{\alpha(\vartheta, \gamma) : \gamma \in L_e(\vartheta)\} = 1$. So, $\alpha_*(\vartheta, K_o(\vartheta)) = 1$ and $\alpha_*(\vartheta, L_e(\vartheta)) = 1$ for each $\vartheta \in B(\vartheta_0, r)$. Since $\{K_o : o \in N^o\}$ and $\{L_e : e \in N^e\}$ are discrete families of α_*-dominated multi-maps from Υ to $P(\Upsilon)$ on $B(\vartheta_0, r)$. Moreover, inequality (3.14) can be re-written as

$$\tau + \mathcal{F}(H(K_o(\vartheta), L_e(\gamma))) \leq \mathcal{F}(\mho_b(\Xi_{(o,e)}(\vartheta, \gamma))),$$

for all $\vartheta, \gamma \in B(\vartheta_0, r) \cap \{L_e K_o(\vartheta_n)\}$, $\alpha(\vartheta, \gamma) \geq 1$. Also inequality (3.15) holds. Then from Theorem 1, we get a sequence $\{L_e K_o(\vartheta_n)\}$ in $B(\vartheta_0, r)$ and $\{L_e K_o(\vartheta_n)\} \to \vartheta^* \in B(\vartheta_0, r)$. Now, $\vartheta_n, \vartheta^* \in B(\vartheta_0, r)$ for all $n \in N$, and either $\vartheta^* \preceq \vartheta_n$ or $\vartheta_n \preceq \vartheta^*$ implies that $\alpha(\vartheta_n, \vartheta^*) \geq 1$ or $\alpha(\vartheta^*, \vartheta_n) \geq 1$.

Consequently all the conditions of Theorem 1 are satisfied. Hence, by Theorem 1 both K_o and L_e have a unique FP ϑ^* in $B(\vartheta_0, r)$ for all $o \in N^o$ and $e \in N^e$.

The outcome was determined by lifting the constraint of the closed ball from the ordered complete sbMLS. If we use single set-valued map as a substitute by the couple and vanishing the restriction of the closed ball as of the hypothesis 2, we present given outcome.

Corollary 3. *Let $(\Upsilon, \preceq, \Lambda_b)$ be an ordered-complete sbMLS. Let $r > 0$, $\vartheta_0 \in B(\vartheta_0, r)$ and $\{K_o : o \in N^o\}$ be a discrete family of semi $\preceq$-dominated set-valued operator from Υ to $P(\Upsilon)$ on $B(\vartheta_0, r)$. Assume that $\mho_b \in \Psi_b$ there exists $\tau > 0$ and a function $\mathcal{F}$ of strictly-increasing satisfying the given assumptions:*

a $\tau + \mathcal{F}(H(K_o(\vartheta), K_o(\gamma))) \leq \mathcal{F}(\mho_b(\Xi_{(o,o)}(\vartheta, \gamma)))$, *(3.14) where $\vartheta, \gamma \in \{\Upsilon K_o(\vartheta_n)\}$, $\vartheta \preceq \gamma$ and $H(K_o(\vartheta), K_o(\gamma)) > 0$.*

Then $\{\Upsilon K_o(\vartheta_n)\}$ holds in Υ for each $n \in N \cup \{0\}$ with $b \geq 1$ and $\{\Upsilon K_o(\vartheta_n)\} \to \vartheta^* \in \Upsilon$.

b *(3.14) exists for ϑ^* also $\vartheta_n \preceq \vartheta^*$ or $\vartheta^* \preceq \vartheta_n$ for each $n \in N \cup \{0\}$. Then ϑ^* is a multi FP of K_o in Υ for all $o \in N^o$.*

Example 2. *Let $\Upsilon = R^+ \cup \{0\}$ and the mapping $\Lambda_b : \Upsilon \times \Upsilon \to \Upsilon$ be the complete sbMLS on Υ defined by*

$$\Lambda_b(f, g) = (f + g)^2 \quad \text{for all } f, g \in \Upsilon,$$

with $b = 2$. Define, $K_o, L_e : \Upsilon \to P(\Upsilon)$ be two discrete families of multi-maps defined as

$$K_o(\vartheta) = \begin{cases} \left[\frac{\vartheta}{3m}, \frac{2\vartheta}{5m}\right], & \text{if } \vartheta \in [1,4] \cap \Upsilon, \\ [3\vartheta m, 7\vartheta m], & \text{if } \vartheta \in (4,\infty) \cap \Upsilon, \end{cases} \quad \text{where } m = 1, 2, 3, \ldots,$$

and

$$L_e(\vartheta) = \begin{cases} \left[\frac{\vartheta}{4n}, \frac{3\vartheta}{5n}\right], & \text{if } \vartheta \in [1,4] \cap \Upsilon, \\ [7\vartheta n, 8\vartheta n], & \text{if } \vartheta \in (4,\infty) \cap \Upsilon, \end{cases} \quad \text{where } n = 1, 2, 3, \ldots,$$

Taking $\vartheta_0 = 1$ and $r = 25$, then $B(\vartheta_0, r) = [1,4] \cap \Upsilon$. Now, considering $\Lambda_b(\vartheta_0, K_1(\vartheta_0)) = \Lambda_b(1, 1/3) = 16/9$. So, $\vartheta_1 = 1/3$, $\Lambda_b(\vartheta_1, L_2(\vartheta_1)) = \Lambda_b(1/3, 1/12)$. As, $\vartheta_2 = 1/12$, $\Lambda_b(\vartheta_2, K_2(\vartheta_2)) = \Lambda_b(1/12, 1/36)$. So, $\vartheta_3 = 1/36$. Hence, we deduce a sequence of the form $\{L_e K_o(\vartheta_n)\} = \{1, 1/3, 1/12, 1/36, \ldots\}$ in Υ generated by ϑ_0. Let $\mho_b(k) = 2k/3$ and $a = 1/2$. The function defined by $\alpha : \Upsilon \times \Upsilon \to R^+$ by

$$\alpha(\vartheta, y) = \begin{cases} 1 & \text{if } \vartheta > y, \\ \frac{1}{3} & \text{otherwise.} \end{cases}$$

Taking $\vartheta, \gamma \in B(\vartheta_0, r) \cap \{L_e K_o(\vartheta_n)\}$ and $\alpha(\vartheta, \gamma) \geq 1$. We deduce that,

$$\begin{aligned} H(K_o(\vartheta), L_e(\gamma)) &= \max\left\{ \sup_{a \in K_o(\vartheta)} \Lambda_b(a, L_e(\gamma)), \sup_{b \in L_e(\gamma)} \Lambda_b(K_o(\vartheta), b) \right\}, \\ &= \max\left\{ \Lambda_b\left(\frac{2\vartheta}{5m}, \left[\frac{\gamma}{4n}, \frac{3\gamma}{5n}\right]\right), \Lambda_b\left(\left[\frac{\vartheta}{3m}, \frac{2\vartheta}{5m}\right], \frac{3\gamma}{5n}\right) \right\}, \\ &= \max\left\{ \Lambda_b\left(\frac{2\vartheta}{5m}, \frac{\gamma}{4n}\right), \Lambda_b\left(\frac{\vartheta}{3m}, \frac{3\gamma}{5n}\right) \right\}, \\ &= \max\left\{ \left(\frac{2\vartheta}{5m} + \frac{\gamma}{4n}\right)^2, \left(\frac{\vartheta}{3m} + \frac{3\gamma}{5n}\right)^2 \right\}, \\ &\leq \mho_b\Bigg(\max\Bigg\{ \Lambda_b(\vartheta, \gamma), \\ &\quad \frac{\Lambda_b\left(\vartheta, \left[\frac{\vartheta}{3m}, \frac{2\vartheta}{5m}\right]\right) \cdot \Lambda_b\left(\gamma, \left[\frac{\gamma}{4n}, \frac{3\gamma}{5n}\right]\right)}{a + \Lambda_b(\vartheta, \gamma)}, \end{aligned}$$

$$\Lambda_b\left(\vartheta, \left[\frac{\vartheta}{3m}, \frac{2\vartheta}{5m}\right]\right), \Lambda_b\left(\gamma, \left[\frac{\gamma}{4n}, \frac{3\gamma}{5n}\right]\right) \Bigg\}\Bigg)$$

$$\leq \mho_b\Bigg(\max\Bigg\{ \Lambda_b(\vartheta, \gamma),$$

$$\frac{\Lambda_b\left(\vartheta, \frac{\vartheta}{3m}\right) \cdot \Lambda_b\left(\gamma, \frac{\gamma}{4n}\right)}{a + \Lambda_b(\vartheta, \gamma)},$$

$$\Lambda_b\left(\vartheta, \frac{\vartheta}{3m}\right), \Lambda_b\left(\gamma, \frac{\gamma}{4n}\right) \Bigg\}\Bigg)$$

$$\leq \mho_b\Bigg(\max\Bigg\{ (\vartheta+\gamma)^2,$$

$$\frac{\left(\vartheta + \frac{\vartheta}{3m}\right)^2 \cdot \left(\gamma + \frac{\gamma}{4n}\right)^2}{\frac{1}{2} + (\vartheta+\gamma)^2},$$

$$\left(\vartheta + \frac{\vartheta}{3m}\right)^2, \left(\gamma + \frac{\gamma}{4n}\right)^2 \Bigg\}\Bigg)$$

$$\leq \mho_b\left(\Lambda_b(\vartheta, \gamma)\right).$$

Case I. *If the points are taken from the closed ball $B(\vartheta_0, r) = [1,4] \cap \Upsilon$. Now letting $1, 2 \in B(\vartheta_0, r) \cap \Upsilon$ and $\alpha(1,2) \geq 1$, we obtain that*

$$H(K_o(\vartheta), L_e(\gamma)) < \mho_b(\Xi_{(o,e)}(\vartheta, \gamma)).$$

Choosing $\tau \in (0, \frac{1}{191})$ and a function $\mathcal{F}$ of strictly increasing defined by $\mathcal{F}(c) = \ln c + c$, we obtain

$$H(K_o(\vartheta), L_e(\gamma)) \cdot e^{H(K_o(\vartheta), L_e(\gamma)) - \mho_b(\Lambda_b(\vartheta,\gamma))} + \tau \leq \mho_b(\Lambda_b(\vartheta, \gamma)),$$
$$\ln(H(K_o(\vartheta), L_e(\gamma))) + H(K_o(\vartheta), L_e(\gamma)) + \tau \leq \ln(\mho_b(\Lambda_b(\vartheta, \gamma))) + \mho_b(\Lambda_b(\vartheta, \gamma)),$$
$$\tau + \mathcal{F}(H(K_o(\vartheta), L_e(\gamma))) \leq \mathcal{F}(\mho_b(\Xi_{(o,e)}(\vartheta, \gamma))).$$

So the contractive inequality (3.1) holds on $B(\vartheta_0, r) \cap \Upsilon$.

Case II. *If the points are taken from the whole space $(4, \infty) \cap \Upsilon$ where $\Upsilon = \mathbb{R}^+ \cup \{0\}$ instead of the closed ball $B(\vartheta_0, r) = [1,4] \cap \Upsilon$. Taking $5, 6 \in \Upsilon$ then $\alpha(5,6) \geq 1$. We have,*

$$\tau + \mathcal{F}(H(K_o(5), L_e(6))) > \mathcal{F}(\mho_b(\Xi_{(o,e)}(5,6))).$$

Hence, the contractive condition (3.1) does not hold on Υ. Moreover, for each $n \in \mathbb{N} \cup \{0\}$,

$$\sum_{i=0}^{n} b^i \mho_b^i(\Lambda_b(\vartheta_0, \vartheta_1)) = 9 \sum_{i=0}^{n} \left(\frac{2}{3}\right)^i < 25 = r.$$

Hence, K_o and L_e satisfy all necessary conditions of Theorem 3.1 for all $\vartheta, \gamma \in B(\vartheta_0, r) \cap \{L_e K_o(\vartheta_n)\}$ with $\alpha(\vartheta, \gamma) \geq 1$. Also, K_o and L_e both have a unique fixed point in $B(\vartheta_0, r)$.

4. Existence Results for Multi-Graph Theory

In the above section, we demonstrated an application derived from Theorem 3.1 in the framework of graph theory. Firstly, Jachymski [16] introduced an important outcome relating to contractive mappings in a distance space associated with graphs. Additionally, Hussain et al. [15], Rasham et al. [29] and Shoaib et al. [34] showed FP theorems for graphic contractions involving graphic structure.

Definition 9 (27). *Let $\Upsilon Neq\varnothing$ and $\mathcal{G} = (W(\mathcal{G}), V(\mathcal{G}))$ be a graph such that $W(\mathcal{G}) = \Upsilon$ and $H \subset \Upsilon$. A family $\{K_o : o \in \mathbb{N}^o\}$ is reported as a graph-dominated multi-map on set H if $(\vartheta, \gamma) \in V(\mathcal{G})$ for all $\gamma \in K_o(\vartheta)$ and $\vartheta \in H$ for every $o \in \mathbb{N}^o$.*

Theorem 3. *Let $b \geq 1$ and (Υ, Λ_b) be a complete sbMLS endowed with a graph $\mathcal{G}$. Let $\{K_o : o \in \mathbb{N}^o\}$ and $\{L_e : e \in \mathbb{N}^e\}$ be two discrete families of α_*-dominated multi-maps from Υ to $P(\Upsilon)$ and $r > 0$, $\vartheta_0 \in B(\vartheta_0, r)$. Assume that for some $\mho_b \in \Psi_b$, the following conditions are satisfied:*

a $\{K_o : o \in \mathbb{N}^o\}$ and $\{L_e : e \in \mathbb{N}^e\}$ are two discrete families of graph-dominated multi-maps on $B(\vartheta_0, r) \cap \{L_e K_o(\vartheta_n)\}$.

b There exists $\tau > 0$ and a strictly increasing mapping $\mathcal{F}$ such that:

$$\tau + \mathcal{F}(H(K_o(\vartheta), L_e(\gamma))) \leq \mathcal{F}(\mho_b(\Xi_{(o,e)}(\vartheta, \gamma))), \tag{4.1}$$

for all $\vartheta, \gamma \in B(\vartheta_0, r) \cap \{L_e K_o(\vartheta_n)\}$, $(\vartheta, \gamma) \in V(\mathcal{G})$ and $H(K_o(\vartheta), L_e(\gamma)) > 0$.

c $\sum_{i=0}^{j} b^i \mho_b^i(\Lambda_b(\vartheta_0, \vartheta_1)) < r$.

Then $\{L_e K_o(\vartheta_n)\}$ is in $B(\vartheta_0, r)$ and $(\vartheta_n, \vartheta_{n+1}) \in V(\mathcal{G})$, moreover $\{L_e K_o(\vartheta_n)\} \to e^*$. Furthermore, if (4.1) holds for e^* and $(\vartheta_n, e^*) \in V(\mathcal{G})$ or $(e^*, \vartheta_n) \in V(\mathcal{G})$ for each $n \in \mathbb{N}$, then e^* is a common FP of K_o and L_e in $B(\vartheta_0, r)$ for all $o \in \mathbb{N}^o$ and $e \in \mathbb{N}^e$.

Proof. The function $\alpha : \Upsilon \times \Upsilon \to \mathbb{R}^+$ is defined by

$$\alpha(\vartheta, \gamma) = \begin{cases} 1, & \text{if } \vartheta \in B(\vartheta_0, r) \text{ and } (\vartheta, \gamma) \in V(\mathcal{G}), \\ 0, & \text{otherwise.} \end{cases}$$

Condition (ii) ensures that $\{K_o : o \in \mathbb{N}^o\}$ and $\{L_e : e \in \mathbb{N}^e\}$ are two discrete families of semi-graph dominated maps on $B(\vartheta_0, r)$: for all $\vartheta \in B(\vartheta_0, r)$ and $(\vartheta, \gamma) \in V(\mathcal{G})$ for every $\gamma \in K_o(\vartheta_n)$ and $(\vartheta, \gamma) \in V(\mathcal{G})$ for each $\gamma \in L_e(\vartheta_n)$. So $\alpha(\vartheta, \gamma) = 1$ for all $\gamma \in K_o(\vartheta_n)$ and hence $\alpha(\vartheta, \gamma) = 1$ for each $\gamma \in L_e(\vartheta_n)$. This means that

$$\inf\{\alpha(\vartheta, \gamma) : \gamma \in K_o(\vartheta_n)\} = 1 \quad \text{and} \quad \inf\{\alpha(\vartheta, \gamma) : \gamma \in L_e(\vartheta_n)\} = 1.$$

Therefore, $\alpha_*(\vartheta, K_o(\vartheta)) = 1$ and $\alpha_*(\vartheta, L_e(\vartheta)) = 1$ for each $\vartheta \in B(\vartheta_0, r)$. Hence, $\{K_o : o \in \mathbb{N}^o\}$ and $\{L_e : e \in \mathbb{N}^e\}$ are a pair of discrete families of semi α_*-dominated set-valued maps from Υ to $P(\Upsilon)$ on $B(\vartheta_0, r)$. Moreover, (4.1) can be written as

$$\tau + \mathcal{F}(H(K_o(\vartheta), L_e(\gamma))) \leq \mathcal{F}(\mho_b(\Xi_{(o,e)}(\vartheta, \gamma))),$$

whenever $\vartheta, \gamma \in B(\vartheta_0, r) \cap \{L_e K_o(\vartheta_n)\}$, $\alpha(\vartheta, \gamma) \geq 1$, $H(K_o(\vartheta), L_e(\gamma)) > 0$. Furthermore, (iii) holds. Then by Theorem 3.1, $\{L_e K_o(\vartheta_n)\}$ exists in $B(\vartheta_0, r)$ and $\{L_e K_o(\vartheta_n)\} \to e^*$ in $B(\vartheta_0, r)$. Now $(\vartheta_n, e^*) \in B(\vartheta_0, r)$ and also $(\vartheta_n, e^*) \in V(\mathcal{G})$ or $(e^*, \vartheta_n) \in V(\mathcal{G})$ signifies that either $\alpha(\vartheta_n, e^*) \geq 1$ or $\alpha(e^*, \vartheta_n) \geq 1$. Hence, all conditions of Theorem 3.1 are verified. So, by Theorem 3.1 both K_o and L_e admit a multi-FP e^* in $B(\vartheta_0, r)$ for all $o \in \mathbb{N}^o$ and $e \in \mathbb{N}^e$.

5. Applications

In this section, we will prove applications on systems of integral equations and fractional differential equations by applying our main hypothesis.

5.1. Volterra Integral Equations

This subsection will provide an application to investigate nonlinear Volterra-type integral equations. More precisely, we shall demonstrate the existence and uniqueness of integral equation solutions. Agarwal et al. [2], Aydi et al. [7], Cosentino et al. [10], Hussain et al. [14], Rasham et al. [30] and Shoaib et al. [35] adopted FP techniques to obtain the unique solution of integral equations. More results involving integral equations applications can be seen in ([26, 31, 33]).

Theorem 4. *Let (Υ, Λ_b) be a sbMLS with coefficient $b > 1$. Let $u \in \Upsilon$ and $K_o, L_e : \Upsilon \to \Upsilon$. Assume there are $\tau > 0$, $\mho_b \in \Psi_b$ and a strictly increasing function $\mathcal{F}$ such that the following assumption is satisfied:*

$$\tau + \mathcal{F}(H(K_o(\mu), L_e(\phi))) \leq \mathcal{F}(\mho_b(\Xi_{(o,e)}(\mu, \phi))), \tag{5.1}$$

whenever $\mu, \phi \in \{\chi_n\}$ and $H(K_o(\mu), L_e(\phi)) > 0$, the sequence $\{\chi_n\} \to \phi \in \Upsilon$. Moreover, if (5.1) holds for ϕ, then ϕ becomes a unique FP of K_o and L_e in Υ.

Proof. The proof of Theorem 5.1 follows a similar approach to that of Theorem 3.1, with the main focus being on establishing uniqueness. Let's denote q as another FP of both L_e and K_o. Assume that $H(K_o(\mu), L_e(\phi)) > 0$, then the following property holds:

$$\tau + \mathcal{F}(H(K_o(\mu), L_e(\phi))) \leq \mathcal{F}(\mho_b(\Xi_{(o,e)}(\mu, \phi))),$$

which further implies that

$$\Lambda_b(\mu, \phi) < \mho_b(\Lambda_b(\mu, \phi)) < b\mho_b(\Lambda_b(\mu, \phi)) < \Lambda_b(\mu, \phi).$$

This is a contradiction. Hence, $H(K_o(\mu), L_e(\phi)) = 0$. So, $\phi = q$.

Now, by considering a system of Volterra integral equations we will demonstrate an application related to Theorem 3.1 to attain the unique solution:

$$\mu(\vartheta) = \int_0^{\vartheta} k_1(\vartheta, \hbar, \mu(\hbar))\, d\hbar, \tag{5.2}$$

$$\phi(\chi) = \int_0^{\vartheta} k_2(\vartheta, \hbar, \phi(\hbar))\, d\hbar, \tag{5.3}$$

for each $\vartheta \in [0,1]$. Now we resolve integrals (5.2) and (5.3). Let $\Upsilon = C([0,1],\mathbb{R}^+)$ denote the set consisting of all continuous functions within the closed interval $[0,1]$ endowed with a complete sbMLS. Taking $\tau > 0$ for any $\mu \in C([0,1],\mathbb{R}^+)$, define a continuous norm as: $\|\mu\|_\tau = \sup_{\vartheta\in[0,1]} |\mu(\vartheta)|e^{-\tau\vartheta}$. Then

$$d_\tau(\mu,\phi) = \sup_{\vartheta\in[0,1]} \left(|\mu(\vartheta)| + |\phi(\vartheta)|\right) e^{-\tau\vartheta} = \|\mu+\phi\|_\tau^2,$$

for every $\mu, \phi \in C([0,1],\mathbb{R}^+)$. With these specifications, $(C([0,1],\mathbb{R}^+), d_\tau)$ becomes a complete sbMLS.

To certify the existence of a unique solution of the integral equation, we prove the upcoming result.

Theorem 5. *Assume the following assumptions hold:*

a $k_1, k_2 : [0,1]\times[0,1]\times C([0,1],\mathbb{R}^+) \to \mathbb{R}$,

b Define the mappings $K_o, L_e : C([0,1],\mathbb{R}^+) \to C([0,1],\mathbb{R}^+)$ *by*

$$K_o(\mu)(\vartheta) = \int_0^{\vartheta} k_1(\vartheta,\hbar,\mu(\hbar))\, d\hbar, \quad L_e(\phi)(\vartheta) = \int_0^{\vartheta} k_2(\vartheta,\hbar,\phi(\hbar))\, d\hbar.$$

Assume there exists $\tau > 0$ *such that*

$$\left(k_1(\vartheta,\hbar,\mu) + k_2(\vartheta,\hbar,\phi)\right)^2 \le \frac{\tau\Xi_{(o,e)}(\mu,\phi)}{\tau\Xi_{(o,e)}(\mu,\phi)+1},$$

for every $\mu,\phi \in C([0,1],\mathbb{R}^+)$, $\vartheta,\hbar \in [0,1]$ *and*

$$\Xi_{(o,e)}(\mu,\phi) = \max\left\{\mho_b\|\mu+\phi\|^2, \|\mu+K_o(\mu)\|^2, \|\phi+L_e(\phi)\|^2, \frac{\|\mu+K_o(\mu)\|^2\cdot\|\phi+L_e(\phi)\|^2}{1+\|\mu+\phi\|^2}\right\}.$$

Then (5.2) and (5.3) have a common solution in $C([0,1],\mathbb{R}^+)$.

Proof. By supposition (ii),

$$\begin{aligned}\|K_o(\mu)+L_e(\phi)\|^2 &= \int_0^{\vartheta} \left(k_1(\vartheta,\hbar,\mu(\hbar)) + k_2(\vartheta,\hbar,\phi(\hbar))\right)^2 d\hbar \\ &\le \int_0^{\vartheta} \frac{\tau\Xi_{(o,e)}(\mu,\phi)}{\tau\Xi_{(o,e)}(\mu,\phi)+1} e^{\tau\hbar}\, d\hbar \\ &\le \frac{\Xi_{(o,e)}(\mu,\phi)}{\tau\Xi_{(o,e)}(\mu,\phi)+1}\int_0^{\vartheta} e^{\tau\hbar}\, d\hbar\end{aligned}$$

$$\leq \frac{\Xi_{(o,e)}(\mu,\phi)}{\tau\Xi_{(o,e)}(\mu,\phi)+1}e^{\tau\vartheta}.$$

This implies,

$$\|K_o(\mu)+L_e(\phi)\|^2 e^{-\tau\vartheta} \leq \frac{\Xi_{(o,e)}(\mu,\phi)}{\tau\Xi_{(o,e)}(\mu,\phi)+1},$$

$$\|K_o(\mu)+L_e(\phi)\|_\tau^2 \leq \frac{\Xi_{(o,e)}(\mu,\phi)}{\tau\Xi_{(o,e)}(\mu,\phi)+1},$$

$$\frac{\tau\Xi_{(o,e)}(\mu,\phi)+1}{\Xi_{(o,e)}(\mu,\phi)} \leq \frac{1}{\|K_o(\mu)+L_e(\phi)\|_\tau^2},$$

$$\tau+\frac{1}{\Xi_{(o,e)}(\mu,\phi)} \leq \frac{1}{\|K_o(\mu)+L_e(\phi)\|_\tau^2},$$

which further suggests

$$\tau-\frac{1}{\|K_o(\mu)+L_e(\phi)\|_\tau^2} \leq -\frac{1}{\Xi_{(o,e)}(\mu,\phi)}.$$

Hence, all assumptions of Theorem 5.1 are verified for $\mathcal{F}(c)=-\frac{1}{c}$ $(c>0)$ and $d_\tau(\mu,\phi)=\|\mu+\phi\|_\tau^2$. So the integrals given in (5.2) and (5.3) possess a common solution.

5.2. Fractional Differential Equations

We utilize our latest outcomes for the solution of fractional differential equations. Presently, a huge number of mathematicians utilized FP methodologies to attain the common solution of fractional differential equations, as debated in [3, 17, 21, 37].

Theorem 6. *Let $C[0,1]$ represent the space of all continuous functions. Moreover, the distance function $\Lambda_b : C[0,1]\times C[0,1]\to[0,\infty)$ defined by $\Lambda_b(u,v)=\|u+v\|_\infty^2$ for each $u,v\in C[0,1]$. Then $(C[0,1],\Lambda_b)$ is a complete sbMLS.*

Proof.

Let $R_1,R_2 : C[0,1]\times C[0,1]\to\mathbb{R}^+$ be continuous functions. The equations involving Caputo fractional derivatives of order σ will be analyzed:

$$D^\sigma f(n)=R_1(n,f(n)), \tag{5.4}$$

$$D^\sigma q(p)=R_2(p,q(p)), \tag{5.5}$$

with integral boundary conditions $f(0)=0$, $If(1)=f'(0)$, and $q(0)=0$, $Iq(1)=q'(0)$. Here D^σ represents the Caputo fractional derivative of order σ expressed as

$$D^\sigma f(n)=\frac{1}{\Gamma(n-\sigma)}\int_0^n (n-\hbar)^{\sigma-1}f(\hbar)\,d\hbar,$$

where $\sigma - 1 < \sigma < r$ and $\sigma = \sigma + 1$, and $I^\sigma f$ is defined as

$$I^\sigma f(n) = \frac{1}{\Gamma(\sigma)} \int_0^n (n-\hbar)^{\sigma-1} f(\hbar)\, d\hbar, \quad \sigma > 0.$$

Then (5.4) can be written as

$$f(n) = \frac{1}{\Gamma(\sigma)} \int_0^n (n-\hbar)^{\sigma-1} R_1(\hbar, f(\hbar))\, d\hbar + \frac{2n}{\Gamma(\sigma)} \int_0^b \int_0^\hbar (\hbar - z)^{\sigma-1} R_1(z, f(z))\, dz\, d\hbar,$$

and (5.5) can be written as

$$q(p) = \frac{1}{\Gamma(\sigma)} \int_0^p (p-\hbar)^{\sigma-1} R_2(\hbar, q(\hbar))\, d\hbar + \frac{2p}{\Gamma(\sigma)} \int_0^b \int_0^\hbar (\hbar - t)^{\sigma-1} R_2(t, q(t))\, dt\, d\hbar.$$

Theorem 7. *Suppose that:*

a There exists $\tau > 0$ and for each $f, q \in C[0,1]$, we have

$$R_1(n, f(n)) + R_2(p, q(p)) = \frac{e^{-\tau}\Gamma(\sigma+1)}{4}(f+q) \quad \text{and} \quad f, q > 0.$$

b There are $g, h \in C[0,1]$ for every $\varepsilon, s \in C[0,1]$,

$$g(\varepsilon) = \frac{1}{\Gamma(\sigma)} \int_0^\varepsilon (\varepsilon-\hbar)^{\sigma-1} R_1(\hbar, f(\hbar))\, d\hbar + \frac{2\varepsilon}{\Gamma(\sigma)} \int_0^b \int_0^\hbar (\hbar - z)^{\sigma-1} R_1(z, f(z))\, dz\, d\hbar,$$

and

$$h(s) = \frac{1}{\Gamma(\sigma)} \int_0^s (s-\hbar)^{\sigma-1} R_2(\hbar, q(\hbar))\, d\hbar + \frac{2s}{\Gamma(\sigma)} \int_0^b \int_0^\hbar (\hbar - t)^{\sigma-1} R_2(t, q(t))\, dt\, d\hbar.$$

Then, (5.4) and (5.5) retain a common solution in $C[0,1]$.

Proof. The mappings $K_o, L_e : C[0,1] \to C[0,1]$ are defined by

$$K_o(g)(\varepsilon) = \frac{1}{\Gamma(\sigma)} \int_0^\varepsilon (\varepsilon-\hbar)^{\sigma-1} R_1(\hbar, f(\hbar))\, d\hbar + \frac{2\varepsilon}{\Gamma(\sigma)} \int_0^b \int_0^\hbar (\hbar - z)^{\sigma-1} R_1(z, f(z))\, dz\, d\hbar,$$
$$L_e(h)(\varepsilon) = \frac{1}{\Gamma(\sigma)} \int_0^\varepsilon (\varepsilon-\hbar)^{\sigma-1} R_2(v, q(\hbar))\, d\hbar + \frac{2\varepsilon}{\Gamma(\sigma)} \int_0^b \int_0^\hbar (\hbar - t)^{\sigma-1} R_2(t, q(t))\, dt\, d\hbar.$$

Then from (ii), $g, h \in C[0,1]$ so that $K_o(g)(\varepsilon) = g(\varepsilon)$ and $L_e(h)(\varepsilon) = h(\varepsilon)$. The continuity of R_1 and R_2 indicates that the discrete families K_o and L_e are also continuous in $C[0,1]$. Now, we will try to prove the contraction restriction of Theorem 3.1. We deduce that:

$$\|K_o(g)(\varepsilon) + L_e(h)(\varepsilon)\|^2$$
$$= \left\| \frac{1}{\Gamma(\sigma)} \int_0^\varepsilon (\varepsilon-\hbar)^{\sigma-1} R_1(\hbar, f(\hbar))\, d\hbar + \frac{2\varepsilon}{\Gamma(\sigma)} \int_0^b \int_0^\hbar (\hbar - t)^{\sigma-1} R_1(t, f(t))\, dt\, d\hbar \right.$$

$$
\begin{aligned}
&+ \frac{1}{\Gamma(\sigma)}\int_0^{\varepsilon}(\varepsilon-\hbar)^{\sigma-1}R_2(\hbar,q(\hbar))\,d\hbar + \frac{2\varepsilon}{\Gamma(\sigma)}\int_0^b\int_0^{\hbar}(\hbar-t)^{\sigma-1}R_2(t,q(t))\,dt\,d\hbar\Bigg\|^2\\
&\leq \Bigg\|\frac{1}{\Gamma(\sigma)}\int_0^{\varepsilon}(\varepsilon-\hbar)^{\sigma-1}R_1(\hbar,f(\hbar))\,d\hbar + \frac{1}{\Gamma(\sigma)}\int_0^{\varepsilon}(\varepsilon-\hbar)^{\sigma-1}R_2(\hbar,q(\hbar))\,d\hbar\\
&+ \frac{2\varepsilon}{\Gamma(\sigma)}\int_0^b\int_0^{\hbar}(\hbar-t)^{\sigma-1}R_1(t,f(t))\,dt\,d\hbar + \frac{2\varepsilon}{\Gamma(\sigma)}\int_0^b\int_0^{\hbar}(\hbar-t)^{\sigma-1}R_2(t,q(t))\,dt\,d\hbar\Bigg\|^2\\
&\leq \frac{1}{\Gamma(\sigma)}\int_0^{\varepsilon}(\varepsilon-\hbar)^{\sigma-1}\|R_1(\hbar,f(\hbar))\|\,d\hbar + \frac{1}{\Gamma(\sigma)}\int_0^{\varepsilon}(\varepsilon-\hbar)^{\sigma-1}\|R_2(\hbar,q(\hbar))\|\,d\hbar\\
&+ \frac{2\varepsilon}{\Gamma(\sigma)}\int_0^b\int_0^{\hbar}(\hbar-t)^{\sigma-1}\|R_1(t,f(t))\|\,dt\,d\hbar + \frac{2\varepsilon}{\Gamma(\sigma)}\int_0^b\int_0^{\hbar}(\hbar-t)^{\sigma-1}\|R_2(t,q(t))\|\,dt\,d\hbar\\
&\leq \frac{1}{\Gamma(\sigma)}\frac{e^{-\tau}\Gamma(\sigma+1)}{4}\int_0^{\varepsilon}(\varepsilon-\hbar)^{\sigma-1}\|g(\hbar)+h(\hbar)\|\,d\hbar\\
&+ \frac{2\varepsilon}{\Gamma(\sigma)}\frac{e^{-\tau}\Gamma(\sigma+1)}{4}\int_0^b\int_0^{\hbar}(\hbar-t)^{\sigma-1}\|g(t)+h(t)\|\,dt\,d\hbar\\
&\leq \frac{1}{\Gamma(\sigma)}\frac{e^{-\tau}\Gamma(\sigma+1)}{4}\|g+h\|\int_0^{\varepsilon}(\varepsilon-\hbar)^{\sigma-1}\,d\hbar\\
&+ \frac{2\varepsilon}{\Gamma(\sigma)}\frac{e^{-\tau}\Gamma(\sigma+1)}{4}\|g+h\|\int_0^b\int_0^{\hbar}(\hbar-t)^{\sigma-1}\,dt\,d\hbar\\
&\leq \frac{1}{\Gamma(\sigma)}\frac{e^{-\tau}\Gamma(\sigma+1)}{4}\|g+h\|\frac{\varepsilon^{\sigma}}{\sigma}\\
&+ \frac{2\varepsilon}{\Gamma(\sigma)}\frac{e^{-\tau}\Gamma(\sigma+1)}{4}\|g+h\|\int_0^b\frac{\hbar^{\sigma}}{\sigma}\,d\hbar\\
&\leq \frac{e^{-\tau}\Gamma(\sigma+1)}{4\Gamma(\sigma)\sigma}\|g+h\|\varepsilon^{\sigma} + \frac{2\varepsilon e^{-\tau}\Gamma(\sigma+1)}{4\Gamma(\sigma)\sigma}\|g+h\|\frac{b^{\sigma+1}}{\sigma+1}\\
&\leq \frac{e^{-\tau}}{4}\|g+h\| + \frac{2e^{-\tau}\beta(\sigma+1,1)}{4}\|g+h\|\\
&\leq \frac{e^{-\tau}}{4}\|g+h\| + \frac{e^{-\tau}}{2}\|g+h\| \leq e^{-\tau}\|g+h\|.
\end{aligned}
$$

Since β represents the beta function. So, we obtain

$$\|K_o(g)(\varepsilon)+L_e(h)(\varepsilon)\|^2 \leq e^{-\tau}\|g+h\|.$$

Taking the square of both sides, we have

$$\|K_o(g)(\varepsilon)+L_e(h)(\varepsilon)\|^2 \leq e^{-2\tau}\|g+h\|^2.$$

Since $\tau>0$ is chosen arbitrarily, it signifies that $e^{-2\tau}<e^{-\tau}$, then the last inequality can be expressed as

$$\|K_o(g)(\varepsilon)+L_e(h)(\varepsilon)\|^2 \leq e^{-\tau}\|g+h\|^2. \tag{5.6}$$

Now, (5.6) can be written as

$$\Lambda_b(K_o(g)(\varepsilon), L_e(h)(\varepsilon)) \leq e^{-\tau}\Lambda_b(g,h),$$
$$e^{\tau}\Lambda_b(K_o(g)(\varepsilon), L_e(h)(\varepsilon)) \leq \Lambda_b(g,h). \tag{5.7}$$

Defining $\mathcal{F}(x) = \ln x$, we have $\mho_b(\Xi_{(o,e)}(g,h)) = \Lambda_b(g,h)$. Moreover, equation (5.7) can be rewritten as:

$$\begin{aligned} \ln\left(e^{\tau}\Lambda_b(K_o(g)(\varepsilon), L_e(h)(\varepsilon))\right) &\leq \ln(\mho_b(\Xi_{(o,e)}(g,h))), \\ \ln(e^{\tau}) + \ln(\Lambda_b(K_o(g)(\varepsilon), L_e(h)(\varepsilon))) &\leq \ln(\mho_b(\Xi_{(o,e)}(g,h))), \\ \tau + \mathcal{F}(\Lambda_b(K_o(g), L_e(h))) &\leq \mathcal{F}(\mho_b(\Xi_{(o,e)}(g,h))). \end{aligned}$$

All assumptions of Theorem 3.1 are satisfied. Therefore, (5.4) and (5.5) possess a common solution.

6. Conclusion

In this paper, we establish some FP findings for discrete families of multivalued dominated nonlinear operators fulfilling generalized locally contractions in the framework of complete sbMLS. Innovative FP theorems for discrete families of ordered multi-maps are established in an ordered complete sbMLS. Moreover, the concept of discrete families of multi-graph-dominated operators is illustrated on a closed ball in these spaces, along with some original discoveries regarding graph contraction involving graph-dominated structure. Some definitions and examples are provided to substantiate our obtained outcomes. Lastly, to demonstrate the uniqueness of our findings, we present their application in solving nonlinear fractional and integral equations.

In the future, our work can be enhanced by examining families of fuzzy maps, L-fuzzy maps, bipolar fuzzy maps and intuitionistic fuzzy maps. This idea can be used in future studies to examine new FP results in various new metric spaces such as intuitionistic fuzzy metric-like spaces, strong fuzzy b-metric spaces and quasi strong fuzzy b-metric spaces.

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