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$\mathcal{Q}$-ideals of Almost Distributive Lattices

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Abstract. In Almost Distributive Lattices (ADLs), the idea of $\mathcal{Q}$-ideals is defined, and various properties of these ideals are investigated. characterizations are established that determine precisely when a λ-ideal in an ADL qualifies as a $\mathcal{Q}$-ideal. Furthermore, equivalent conditions are established for when an $\mathcal{E}$-ideal in an ADL can be recognized as a $\mathcal{Q}$-ideal. The characterization of $\mathcal{E}$-complemented ADLs is achieved through the use of $\mathcal{Q}$-ideals.

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1. Introduction

In the note [1], the authors introduced the concepts of dual annihilators and μ-filters in ADLs. Certain topological properties of prime μ-filters are also investigated in this paper. In [2], the authors investigated certain important properties of prime $\mathcal{E}$-ideals of ADLs. In their recent contribution [3], Rafi et al. established the theory of ν-ideals in ADLs and obtained a characterization based on minimal prime $\mathcal{E}$-ideals. In [4], the authors introduced the concepts of $\mathcal{R}-$ideals and λ-ideals of an ADLs.

The primary objective of this paper is to provide a characterization of $\mathcal{E}$-complemented ADLs using a specific type of $\mathcal{E}$-ideals found in ADLs. The paper develops the concept of $\mathcal{Q}$-ideals and explores several of their structural features using maximal ideals together

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with minimal prime $\mathcal{E}$-ideals in ADLs. Initially, $\mathcal{E}$-quasi-complemented ADLs are characterized through their prime $\mathcal{E}$-ideals. It is noted that every $\mathcal{Q}$-ideal in an ADL is also a λ-ideal. A set of equivalent conditions is provided to establish when a λ-ideal of an ADL qualifies as a $\mathcal{Q}$-ideal. Moreover, it is observed that while every proper $\mathcal{Q}$-ideal of an ADL is a ν-ideal. However, equivalent conditions are given for when a ν-ideal in an ADL can be classified as a $\mathcal{Q}$-ideal. Additional equivalent conditions are outlined for when the set of all $\mathcal{Q}$-ideals forms a sublattice of the lattice of all ideals, leading to a further characterization of $\mathcal{E}$-complemented ADLs. Another theorem is presented which shows that every $\mathcal{E}$-ideal in an $\mathcal{E}$-complemented ADL becomes a $\mathcal{Q}$-ideal. Finally, Boolean algebras are characterized using $\mathcal{Q}$-ideals of ADLs.

2. Preliminaries

The necessary definitions and major results from [5, 6] are summarized here for use throughout the paper.

Definition 1. *[6] We call an algebra* $(\mathcal{L}, \vee, \wedge, 0)$ *of type* $(2, 2, 0)$ *an Almost Distributive Lattice (ADL) with zero if it satisfies the following set of axioms. :*

(1) $(\varsigma \vee \varepsilon) \wedge \zeta = (\varsigma \wedge \zeta) \vee (\varepsilon \wedge \zeta)$;
(2) $\varsigma \wedge (\varepsilon \vee \zeta) = (\varsigma \wedge \varepsilon) \vee (\varsigma \wedge \zeta)$;
(3) $(\varsigma \vee \varepsilon) \wedge \varepsilon = \varepsilon$;
(4) $(\varsigma \vee \varepsilon) \wedge \varsigma = \varsigma$;
(5) $\varsigma \vee (\varsigma \wedge \varepsilon) = \varsigma$;
(6) $0 \wedge \varsigma = 0$, *for any* $\varsigma, \varepsilon, \zeta \in \mathcal{L}$.

For elements $\alpha, \beta \in \mathcal{L}$, the condition

$$\alpha = \alpha \wedge \beta \qquad \text{(equivalently, } \alpha \vee \beta = \beta\text{)}$$

is interpreted as $\alpha \leq \beta$. This relation defines a partial order on the ADL $(\mathcal{L}, \vee, \wedge, 0)$. An element $\mathfrak{m} \in \mathcal{L}$ that is maximal with respect to this order is called a *maximal element*, and the set of all such elements is denoted by $\mathcal{M}_{\text{Max.elts}}$. As noted by Swamy [6], an ADL $\mathcal{L}$ exhibits almost all of the structural properties of a distributive lattice, except for the lack of commutativity between $\vee$ and $\wedge$, and the failure of right distributivity of $\vee$ over $\wedge$. The presence of either of these conditions would make $\mathcal{L}$ a distributive lattice. Let $\mathcal{S}$ be a nonempty subset of $\mathcal{L}$. The set $\mathcal{S}$ is an *ideal* (respectively, a *filter*) if for all $\alpha, \beta \in \mathcal{S}$ and $\mu \in \mathcal{L}$ one has

$$\alpha \vee \beta,\ \alpha \wedge \mu \in \mathcal{S} \quad \text{(respectively, } \alpha \wedge \beta,\ \mu \vee \alpha \in \mathcal{S}\text{)}.$$

A maximal ideal (respectively, maximal filter) contains every ideal (filter) properly contained in it. For any subset $\mathcal{G} \subseteq \mathcal{L}$, the ideal generated by $\mathcal{G}$ is

$$(\mathcal{G}] := \left\{ \left(\bigvee_{i=1}^{n} \alpha_i \right) \wedge x \;\middle|\; \alpha_i \in \mathcal{G},\ x \in \mathcal{L},\ n \in \mathbb{N} \right\}.$$

If $\mathcal{G} = \{\alpha\}$, we write $(\alpha]$ and call it the *principal ideal* generated by α. Likewise, the filter generated by $\mathcal{G}$ is

$$[\mathcal{G}) := \left\{ x \vee \left(\bigwedge_{i=1}^{n} \alpha_i \right) \;\middle|\; \alpha_i \in \mathcal{G},\, x \in \mathcal{L},\, n \in \mathbb{N} \right\},$$

and for $\mathcal{G} = \{\alpha\}$, we write $[\alpha)$ for the principal filter. It is routine to verify that, for all $\alpha, \beta \in \mathcal{L}$,

$$(\alpha] \vee (\beta] = (\alpha \vee \beta], \qquad (\alpha] \cap (\beta] = (\alpha \wedge \beta].$$

Thus the system of principal ideals $(\mathcal{PI}(\mathcal{L}), \vee, \cap)$ forms a sublattice of the distributive lattice $(\mathcal{S}(\mathcal{L}), \vee, \cap)$ of all ideals of $\mathcal{L}$. Similarly, the lattice of all filters $(\mathcal{F}(\mathcal{L}), \vee, \cap)$ is a bounded distributive lattice. Rao [7] established that a prime ideal $\mathcal{A}$ in $\mathcal{L}$ exists exactly when its complement $\mathcal{L} \setminus \mathcal{A}$ is a prime filter of $\mathcal{L}$.

Proposition 1 ([2])**.** *Let $\mathcal{L}$ be an ADL and $\alpha, \beta, \gamma \in \mathcal{L}$. Then:*

(i) If $\alpha \leq \beta$, then $(\beta, \mathcal{E}) \subseteq (\alpha, \mathcal{E})$.

(ii) $(\alpha \vee \beta, \mathcal{E}) = (\alpha, \mathcal{E}) \cap (\beta, \mathcal{E})$.

(iii) $((\alpha \wedge \beta, \mathcal{E}), \mathcal{E}) = ((\alpha, \mathcal{E}), \mathcal{E}) \cap ((\beta, \mathcal{E}), \mathcal{E})$.

(iv) $(\alpha, \mathcal{E}) = \mathcal{L}$ if and only if $\alpha \in \mathcal{E}$.

A prime $\mathcal{E}$-ideal $\mathcal{X}$ of $\mathcal{L}$ is called a *minimal prime $\mathcal{E}$-ideal over $\mathcal{J}$* (where $\mathcal{J}$ is an $\mathcal{E}$-ideal) if

$$\mathcal{J} \subseteq \mathcal{X} \quad \text{and there is no prime } \mathcal{E}\text{-ideal } \mathcal{W} \text{ with } \mathcal{J} \subseteq \mathcal{W} \subsetneq \mathcal{X}.$$

When $\mathcal{J} = \mathcal{E}$, the ideal $\mathcal{X}$ is simply referred to as a *minimal prime $\mathcal{E}$-ideal.* As shown in [2], a prime $\mathcal{E}$-ideal $\mathcal{A}$ is minimal if and only if for every $x \in \mathcal{A}$ there exists $y \notin \mathcal{A}$ such that $x \wedge y \in \mathcal{E}$. An ideal $\mathcal{S}$ of an ADL is called an $\mathcal{R}$-ideal [4] if

$$\mathcal{S} = ((\mathcal{S}, \mathcal{E}), \mathcal{E}).$$

Every ideal of the form $(x, \mathcal{E})$ is an $\mathcal{R}$-ideal. An ideal $\mathcal{S}$ is called a *λ-ideal* [4] if

$$((x, \mathcal{E}), \mathcal{E}) \subseteq \mathcal{S} \qquad \text{whenever } x \in \mathcal{S}.$$

Clearly, every $\mathcal{R}$-ideal is a λ-ideal. For a filter $\mathcal{H}$ of $\mathcal{L}$, define

$$\nu(\mathcal{H}) := \{x \in \mathcal{L} \mid x \wedge a \in \mathcal{E} \text{ for some } a \in \mathcal{H}\}.$$

As shown in [3], $\nu(\mathcal{H})$ is always an $\mathcal{E}$-ideal of $\mathcal{L}$. An ideal of the form $\nu(\mathcal{H})$ is called a *ν-ideal*, and every minimal prime $\mathcal{E}$-ideal of $\mathcal{L}$ is a ν-ideal. An element α of an ADL with maximal elements is said to be *$\mathcal{E}$-complemented* if there exists $\beta \in \mathcal{L}$ such that

$$\alpha \wedge \beta \in \mathcal{E} \qquad \text{and} \qquad \alpha \vee \beta \text{ is a maximal element of } \mathcal{L}.$$

The ADL $\mathcal{L}$ is called an *$\mathcal{E}$-complemented ADL* if every element of $\mathcal{L}$ is $\mathcal{E}$-complemented.

3. $\mathcal{Q}$-ideals

This section develops the notion of $\mathcal{Q}$-ideals in an Almost Distributive Lattice. The correspondence between $\mathcal{Q}$-ideals and ν-ideals is established, and several equivalent formulations are given to characterize $\mathcal{Q}$-ideals among the ideals of an ADL.

Lemma 1. *Every maximal ideal of an ADL is a prime $\mathcal{E}$-ideal.*

Proof. Let $\mathcal{X}$ be a maximal ideal of $\mathcal{L}$, and consider $\mu \in \mathcal{E}$. It is clear that $\mathcal{X}$ is a prime ideal. Assume that $\mu \notin \mathcal{X}$. Since $\mathcal{X}$ is maximal, it follows that $\mathcal{X} \vee (\mu] = \mathcal{L}$. This means that m must belong to $\mathcal{X} \vee (\mu]$. Consequently, there is $\varsigma \in \mathcal{X}$ satisfying $\varsigma \vee \mu = m$. Hence, ς is an element of $(\mu)^+ = \mathcal{M}_{max.elt}$, which contradicts the hypothesis. Hence, we must have that $\mu \in \mathcal{X}$, establishing that $\mathcal{E} \subseteq \mathcal{X}$. Thus, $\mathcal{X}$ is a prime $\mathcal{E}$-ideal of $\mathcal{L}$.

Theorem 1. *The assertions below are equivalent in $\mathcal{L}$*

(1) *$\mathcal{L}$ is $\mathcal{E}$-complemented;*

(2) *every prime $\mathcal{E}$-ideal is maximal;*

(3) *every prime $\mathcal{E}$-ideal is minimal.*

Proof. (1) $\Rightarrow$ (2): Suppose that $\mathcal{L}$ is $\mathcal{E}$-complemented and that $\mathcal{A}$ is a prime $\mathcal{E}$-ideal of $\mathcal{L}$. If there exists a proper ideal $\mathcal{V}$ strictly containing $\mathcal{A}$, then we can select an element $\mu \in \mathcal{V}$ with $\mu \notin \mathcal{A}$. Since $\mathcal{L}$ is $\mathcal{E}$-complemented, there exists an element $\pi \in \mathcal{L}$ for which $\mu \wedge \pi \in \mathcal{E}$ and $\mu \vee \pi \in \mathcal{M}_{max.elt}$. Given that $\mu \notin \mathcal{A}$, it follows that $(\mu, \mathcal{E}) \subseteq \mathcal{A}$. Consequently, $\pi \in (\mu, \mathcal{E}) \subseteq \mathcal{A} \subset \mathcal{V}$. Thus, $\mu \vee \pi \in \mathcal{V}$, leading a contradiction. Hence, we deduce that $\mathcal{A}$ must be a maximal ideal.
(2) $\Rightarrow$ (3): Since every maximal ideal is also a prime $\mathcal{E}$-ideal, this is evident.
(3) $\Rightarrow$ (1): Assume (3). Let μ be an element of $\mathcal{L}$. Suppose that $(\mu] \vee (\mu, \mathcal{E}) \neq \mathcal{L}$. This implies there exists a prime $\mathcal{E}$-ideal $\mathcal{A}$ in $\mathcal{L}$ such that $(\mu] \vee (\mu, \mathcal{E}) \subseteq \mathcal{A}$. Consequently, we have $\mu \in \mathcal{A}$ and $(\mu, \mathcal{E}) \subseteq \mathcal{A}$. Given that $\mathcal{A}$ is minimal and contains $(\mu, \mathcal{E})$, it must follow that $\mu \notin \mathcal{A}$, which yields a contradiction. Hence, we deduce that $(\mu] \vee (\mu, \mathcal{E}) = \mathcal{L}$. Consequently, $m \in (\mu] \vee (\mu, \mathcal{E})$, where $m \in \mathcal{M}_{max.elt}$. Thus, there exists an element $\varepsilon \in (\mu, \mathcal{E})$ such that $\mu \vee \varepsilon \in \mathcal{M}_{max.elt}$. Since $\varepsilon \in (\mu, \mathcal{E})$, it follows that $\varepsilon \wedge \mu \in \mathcal{E}$. Thus, we can conclude that $\mathcal{L}$ is $\mathcal{E}$-complemented.

Definition 2. *Given an ideal $\mathcal{S}$ in $\mathcal{L}$, we define the set $\mathcal{Q}(\mathcal{S})$ as follows:*

$$\mathcal{Q}(\mathcal{S}) = \{\mu \in \mathcal{L} \mid (\mu, \mathcal{E}) \vee \mathcal{S} = \mathcal{L}\}.$$

Clearly $\mathcal{Q}(\mathcal{L}) = \mathcal{L}$. For $\mathcal{S} = \mathcal{E}$, obviously we get $\mathcal{Q}(\mathcal{E}) = \mathcal{E}$.

Lemma 2. *For any ideal $\mathcal{S}$ of an ADL $\mathcal{L}$, $\mathcal{Q}(\mathcal{S})$ is an $\mathcal{E}$-ideal of $\mathcal{L}$.*

Proof. Clearly, $\mathcal{E} \subseteq \mathcal{Q}(\mathcal{S})$. Let $\mu, \pi \in \mathcal{Q}(\mathcal{S})$. Then, we have $(\mu, \mathcal{E}) \vee \mathcal{S} = \mathcal{L}$ and $(\pi, \mathcal{E}) \vee \mathcal{S} = \mathcal{L}$. Consequently, $(\mu \vee \pi, \mathcal{E}) \vee \mathcal{S} = \big\{(\mu, \mathcal{E}) \cap (\pi, \mathcal{E})\big\} \vee \mathcal{S} = \big\{(\mu, \mathcal{E}) \vee \mathcal{S}\big\} \cap \big\{(\pi, \mathcal{E}) \vee \mathcal{S}\big\} = \mathcal{L} \cap \mathcal{L} = \mathcal{L}$. Thus, $\mu \vee \pi \in \mathcal{Q}(\mathcal{S})$. Now let $\mu \in \mathcal{Q}(\mathcal{S})$ and $\pi \in \mathcal{L}$. Then, we have $(\mu,\mathcal{E}) \subseteq (\mu\wedge\pi,\mathcal{E})$, which implies that $\mathcal{L} = (\mu,\mathcal{E})\vee\mathcal{S} \subseteq (\mu\wedge\pi,\mathcal{E})\vee\mathcal{S}$. Therefore, $\mu\wedge\pi \in \mathcal{Q}(\mathcal{S})$. This shows that $\mathcal{Q}(\mathcal{S})$ is an $\mathcal{E}$-ideal in $\mathcal{L}$.

The subsequent result derived various basic properties of $\mathcal{Q}(\mathcal{S})$.

Lemma 3. *For any two ideals $\mathcal{S}$ and $\mathcal{T}$ in $\mathcal{L}$, the following statement is true:*

(1) $\mathcal{E} \subseteq \mathcal{S}$ *iff* $\mathcal{Q}(\mathcal{S}) \subseteq \mathcal{S}$,
(2) $\mathcal{S} \subseteq \mathcal{T}$ *implies* $\mathcal{Q}(\mathcal{S}) \subseteq \mathcal{Q}(\mathcal{T})$,
(3) $\mathcal{Q}(\mathcal{S}\cap\mathcal{T}) = \mathcal{Q}(\mathcal{S})\cap\mathcal{Q}(\mathcal{T})$,
(4) $\mathcal{Q}(\mathcal{S})\vee\mathcal{Q}(\mathcal{T}) \subseteq \mathcal{Q}(\mathcal{S}\vee\mathcal{T})$.

Proof. (1) Assume that $\mathcal{E} \subseteq \mathcal{S}$. Let μ be an element of $\mathcal{Q}(\mathcal{S})$. Then, we have $(\mu,\mathcal{E})\vee\mathcal{S} = \mathcal{L}$. Consequently, it follows that $\mu \in (\mu,\mathcal{E})\vee\mathcal{S}$. Thus, we can write $\mu = \varsigma\vee\varepsilon$ for some $\varsigma \in (\mu,\mathcal{E})$ and $\varepsilon \in \mathcal{S}$. Since ς belongs to $(\mu,\mathcal{E})$, it follows that $\varsigma\wedge\mu \in \mathcal{E}$. Therefore, there exists $e \in \mathcal{E}$ such that $\varsigma\wedge\mu = e$. This allows us to express μ as:

$$\mu = \mu\wedge\mu = (\varsigma\vee\varepsilon)\wedge\mu = (\varsigma\wedge\mu)\vee(\varepsilon\wedge\mu) = e\vee(\varepsilon\wedge\mu) \in \mathcal{E}\vee\mathcal{S} = \mathcal{S},$$

because $\varepsilon\wedge\mu \in \mathcal{S}$. Thus, we conclude that $\mathcal{Q}(\mathcal{S}) \subseteq \mathcal{S}$. The converse is straightforward, given that $\mathcal{E} \subseteq \mathcal{Q}(\mathcal{S})$.
(2) Assume that $\mathcal{S} \subseteq \mathcal{T}$. Let μ be an element of $\mathcal{Q}(\mathcal{S})$. Then, we have $\mathcal{L} = (\mu,\mathcal{E})\vee\mathcal{S} \subseteq (\mu,\mathcal{E})\vee\mathcal{T}$. Thus, it follows that $\mu \in \mathcal{Q}(\mathcal{T})$.
(3) It is evident that $\mathcal{Q}(\mathcal{S}\cap\mathcal{T}) \subseteq \mathcal{Q}(\mathcal{S})\cap\mathcal{Q}(\mathcal{T})$. Conversely, let μ be an element of $\mathcal{Q}(\mathcal{S})\cap\mathcal{Q}(\mathcal{T})$. Then we have $(\mu,\mathcal{E})\vee\mathcal{S} = (\mu,\mathcal{E})\vee\mathcal{T} = \mathcal{L}$. Now, consider $(\mu,\mathcal{E})\vee(\mathcal{S}\cap\mathcal{T}) = \{(\mu,\mathcal{E})\vee\mathcal{S}\}\cap\{(\mu,\mathcal{E})\vee\mathcal{T}\} = \mathcal{L}\cap\mathcal{L} = \mathcal{L}$. Therefore, it follows that $\mu \in \mathcal{Q}(\mathcal{S}\cap\mathcal{T})$. Hence, we conclude that $\mathcal{Q}(\mathcal{S})\cap\mathcal{Q}(\mathcal{T}) \subseteq \mathcal{Q}(\mathcal{S}\cap\mathcal{T})$. Thus, we have $\mathcal{Q}(\mathcal{S}\cap\mathcal{T}) = \mathcal{Q}(\mathcal{S})\cap\mathcal{Q}(\mathcal{T})$.
(4) This is a consequence of (2).

Definition 3. *An ideal $\mathcal{S}$ of an ADL $\mathcal{L}$ is referred as a $\mathcal{Q}$-ideal if $\mathcal{S} = \mathcal{Q}(\mathcal{S})$.*

It is evident that $\mathcal{E}$ and $\mathcal{L}$ are $\mathcal{Q}$-ideals within $\mathcal{L}$. In [3], the set of all $\mathcal{R}$-ideals in $\mathcal{L}$ is characterized using $\mathcal{E}$-annulets of an ADL. In the subsequent theorem, it is demonstrated that the collection of all $\mathcal{R}$-ideals of an ADL $\mathcal{L}$ properly includes the collection of all $\mathcal{Q}$-ideals of $\mathcal{L}$.

Proposition 2. *Every $\mathcal{Q}$-ideal of an ADL is an $\mathcal{R}$-ideal.*

Proof. Let $\mathcal{S}$ be a $\mathcal{Q}$-ideal within ADL $\mathcal{L}$. This means that $\mathcal{Q}(\mathcal{S}) = \mathcal{S}$. Let μ be an element of $\mathcal{S}$. Then, we know that $(\mu,\mathcal{E})\vee\mathcal{S} = \mathcal{L}$. Now, consider ν belonging to $((\mu,\mathcal{E}),\mathcal{E})$. Since $(\mu,\mathcal{E}) \subseteq (\nu,\mathcal{E})$, it follows that $\mathcal{L} = (\mu,\mathcal{E})\vee\mathcal{S} \subseteq (\nu,\mathcal{E})\vee\mathcal{S}$. Hence, ν is an element of $\mathcal{Q}(\mathcal{S}) = \mathcal{S}$, which leads to the conclusion that $((\mu,\mathcal{E}),\mathcal{E}) \subseteq \mathcal{S}$. Therefore, $\mathcal{S}$ qualifies as an $\mathcal{R}$-ideal of $\mathcal{L}$.

The next theorem provides a set of equivalent conditions that must be satisfied for every $\mathcal{R}$-ideal in ADL to be classified as a $\mathcal{Q}$-ideal.

Theorem 2. *The subsequent statements are equivalent in an ADL $\mathcal{L}$:*

(1) *every λ-ideal is a $\mathcal{Q}$-ideal;*

(2) *every $\mathcal{R}-$ideal is a $\mathcal{Q}$-ideal;*

(3) *for each $\mu \in \mathcal{L}$, $((\mu, \mathcal{E}), \mathcal{E})$ is a $\mathcal{Q}$-ideal;*

(4) *for each $\mu \in \mathcal{L}$, $(\mu, \mathcal{E}) \vee ((\mu, \mathcal{E}), \mathcal{E}) = \mathcal{L}$.*

Proof. (1) $\Rightarrow$ (2): It is straightforward.
(2) $\Rightarrow$ (3): Since every $((\mu, \mathcal{E}), \mathcal{E})$ is an $\mathcal{R}-$ideal, it is straightforward.
(3) $\Rightarrow$ (4): Assuming condition (3), let μ be an element of $\mathcal{L}$. Since $((\mu, \mathcal{E}), \mathcal{E})$ constitutes a $\mathcal{Q}$-ideal within $\mathcal{L}$, it follows that $((\mu, \mathcal{E}), \mathcal{E}) = \mathcal{Q}\big(((\mu, \mathcal{E}), \mathcal{E})\big)$. It follows immediately that μ is included in $((\mu, \mathcal{E}), \mathcal{E}) = \mathcal{Q}\big(((\mu, \mathcal{E}), \mathcal{E})\big)$. Thus, we conclude that $(\mu, \mathcal{E}) \vee ((\mu, \mathcal{E}), \mathcal{E}) = \mathcal{L}$.
(4) $\Rightarrow$ (1): Assume that for every $\mu \in \mathcal{L}$, we have $(\mu, \mathcal{E}) \vee ((\mu, \mathcal{E}), \mathcal{E}) = \mathcal{L}$. Let $\mathcal{S}$ be a λ-ideal in $\mathcal{L}$. It is evident that $\mathcal{Q}(\mathcal{S}) \subseteq \mathcal{S}$. Conversely, let $\mu \in \mathcal{S}$. As $\mathcal{S}$ is a λ-ideal, it follows that $((\mu, \mathcal{E}), \mathcal{E}) \subseteq \mathcal{S}$. Therefore, we have $\mathcal{L} = (\mu, \mathcal{E}) \vee ((\mu, \mathcal{E}), \mathcal{E}) \subseteq (\mu, \mathcal{E}) \vee \mathcal{S}$. Consequently, $\mu \in \mathcal{Q}(\mathcal{S})$. Thus, we conclude that $\mathcal{S}$ is a $\mathcal{Q}$-ideal of $\mathcal{L}$.

As shown in [3], a ν-ideal in an ADL coincides with the intersection of all minimal prime $\mathcal{E}$-ideals that contain it. The subsequent discussion demonstrates that the class of $\mathcal{Q}$-ideals is properly contained in the class of ν-ideals.

Theorem 3. *Every proper $\mathcal{Q}$-ideal of $\mathcal{L}$ with maximal element m is an ν-ideal.*

Proof. Let $\mathcal{S}$ be a proper $\mathcal{Q}$-ideal within an ADL $\mathcal{L}$, implying that $\mathcal{Q}(\mathcal{S}) = \mathcal{S}$. Consider the set defined as $\mathcal{I} = \{\mu \in \mathcal{L} \mid ((\mu, \mathcal{E}), \mathcal{E}) \vee \mathcal{S} = \mathcal{L}\}$. First, we will establish that $\mathcal{I}$ is an ideal of $\mathcal{L}$ such that $\mathcal{I} \cap \mathcal{E} = \emptyset$. It is clear that $m \in \mathcal{I}$. Let μ and π be elements of $\mathcal{I}$. Then we can write:

$$((\mu \wedge \pi, \mathcal{E}), \mathcal{E}) \vee \mathcal{S} = \{((\mu, \mathcal{E}), \mathcal{E}) \cap ((\pi, \mathcal{E}), \mathcal{E})\} \vee \mathcal{S} = \{((\mu, \mathcal{E}), \mathcal{E}) \vee \mathcal{S}\} \cap \{((\pi, \mathcal{E}), \mathcal{E}) \vee \mathcal{S}\} = \mathcal{L} \cap \mathcal{L} = \mathcal{L}.$$

Thus, it follows that $\mu \wedge \pi \in \mathcal{I}$. Next, let $\mu \in \mathcal{I}$ and $\pi \leq \mu$. Since $\mathcal{L} = ((\mu, \mathcal{E}), \mathcal{E}) \vee \mathcal{S} \subseteq ((\pi, \mathcal{E}), \mathcal{E}) \vee \mathcal{S}$, it follows that $\pi \in \mathcal{I}$. Consequently, $\mathcal{I}$ forms a filter in $\mathcal{L}$. Now, assume $\mu \in \mathcal{I} \cap \mathcal{E}$. This gives us $((\mu, \mathcal{E}), \mathcal{E}) \vee \mathcal{S} = \mathcal{L}$ and $((\mu, \mathcal{E}), \mathcal{E}) = \mathcal{E}$. Hence, this implies $\mathcal{S} = \mathcal{E} \vee \mathcal{S} = \mathcal{L}$, which produces a contradiction. Therefore, we can conclude that $\mathcal{I} \cap \mathcal{E} = \emptyset$. Finally, we demonstrate that $\mathcal{S} = \nu(\mathcal{I})$. If $\mu \in \nu(\mathcal{I})$, then there exists an element $\pi \in \mathcal{I}$ such that $\mu \wedge \pi \in \mathcal{E}$. Now

$$\begin{aligned} \mu \wedge \pi \in \mathcal{E} &\Rightarrow \pi \in (\mu, \mathcal{E}) \\ &\Rightarrow ((\pi, \mathcal{E}), \mathcal{E}) \subseteq (\mu, \mathcal{E}) \\ &\Rightarrow \mathcal{L} = ((\pi, \mathcal{E}), \mathcal{E}) \vee \mathcal{S} \subseteq (\mu, \mathcal{E}) \vee \mathcal{S} \qquad \text{since } y \in S \\ &\Rightarrow \mu \in \mathcal{Q}(\mathcal{S}) = \mathcal{S} \qquad \text{since } \mathcal{S} \text{ is a } \mathcal{Q}\text{-ideal} \end{aligned}$$

This leads us to conclude that $\nu(\mathcal{I}) \subseteq \mathcal{S}$. Now, let's consider the opposite direction. Let μ be an element of $\mathcal{S}$, which we know is equal to $\mathcal{Q}(\mathcal{S})$. From this, it follows that $(\mu, \mathcal{E}) \vee \mathcal{Q}(\mathcal{S}) = \mathcal{L}$. As a result, we find that m belongs to $(\mu, \mathcal{E}) \vee \mathcal{Q}(\mathcal{S})$. Thus, we can write m as $m = \varsigma \vee \varepsilon$, where ς is an element of $(\mu, \mathcal{E})$ and ε is an element of $\mathcal{Q}(\mathcal{S})$. Consequently, we have $\varsigma \wedge \mu \in \mathcal{E}$, and we can also assert that $(\varepsilon, \mathcal{E}) \vee \mathcal{S} = \mathcal{L}$. Now

$$\begin{aligned}
\varsigma \vee \varepsilon = m \quad &\Rightarrow \quad (\varsigma \vee \varepsilon, \mathcal{E}) = (m, \mathcal{E}) = \mathcal{E} \\
&\Rightarrow \quad (\varsigma, \mathcal{E}) \cap (\varepsilon, \mathcal{E}) = \mathcal{E} \\
&\Rightarrow \quad (\varepsilon, \mathcal{E}) \subseteq ((\varsigma, \mathcal{E}), \mathcal{E}) \\
&\Rightarrow \quad \mathcal{L} = (\varepsilon, \mathcal{E}) \vee \mathcal{S} \subseteq ((\varsigma, \mathcal{E}), \mathcal{E}) \vee \mathcal{S} \qquad \text{since } b \in \mathcal{Q}(I) \\
&\Rightarrow \quad \varsigma \in \mathcal{I} \text{ and } \varsigma \wedge \mu \in \mathcal{E} \\
&\Rightarrow \quad \mu \in \nu(\mathcal{I})
\end{aligned}$$

This establishes that $\mathcal{S} = \mathcal{Q}(\mathcal{S}) \subseteq \nu(\mathcal{I})$. Consequently, we conclude that $\mathcal{S} = \nu(\mathcal{I})$. Therefore, $\mathcal{S}$ is identified as a ν-ideal of $\mathcal{L}$.

Proposition 3. *For each $\varsigma \in \mathcal{L} - \mathcal{E}$, $(\varsigma, \mathcal{E})$ is a ν-ideal of an ADL $\mathcal{L}$.*

Proof. Let $\varsigma \in \mathcal{L} - \mathcal{E}$. It is evident that $[\varsigma) \cap \mathcal{E} = \emptyset$. We will prove that $(\varsigma, \mathcal{E}) = \nu([\varsigma))$. First, assume $\mu \in (\varsigma, \mathcal{E})$. This indicates that $\mu \wedge \varsigma \in \mathcal{E}$. Since ς belongs to $[\varsigma)$, it follows that $\mu \in \nu([\varsigma))$. Therefore, we have $(\varsigma, \mathcal{E}) \subseteq \nu([\varsigma))$. Let $\mu \in \nu([\varsigma))$. This means that there exists some $\nu \in [\varsigma)$ such that $\mu \wedge \nu \in \mathcal{E}$. Given that $\mu \wedge \nu \leq \mu \wedge \varsigma$, we conclude that $\mu \wedge \varsigma \in \mathcal{E}$. Hence, $\mu \in (\varsigma, \mathcal{E})$. This demonstrates that $\nu([\varsigma)) \subseteq (\varsigma, \mathcal{E})$. Thus, we can conclude that $(\varsigma, \mathcal{E}) = \nu([\varsigma))$.

Proposition 4. *Every prime $\mathcal{Q}$-ideal is a minimal prime $\mathcal{E}$-ideal.*

Proof. Let $\mathcal{A}$ be a prime $\mathcal{Q}$-ideal of an ADL $\mathcal{L}$. This implies that $\mathcal{A} = \mathcal{Q}(\mathcal{A})$. For any $\mu \in \mathcal{A}$, since μ is in $\mathcal{Q}(\mathcal{A})$, we can conclude that $(\mu, \mathcal{E}) \vee \mathcal{A} = \mathcal{L}$. Consequently, it follows that m belongs to $(\mu, \mathcal{E}) \vee \mathcal{A}$. Thus, there exist elements $\varsigma \in (\mu, \mathcal{E})$ and $\varepsilon \in \mathcal{A}$ such that $\varsigma \wedge \varepsilon$ is maximal. Since ς is in $(\mu, \mathcal{E})$, it follows that $\varsigma \wedge \mu \in \mathcal{E}$. Now, suppose for contradiction that ς also belongs to $\mathcal{A}$. Then, since both ς and ε are in $\mathcal{A}$, we would have $\varsigma \wedge \varepsilon \in \mathcal{A}$, leading to a contradiction regarding the maximality. Therefore, for every $\mu \in \mathcal{A}$, there exists an $\varsigma \notin \mathcal{A}$ such that $\mu \wedge \varsigma \in \mathcal{E}$. Consequently, by utilizing Lemma (2), we can conclude that $\mathcal{A}$ is a minimal prime $\mathcal{E}$-ideal of $\mathcal{L}$.

The theorem below establishes equivalent criteria for a minimal prime $\mathcal{E}$-ideal of an ADL to be a prime $\mathcal{Q}$-ideal.

Theorem 4. *The subsequent statements are equivalent in an ADL $\mathcal{L}$:*

(1) *every minimal prime $\mathcal{E}$-ideal is a prime $\mathcal{Q}$-ideal;*
(2) *for each $\mu \in \mathcal{L}$, $(\mu, \mathcal{E}) \vee ((\mu, \mathcal{E}), \mathcal{E}) = \mathcal{L}$;*
(3) *every ν-ideal is a $\mathcal{Q}$-ideal;*
(4) *every prime ν-ideal is a $\mathcal{Q}$-ideal.*

Proof. (1) $\Rightarrow$ (2):Assume that every minimal prime $\mathcal{E}$-ideal qualifies as a prime $\mathcal{Q}$-ideal. Let μ be an element in $\mathcal{L}$. If $(\mu, \mathcal{E}) \vee ((\mu, \mathcal{E}), \mathcal{E}) \neq \mathcal{L}$, then there exists a maximal filter $\mathcal{X}$

such that $\{(\mu,\mathcal{E})\vee((\mu,\mathcal{E}),\mathcal{E})\}\cap\mathcal{X}=\emptyset$. Given that $\mathcal{E}$ is contained within $(\mu,\mathcal{E})\vee((\mu,\mathcal{E}),\mathcal{E})$, it follows that $\mathcal{X}$ does not intersect with $\mathcal{E}$. Therefore, $\mathcal{L}-\mathcal{X}$ forms a minimal prime $\mathcal{E}$-ideal in $\mathcal{L}$. According to our assumption, $\mathcal{L}-\mathcal{X}$ is also a $\mathcal{Q}$-ideal. Now, suppose $\mu\in\mathcal{X}$. Since μ is an element of $((\mu,\mathcal{E}),\mathcal{E})$, we conclude that μ belongs to the intersection $\{(\mu,\mathcal{E})\vee((\mu,\mathcal{E}),\mathcal{E})\}\cap\mathcal{X}$, which results in a contradiction. Thus, we have $\mu\notin\mathcal{X}$, leading to the conclusion that $\mu\in\mathcal{L}-\mathcal{X}=\mathcal{Q}(\mathcal{L}-\mathcal{X})$. Consequently, it follows that $(\mu,\mathcal{E})\vee(\mathcal{L}-\mathcal{X})=\mathcal{L}$. This implies that for some $\varsigma\in(\mu,\mathcal{E})$ and $\varepsilon\in\mathcal{L}-\mathcal{X}$, the expression $\varsigma\vee\varepsilon$ is maximal within $\mathcal{X}$. Since ε is not an element of $\mathcal{X}$ and $\mathcal{X}$ is a prime filter, we must conclude that ς is an element of $\mathcal{X}$. This leads us to the situation where ς is also part of the intersection $\{(\mu,\mathcal{E})\vee((\mu,\mathcal{E}),\mathcal{E})\}\cap\mathcal{X}$, resulting in yet another contradiction. Therefore, we can conclude that $(\mu,\mathcal{E})\vee((\mu,\mathcal{E}),\mathcal{E})=\mathcal{L}$ for all μ within $\mathcal{L}$.

(2) $\Rightarrow$ (3):Assume that condition (2) is satisfied. Let $\mathcal{S}$ be a ν-ideal of $\mathcal{L}$. It follows directly that $\mathcal{Q}(\mathcal{S})\subseteq\mathcal{S}$. Now, to prove the converse, consider any element $\mu\in\mathcal{S}$. Because $\mathcal{S}$ is a ν-ideal, we obtain $((\mu,\mathcal{E}),\mathcal{E})\subseteq\mathcal{S}$. Consequently, we have $\mathcal{L}=(\mu,\mathcal{E})\vee((\mu,\mathcal{E}),\mathcal{E})\subseteq(\mu,\mathcal{E})\vee\mathcal{S}$. This indicates that μ is an element of $\mathcal{Q}(\mathcal{S})$. Thus, we can conclude that $\mathcal{S}$ is indeed a $\mathcal{Q}$-ideal of $\mathcal{L}$.

(3) $\Rightarrow$ (4): It is obvious.

(4) $\Rightarrow$ (1): Since every minimal prime $\mathcal{E}$-ideal is a prime ν-ideal, it is straightforward.

Definition 4. *For every proper ideal $\mathcal{S}$ in an ADL $\mathcal{L}$, we establish $\Lambda(\mathcal{S})$ as the set $\{\mu\in\mathcal{L}\mid$ it is not true that $(\mu,\mathcal{E})\subseteq\mathcal{S}\}$.*

Proposition 5. *Let $\mathcal{L}$ be an ADL and $\mathcal{X}$ be a maximal ideal of $\mathcal{L}$. Then the set $\Lambda(\mathcal{X})$ is an $\mathcal{E}$-ideal of $\mathcal{L}$ such that $\Lambda(\mathcal{X})\subseteq\mathcal{X}$.*

Proof. Let $\mathcal{X}$ be a maximal ideal. It is evident that $\mathcal{E}\subseteq\mathcal{X}$. Since $\mathcal{X}$ is a proper ideal, for any $e\in\mathcal{E}$, we have $(e,\mathcal{E})\nsubseteq\mathcal{X}$. Therefore, $\mathcal{E}\subseteq\Lambda(\mathcal{X})$. Now, let $\mu,\pi\in\Lambda(\mathcal{X})$. Then $(\mu,\mathcal{E})\nsubseteq\mathcal{X}$ and $(\pi,\mathcal{E})\nsubseteq\mathcal{X}$. Consequently, we get $\mathcal{X}\subset\mathcal{X}\vee(\mu,\mathcal{E})$ and $\mathcal{X}\subset\mathcal{X}\vee(\pi,\mathcal{E})$. Given that $\mathcal{X}$ is maximal, it follows that $\mathcal{X}\vee(\mu,\mathcal{E})=\mathcal{L}$ and $\mathcal{X}\vee(\pi,\mathcal{E})=\mathcal{L}$. Thus, we conclude

$$\mathcal{X}\vee(\mu\vee\pi,\mathcal{E})=\mathcal{X}\vee\{(\mu,\mathcal{E})\cap(\pi,\mathcal{E})\}=\{\mathcal{X}\vee(\mu,\mathcal{E})\}\cap\{\mathcal{X}\vee(\pi,\mathcal{E})\}=\mathcal{L}\cap\mathcal{L}=\mathcal{L}.$$

If $(\mu\vee\pi,\mathcal{E})\subseteq\mathcal{X}$, then it follows that $\mathcal{X}=\mathcal{L}$, which is a contradiction. Consequently, $(\mu\vee\pi,\mathcal{E})\nsubseteq\mathcal{X}$. This means that $\mu\vee\pi\in\Lambda(\mathcal{X})$. Next, let $\mu\in\Lambda(\mathcal{X})$ and assume $\mu\leq\pi$. Since $(\mu,\mathcal{E})\nsubseteq\mathcal{X}$ and $\mu\leq\pi$, it follows that $(\mu,\mathcal{E})\subseteq(\pi,\mathcal{E})$. Therefore, $(\pi,\mathcal{E})\nsubseteq\mathcal{X}$, which implies that $\pi\in\Lambda(\mathcal{X})$. Thus, $\Lambda(\mathcal{X})$ is an $\mathcal{E}$-ideal of $\mathcal{L}$. Now, let $\mu\in\Lambda(\mathcal{X})$. This indicates that $(\mu,\mathcal{E})\nsubseteq\mathcal{X}$. Thus, there exists some $\varsigma\in(\mu,\mathcal{E})$ such that $\varsigma\notin\mathcal{X}$. Since $\varsigma\in(\mu,\mathcal{E})$, we have $\varsigma\wedge\mu\in\mathcal{E}$, which implies that $(\varsigma\wedge\mu]\subseteq\mathcal{E}$. Assume for contradiction that $\mu\notin\mathcal{X}$. This would imply that $\mathcal{X}\vee(\mu]=\mathcal{L}$. Since $\varsigma\notin\mathcal{X}$, we have $\mathcal{X}\vee(\varsigma]=\mathcal{L}$. Consequently, we find that

$$\mathcal{L}=\mathcal{X}\vee\{(\varsigma]\cap(\mu]\}=\mathcal{X}\vee(\varsigma\wedge\mu]\subseteq\mathcal{X}\vee\mathcal{E}=\mathcal{X},$$

which leads to a contradiction. Hence, it must be true that $\mu\in\mathcal{X}$. Therefore, we conclude that $\Lambda(\mathcal{X})\subseteq\mathcal{X}$.

Proposition 6. *Let $\mathcal{A}$ be a prime $\mathcal{E}$-ideal of $\mathcal{L}$. Then we have*

(1) $\mathcal{Q}(\mathcal{A}) \subseteq \Lambda(\mathcal{A})$,

(2) *if $\mathcal{A}$ is maximal, then* $\mathcal{Q}(\mathcal{A}) = \Lambda(\mathcal{A})$.

Proof. (1) Let μ belong to $\mathcal{Q}(\mathcal{A})$. This implies that $(\mu, \mathcal{E}) \vee \mathcal{A} = \mathcal{L}$. Suppose, for contradiction, that $(\mu, \mathcal{E})$ is contained within $\mathcal{A}$. In that case, it would lead to $\mathcal{A} = \mathcal{L}$, which is a contradiction. Hence, we must conclude that $(\mu, \mathcal{E}) \nsubseteq \mathcal{A}$. As a result, it follows that μ is an element of $\Lambda(\mathcal{A})$. Therefore, we can state that $\mathcal{Q}(\mathcal{A}) \subseteq \Lambda(\mathcal{A})$.
(2) From (1), we conclude that $\mathcal{Q}(\mathcal{A}) \subseteq \Lambda(\mathcal{A})$. Conversely, let μ be an element of $\Lambda(\mathcal{A})$. This implies that $(\mu, \mathcal{E}) \nsubseteq \mathcal{A}$. Since $\mathcal{A}$ is a maximal ideal, we have $(\mu, \mathcal{E}) \vee \mathcal{A} = \mathcal{L}$. Therefore, it follows that μ is an element of $\mathcal{Q}(\mathcal{A})$. Consequently, we can state that $\Lambda(\mathcal{A}) = \mathcal{Q}(\mathcal{A})$.

Let $Max\mathcal{L}$ represent the set of all maximal ideals in an ADL $\mathcal{L}$. For any ideal $\mathcal{S}$ of $\mathcal{L}$, we define $\mathcal{F}(\mathcal{S} = \{M \in Max\mathcal{L} \,|\, \mathcal{S} \subseteq \mathcal{X}\}$.

Theorem 5. *For any ideal $\mathcal{S}$ of an ADL $\mathcal{L}$,* $\mathcal{Q}(\mathcal{S}) = \bigcap\limits_{\mathcal{X} \in \mathcal{F}(\mathcal{S})} \Lambda(\mathcal{X})$.

Proof. Let $\mu \in \mathcal{Q}(\mathcal{S})$ and suppose $\mathcal{S} \subseteq \mathcal{X}$ for some $\mathcal{X} \in Max\mathcal{L}$. Then we have $\mathcal{L} = (\mu, \mathcal{E}) \vee \mathcal{S} \subseteq (\mu, \mathcal{E}) \vee \mathcal{X}$. If we assume $(\mu, \mathcal{E}) \subseteq \mathcal{X}$, it follows that $\mathcal{X} = \mathcal{L}$, which leads to a contradiction. Thus, $(\mu, \mathcal{E}) \nsubseteq \mathcal{X}$, meaning $\mu \in \Lambda(\mathcal{X})$ for every $\mathcal{X} \in \mathcal{F}(\mathcal{S})$. Therefore, we conclude that $\mathcal{Q}(\mathcal{S}) \subseteq \bigcap_{\mathcal{X} \in \mathcal{F}(\mathcal{S})} \Lambda(\mathcal{X})$. Conversely, let $\mu \in \bigcap_{\mathcal{X} \in \mathcal{F}(\mathcal{S})} \Lambda(\mathcal{X})$. This implies $\mu \in \Lambda(\mathcal{X})$ for every $\mathcal{X} \in \mathcal{F}(\mathcal{S})$. Now, suppose $(\mu, \mathcal{E}) \vee \mathcal{S} \neq \mathcal{L}$. Then there exists a maximal ideal $\mathcal{X}_0$ such that $(\mu, \mathcal{E}) \vee \mathcal{S} \subseteq \mathcal{X}_0$. This leads to $(\mu, \mathcal{E}) \subseteq \mathcal{X}_0$ and $\mathcal{S} \subseteq \mathcal{X}_0$. Given that $\mathcal{S} \subseteq \mathcal{X}_0$, we find that $\mu \in \Lambda(\mathcal{X}_0)$. Hence, $(\mu, \mathcal{E}) \nsubseteq \mathcal{X}_0$, resulting in another contradiction. Therefore, we conclude that $(\mu, \mathcal{E}) \vee \mathcal{S} = \mathcal{L}$, which implies $\mu \in \mathcal{Q}(\mathcal{S})$. As a result, we have $\bigcap_{\mathcal{X} \in \mathcal{F}(\mathcal{S})} \Lambda(\mathcal{X}) \subseteq \mathcal{Q}(\mathcal{S})$.

From the previous theorem, it is clear that $\mathcal{Q}(\mathcal{S})$ is a subset of $\Lambda(\mathcal{X})$ for every $\mathcal{X}$ belonging to $\mathcal{F}(\mathcal{S})$. In what follows, we will establish a series of equivalent conditions that characterize when the collection of $\mathcal{E}$-ideals of the form $\mathcal{Q}(\mathcal{S})$ constitutes a sublattice within the lattice $\mathcal{I}(\mathcal{L})$ of all ideals in $\mathcal{L}$. This will lead us to a characterization of a $\mathcal{E}$-complemented ADL.

Theorem 6. *Let $\mathcal{L}$ be an ADL. Then the following assertions are equivalent:*

(1) *$\mathcal{L}$ is $\mathcal{E}$-complemented;*

(2) *for any $\mathcal{X} \in Max\mathcal{L}$, $\Lambda(\mathcal{X})$ is maximal;*

(3) *for any $\mathcal{S}, \mathcal{T} \in \mathcal{I}(\mathcal{L})$, $\mathcal{S} \vee \mathcal{T} = \mathcal{L}$ implies $\mathcal{Q}(\mathcal{S}) \vee \mathcal{Q}(\mathcal{T}) = \mathcal{L}$;*

(4) *for any $\mathcal{S}, \mathcal{T} \in \mathcal{I}(\mathcal{L})$, $\mathcal{Q}(\mathcal{S}) \vee \mathcal{Q}(\mathcal{S}) = \mathcal{Q}(\mathcal{S} \vee \mathcal{T})$;*

(5) *for any two distinct maximal ideals $\mathcal{X}$ and $\mathcal{W}$, $\Lambda(\mathcal{X}) \vee \Lambda(\mathcal{W}) = \mathcal{L}$;*

(6) *for any $\mathcal{X} \in Max\mathcal{L}$, $\mathcal{X}$ is the unique member of $Max\mathcal{L}$ such that $\Lambda(\mathcal{X}) \subseteq \mathcal{X}$.*

Proof. (1) $\Rightarrow$ (2) : Assume that $\mathcal{L}$ is $\mathcal{E}$-complemented. Let $\mathcal{X}$ be a maximal ideal in $\mathcal{L}$. We need to establish that $\Lambda(\mathcal{X}) = \mathcal{X}$. First, it is clear that $\Lambda(\mathcal{X}) \subseteq \mathcal{X}$. Now, consider an element μ belonging to $\mathcal{X}$. Given that $\mathcal{L}$ is $\mathcal{E}$-complemented, there exists an element π in

$\mathcal{L}$ such that $\mu\wedge\pi \in \mathcal{E}$ and $\mu\vee\pi$ is a maximal element. This indicates that π is included in $(\mu,\mathcal{E})$. Suppose π is also in $\mathcal{X}$. Then, $\mu\vee\pi$ would be in $\mathcal{X}$, which creates a contradiction. Therefore, π cannot be in $\mathcal{X}$, leading to the conclusion that $(\mu,\mathcal{E}) \nsubseteq \mathcal{X}$. This means that μ belongs to $\Lambda(\mathcal{X})$. Consequently, we conclude that $\mathcal{X} \subseteq \Lambda(\mathcal{X})$.

(2) $\Rightarrow$ (3) : Assume that condition (2) holds. It is evident that $\Lambda(\mathcal{X}) = \mathcal{X}$ for every maximal ideal $\mathcal{X}$ in $Max\mathcal{L}$. Let $\mathcal{S}$ and $\mathcal{T}$ be ideals in $\mathcal{I}(\mathcal{L})$ such that $\mathcal{S}\vee\mathcal{T} = \mathcal{L}$. Now, suppose that $\mathcal{Q}(\mathcal{S})\vee\mathcal{Q}(\mathcal{T}) \neq \mathcal{L}$. This implies that there exists a maximal ideal $\mathcal{X}$ such that $\mathcal{Q}(\mathcal{S})\vee\mathcal{Q}(\mathcal{T}) \subseteq \mathcal{X}$. Consequently, we have $\mathcal{Q}(\mathcal{S}) \subseteq \mathcal{X}$ and $\mathcal{Q}(\mathcal{T}) \subseteq \mathcal{X}$. Now

$$\begin{aligned} \mathcal{Q}(\mathcal{S}) \subseteq \mathcal{X} \quad &\Rightarrow \quad \bigcap_{\mathcal{X}_i\in\mathcal{F}(\mathcal{S})} \Lambda(\mathcal{X}_i) \subseteq \mathcal{X} \\ &\Rightarrow \quad \Lambda(\mathcal{X}_i) \subseteq \mathcal{X} \ \text{ for some } \mathcal{X}_i \in \mathcal{F}(\mathcal{S}) \text{ (since } \mathcal{X} \text{ is prime)} \\ &\Rightarrow \quad \mathcal{X}_i \subseteq \mathcal{X} \qquad \text{By (2)} \\ &\Rightarrow \quad \mathcal{S} \subseteq \mathcal{X} \qquad \text{since } \mathcal{S} \subseteq \mathcal{X}_i \end{aligned}$$

In a similar manner, we can conclude that $\mathcal{T} \subseteq \mathcal{X}$. Consequently, we have $\mathcal{L} = \mathcal{S}\vee\mathcal{T} \subseteq \mathcal{X}$, which contradicts the maximality of $\mathcal{X}$. Thus, it follows that $\mathcal{Q}(\mathcal{S})\vee\mathcal{Q}(\mathcal{T}) = \mathcal{L}$.

(3) $\Rightarrow$ (4) : Assume condition (3) holds. For any ideals $\mathcal{S}$ and $\mathcal{T}$ in $\mathcal{I}(\mathcal{L})$, it is evident that $\mathcal{Q}(\mathcal{S})\vee\mathcal{Q}(\mathcal{T}) \subseteq \mathcal{Q}(\mathcal{S}\vee\mathcal{T})$. Conversely, let $\mu \in \mathcal{Q}(\mathcal{S}\vee\mathcal{T})$. Then we have:

$$\{(\mu,\mathcal{E})\vee\mathcal{S}\}\vee\{(\mu,\mathcal{E})\vee\mathcal{T}\} = (\mu,\mathcal{E})\vee\mathcal{S}\vee\mathcal{T} = \mathcal{L}.$$

By the assumption of condition (3), it follows that $\mathcal{Q}((\mu,\mathcal{E})\vee\mathcal{S})\vee\mathcal{Q}((\mu,\mathcal{E})\vee\mathcal{T}) = \mathcal{L}$. Hence, we conclude that $\mu \in \mathcal{Q}((\mu,\mathcal{E})\vee\mathcal{S})\vee\mathcal{Q}((\mu,\mathcal{E})\vee\mathcal{T})$. This implies that μ can be expressed as $\mu = \tau\vee\omega$ for some $\tau \in \mathcal{Q}((\mu,\mathcal{E})\vee\mathcal{S})$ and $\omega \in \mathcal{Q}((\mu,\mathcal{E})\vee\mathcal{T})$. Now

$$\begin{aligned} \tau \in \mathcal{Q}((\mu,\mathcal{E})\vee\mathcal{S}) \quad &\Rightarrow \quad (\tau,\mathcal{E})\vee\{(\mu,\mathcal{E})\vee\mathcal{S}\} = \mathcal{L} \\ &\Rightarrow \quad \mathcal{L} = \{(\tau,\mathcal{E})\vee(\mu,\mathcal{E})\}\vee\mathcal{S} \subseteq (\tau\wedge\mu,\mathcal{E})\vee\mathcal{S} \\ &\Rightarrow \quad (\tau\wedge\mu,\mathcal{E})\vee\mathcal{S} = \mathcal{L} \\ &\Rightarrow \quad \tau\wedge\mu \in \mathcal{Q}(\mathcal{S}) \end{aligned}$$

In a similar manner, we can deduce that $\omega\wedge\mu \in \mathcal{Q}(\mathcal{T})$. As a result, we arrive at the following consequence:

$$\begin{aligned} \mu &= \mu\wedge\mu \\ &= (\tau\vee\omega)\wedge\mu \\ &= (\tau\wedge\mu)\vee(\omega\wedge\mu) \end{aligned}$$

where $\tau\wedge\mu \in \mathcal{Q}(\mathcal{S})$ and $\omega\wedge\mu \in \mathcal{Q}(\mathcal{T})$. It gives $\mu \in \mathcal{Q}(\mathcal{S})\vee\mathcal{Q}(\mathcal{T})$. Hence $\mathcal{Q}(\mathcal{S}\vee\mathcal{T}) \subseteq \mathcal{Q}(\mathcal{S})\vee\mathcal{Q}(\mathcal{T})$. It concludes that $\mathcal{Q}(\mathcal{S})\vee\mathcal{Q}(\mathcal{T}) = \mathcal{Q}(\mathcal{S}\vee\mathcal{T})$.

(4) $\Rightarrow$ (5) : Suppose condition (4) holds. Let $\mathcal{X}$ and $\mathcal{W}$ be two distinct maximal ideals of $\mathcal{L}$. Select $\mu \in \mathcal{X}\setminus\mathcal{W}$ and $\pi \in \mathcal{W}\setminus\mathcal{X}$. Since $\mu \notin \mathcal{W}$, we have $\mathcal{W}\vee(\mu] = \mathcal{L}$. Similarly, as $\pi \notin \mathcal{X}$, we obtain $\mathcal{X}\vee(\pi] = \mathcal{L}$. Now, we get

$$\mathcal{L} = \mathcal{Q}(\mathcal{L})$$

$$\begin{aligned}
&= \mathcal{Q}(\mathcal{L} \vee \mathcal{L}) \\
&= \mathcal{Q}\big(\{\mathcal{W} \vee (\mu]\} \vee \{\mathcal{X} \vee (\pi]\}\big) \\
&= \mathcal{Q}\big(\{\mathcal{X} \vee (\mu]\} \vee \{\mathcal{W} \vee (\pi]\}\big) \\
&= \mathcal{Q}(\mathcal{X} \vee \mathcal{W}) \qquad \text{since } \mu \in \mathcal{X} \text{ and } \pi \in \mathcal{W} \\
&= \mathcal{Q}(\mathcal{X}) \vee \mathcal{Q}(\mathcal{W}) \qquad \text{By condition (4)} \\
&\subseteq \Lambda(\mathcal{X}) \vee \Lambda(\mathcal{W}) \qquad \text{By Proposition 6(1)}
\end{aligned}$$

Therefore $\Lambda(\mathcal{X}) \vee \Lambda(\mathcal{W}) = \mathcal{L}$.

(5) $\Rightarrow$ (6) : Assume condition (5) is satisfied. Let $\mathcal{X}$ belong to $Max\mathcal{L}$. Now, suppose there exists $\mathcal{W} \in Max\mathcal{L}$ such that $\mathcal{W} \neq \mathcal{X}$ and $\Lambda(\mathcal{W}) \subseteq \mathcal{X}$. Since $\Lambda(\mathcal{X}) \subseteq \mathcal{X}$ by the given assumption, we conclude that $\mathcal{L} = \Lambda(\mathcal{X}) \vee \Lambda(\mathcal{W}) = \mathcal{X}$, which results in a contradiction. Thus, $\mathcal{X}$ must be the only maximal ideal where $\Lambda(\mathcal{X}) \subseteq \mathcal{X}$.

(6) $\Rightarrow$ (1): Assume condition (6) holds. Let $\mu \in \mathcal{L}$. Suppose $m \notin (\mu] \vee (\mu, \mathcal{E})$. Then, there exists a maximal ideal $\mathcal{X}$ such that $(\mu] \vee (\mu, \mathcal{E}) \subseteq \mathcal{X}$. Hence, $\mu \in \mathcal{X}$ and $(\mu, \mathcal{E}) \subseteq \mathcal{X}$, which implies that $\mu \in \mathcal{X}$ and $\mu \notin \Lambda(\mathcal{X})$. Since $\mu \notin \Lambda(\mathcal{X})$, there must exist another maximal ideal $\mathcal{X}_0$ where $\mu \notin \mathcal{X}_0$ and $\Lambda(\mathcal{X}) \subseteq \mathcal{X}_0$. Given the uniqueness of $\mathcal{X}$, we conclude that $\mathcal{X} = \mathcal{X}_0$, resulting in $\mu \notin \mathcal{X}_0 = \mathcal{X}$, which leads to a contradiction. Thus, $m \in (\mu] \vee (\mu, \mathcal{E})$, which implies that $m = \mu \vee \varsigma$ for some $\varsigma \in (\mu, \mathcal{E})$. Therefore, $\mu \vee \varsigma = m$ and $\mu \wedge \varsigma \in \mathcal{E}$, establishing that $\mathcal{L}$ is $\mathcal{E}$-complemented.

Theorem 7. *Following assertions are equivalent in an ADL $\mathcal{L}$:*

(1) *$\mathcal{L}$ is $\mathcal{E}$-complemented;*

(2) *every $\mathcal{E}$-ideal is a $\mathcal{Q}$-ideal;*

(3) *every prime $\mathcal{E}$-ideal is a $\mathcal{Q}$-ideal;*

(4) *every prime $\mathcal{E}$-ideal is minimal.*

Proof. (1) $\Rightarrow$ (2): Suppose $\mathcal{L}$ is $\mathcal{E}$-complemented. Let $\mathcal{S}$ be an $\mathcal{E}$-ideal in $\mathcal{L}$. Obviously, $\mathcal{Q}(\mathcal{S})$ lies within $\mathcal{S}$. Now, take any $\mu \in \mathcal{S}$. As $\mathcal{L}$ is $\mathcal{E}$-complemented, there exists some $\pi \in \mathcal{L}$ such that $\mu \wedge \pi \in \mathcal{E}$ and $\mu \vee \pi$ is maximum. Assume that $(\mu, \mathcal{E}) \vee \mathcal{S}$ is not equal to $\mathcal{L}$. Then, there is a prime ideal $\mathcal{A}$ such that $(\mu, \mathcal{E}) \vee \mathcal{S} \subseteq \mathcal{A}$, which implies $(\mu, \mathcal{E}) \subseteq \mathcal{A}$ and $\mu \in \mathcal{S} \subseteq \mathcal{A}$. If $\pi \in \mathcal{A}$, then $\mu \vee \pi$ would also be in $\mathcal{A}$, contradicting the fact that $\mu \vee \pi$ is maximal. Hence, π cannot belong to $\mathcal{A}$. Given that $\mu \wedge \pi \in \mathcal{E}$, we deduce that $\pi \in (\mu, \mathcal{E}) \subseteq \mathcal{A}$, creating a contradiction. Thus, we conclude that $(\mu, \mathcal{E}) \vee \mathcal{S} = \mathcal{L}$, meaning $\mu \in \mathcal{Q}(\mathcal{S})$. Therefore, $\mathcal{S}$ is equal to $\mathcal{Q}(\mathcal{S})$, confirming that $\mathcal{S}$ is a $\mathcal{Q}$-ideal of $\mathcal{L}$.

(2) $\Rightarrow$ (3): It is obvious.

(3) $\Rightarrow$ (4): Suppose that each prime $\mathcal{E}$-ideal is a $\mathcal{Q}$-ideal. Let $\mathcal{A}$ be a prime $\mathcal{E}$-ideal of $\mathcal{L}$. Since $\mathcal{A}$ is a proper ideal, there exists an element $\zeta \in \mathcal{L}$ such that $\zeta \notin \mathcal{A}$. According to condition (3), $\mathcal{A}$ must be a $\mathcal{Q}$-ideal of $\mathcal{L}$, so $\mathcal{Q}(\mathcal{A}) = \mathcal{A}$. Now, let $\mu \in \mathcal{A} = \mathcal{Q}(\mathcal{A})$. This implies that $(\mu, \mathcal{E}) \vee \mathcal{A} = \mathcal{L}$, and hence $\zeta \in (\mu, \mathcal{E}) \vee \mathcal{A}$. Therefore, $\zeta = \varsigma \vee \varepsilon$ for some $\varsigma \in (\mu, \mathcal{E})$ and $\varepsilon \in \mathcal{A}$. Since $\varsigma \in (\mu, \mathcal{E})$, we have $\mu \wedge \varsigma \in \mathcal{E}$. Assume now that $\varsigma \in \mathcal{A}$. Given that $\mathcal{A}$ is a prime ideal and $\varepsilon \in \mathcal{A}$, it follows that $\zeta = \varsigma \vee \varepsilon \in \mathcal{A}$, which contradicts our

earlier assumption that $\zeta \notin \mathcal{A}$. Thus, ς must not belong to $\mathcal{A}$. Consequently, $\mu \wedge \varsigma \in \mathcal{E}$ for some $\varsigma \notin \mathcal{A}$, indicating that $\mathcal{A}$ is minimal.
(4) $\Rightarrow$ (1): From Theorem 1, its clear.

It is evident that every filter within a Boolean algebra qualifies as an $\mathcal{E}$-ideal. Additionally, it can be readily observed that every Boolean algebra is $\mathcal{E}$-complemented. Consequently, we can derive the following:

Theorem 8. *Following assertions are equivalent in an ADL $\mathcal{L}$:*

(1) *$\mathcal{L}$ is a Boolean algebra;*
(2) *every ideal is a $\mathcal{Q}$-ideal;*
(3) *every prime ideal is a $\mathcal{Q}$-ideal;*
(4) *every prime ideal is minimal.*

Proof. The implications (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3), and (3) $\Rightarrow$ (4) have been established.
(4) $\Rightarrow$ (1): Assume that every prime ideal of $\mathcal{L}$ is minimal. Let $\mu \in \mathcal{L}$. Suppose $m \notin (\mu] \vee (\mu)^*$. Then there exists a prime ideal $\mathcal{A}$ such that $(\mu] \vee (\mu)^* \subseteq \mathcal{A}$. Hence $\mu \in \mathcal{A}$ and $(\mu)^* \subseteq \mathcal{A}$. Since $\mathcal{A}$ is minimal and $(\mu)^* \subseteq \mathcal{A}$, we get $\mu \notin \mathcal{A}$ which is a contraction. Hence $m \in (\mu] \vee (\mu)^*$. Then there exist $\varsigma \in (\mu)^*$ such that $\varsigma \vee \mu = m$. Since $\varsigma \in (\mu)^*$, we get $\mu \wedge \varsigma = 0$. Hence ς is the complement of μ. Therefore $\mathcal{L}$ is a Boolean algebra.

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