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Best Proximity Points of $(\mathcal{A}, \mathcal{S}, p)$-Contractions in the Framework of w-Distance

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Abstract. Shahzad et al. [RACSAM 111 (2017) 307-324] introduced the notion of $(\mathcal{A}, \mathcal{S})$- contractions which unifies several well known nonlinear type contractions (e.g. $\mathcal{R}$-contractions, $\mathcal{Z}$- contractions, $\mathcal{L}$-contractions etc.) in one go. In this paper we extend the notion of $(\mathcal{A}, \mathcal{S})$-contractions from Shahzad et al. [1] to non-self mappings in a metric space with a w-distance. Then we prove some new proximity point theorems for the aforementioned type of mappings. This paper is a continuation of our previous papers [2, 3].

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1. Introduction and preliminaries

Metric fixed point theory is one of the main branches of mathematical analysis, and it has wide applications in natural sciences, economy, etc. The well-known Banach contraction principle [4] represents the founding result of this theory. Since then, many new results have been obtained by generalizing the contractive definitions or by generalizing the metric structure. One of the vivid research areas is the best proximity theory for non-self mappings, which deals with the problem of finding a point which is in a certain sense closest to its image under the mapping considered. For a comprehensive treatment of these subjects, we refer the reader to [5].

In [1] the authors introduced a new class of contractivity conditions for mappings from a metric space into itself endowed with a binary relation.

These conditions unify several kinds of contractive operators (e.g. $\mathcal{R}$-contractions, $\mathcal{Z}$-contractions, $\mathcal{L}$-contractions etc.) in one go. And the authors presented some results about existence and uniqueness of fixed points that extend and generalized many theorems in the field of fixed point theory. In this paper we extend the notion of $(\mathcal{A}, \mathcal{S})$-contractions

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from Shahzad et al. [1] to non-self mappings in a metric space with a w-distance. Then we prove some new proximity point theorems for the aforementioned type of mappings. This paper is a continuation of our previous papers [2, 3].

Kada et al. [6] (for more results see e.g. [7–10]) have introduced the concept of w-distance on a metric space to generalize many important fixed point theorems.

Definition 1.1. Let (X, d) be a metric space. Then a function $p : X \times X \to [0, \infty)$ is called a *w-distance* on X if the following are satisfied:

(P1) $p(x, z) \leq p(x, y) + p(y, z)$, for any $x, y, z \in X$,

(P2) for any $x \in X$, function $p(x, \cdot) : X \to [0, \infty)$ is lower semicontinuous,

(P3) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \varepsilon$.

Example 1.1. Let X be a normed space with norm $\|\cdot\|$. Then a function $p : X \times X \to [0, \infty)$ defined by

$$p(x, y) = \|x\| + \|y\| \quad \text{for every } x, y \in X$$

is a w-distance.

Example 1.2. ([9]) Let $X = \mathbb{R}$ be endowed with the Euclidean metric $d = |\cdot|$, $k, m \in \mathbb{R}_+$, be a positive constants and $p : X \times X \to \mathbb{R}_+$ be defined by

$$p(x, y) = |x|^k + |y|^m \quad \text{for every } x, y \in X.$$

Then p is a w-distance in X.

The following example shows that we can construct another w-distance from a given w-distance under certain conditions.

Example 1.3. ([11]) Let $x_0 \in X$, p a w-distance on X, and $h : [0, +\infty) \to [0, +\infty$ a nondecreasing function. If, for each $r > 0$,

$$\inf_{x \in X} \int_{p(x_0,x)}^{px_0,x)+r} \frac{dt}{1 + h(t)} > 0,$$

then the function q defined by

$$q(x, y) = \int_{p(x_0,x)}^{p(x_0,x)+p(x,y)} \frac{dt}{1 + h(t)} \quad \text{for all } x, y \in X$$

is a w-distance. In particular, if p is bounded on $X \times X$, then q is a w-distance

The present authors have introduced the concept of w_0-distance to study best proximity points, see [2, 3].

Definition 1.2. A w-distance on a metric space (X, d) is called w_0-distance if the function $p(\cdot, y) : X \to \mathbb{R}$ is lower semicontinuous for all $y \in X$.

The following very useful lemma has been proven in [6]

Lemma 1.1 ([6]). *Let X be a metric space with metric d and let p be a w-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X, let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0, +\infty)$ converging to 0, and let x, y, $z \in X$. Then the following hold:*

(i) If $p(x_n, y) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $y = z$. In particular, if $p(x, y) = 0$ and $p(x, z) = 0$, then $y = z$;

(ii) if $p(x_n, y_n) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then y_n converges to z;

(iii) if $p(x_n, x_m) \leq \alpha_n$ for any $n, m \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence.

(iv) if $p(y, x_n) \leq \alpha_n$ for any $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence.

The following lemma from [2] can be very useful to show that a given sequence in a metric space with a w-distance is Cauchy. We denote

$$p_\alpha(x, y) := \max\{p(x, y), p(y, x)\}.$$

Lemma 1.2. *Let (X, d) be a metric space with w_0-distance p, and let $\{x_n\}$ be a sequence in X such that*

$$\lim_{n\to\infty} p_\alpha(x_n, x_{n+1}) = 0.$$

Then one of the following conditions is fulfilled:

(i) $\lim_{m,n\to\infty} p_\alpha(x_n, x_m) = 0$

(ii) there exist an $\varepsilon_0 > 0$ and two subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ with $m_k > n_k$ for all $k \in \mathbb{N}$ such that $p_\alpha(x_{n_k}, x_{m_k}) \geq \varepsilon_0$ for all $k \in \mathbb{N}$ and

$$\lim_{k\to\infty} p_\alpha(x_{n_k}, x_{m_k}) = \lim_{k\to\infty} p_\alpha(x_{n_k-1}, x_{m_k-1}) = \varepsilon_0.$$

Let (X, d) be a metric space and $A, B \subseteq X$ two nonempty subsets. In this paper, by using w-distances we extend the notion of $(\mathcal{A}, \mathcal{S})$-contractions from [1] to non-self mappings of the from $T : A \to B$, the so called $(\mathcal{A}, \mathcal{S}, p)$-contractions. Then we prove our main results, which deal with existence of best proximity points for $(\mathcal{A}, \mathcal{S}, p)$-contractions, that is, the points $x \in A$ such that $d(x, Tx) = d(A, B)$. We recall the following notations, see e.g. [2, 3, 12, 13]:

$$\begin{aligned} d(A, B) &= \inf\{d(x, y) : x \in A, y \in B\} \\ A_0 &= \{x \in A : (\exists y \in B)\ d(x, y) = d(A, B)\} \\ B_0 &= \{y \in B : (\exists x \in A)\ d(x, y) = d(A, B)\}. \end{aligned}$$

2. Main results

In the sequel:

- (X, d) is a metric space with a w_0-distance p,
- $\mathcal{S}$ is a relation on X,
- A and B are two nonempty subsets of X, and
- $T : A \to B$ is a mapping.

Let us recall and modify some basic notions from [1]. For all $x, y \in X$, $x\mathcal{S}^*y$ means that $x\mathcal{S}y$ and $x \neq y$.

Definition 2.1. A sequence $\{x_n\} \subseteq X$ is $\mathcal{S}$-strictly-increasing if $x_n\mathcal{S}^*x_m$ for all $m, n \in \mathbb{N}$ with $n < m$. A metric space (X, d) is $\mathcal{S}$-strictly-increasing-regular if for every $\mathcal{S}$-strictly-increasing sequence $\{x_n\}$ converging to $z \in X$ we have $x_n\mathcal{S}z$ for all $n \in \mathbb{N}$. A nonempty subset $Y \subseteq X$ is $(\mathcal{S}, d)$-strictly-increasing-complete if every d-Cauchy $\mathcal{S}$-strictly-increasing sequence converges to a point in Y.

Definition 2.2. Let $\{a_n\}$ and $\{b_n\}$ be two sequences in $[0, \infty)$. Then, $\{(a_n, b_n)\}$ is a $(T, \mathcal{S}, p)$-sequence (respectively, $(T, \mathcal{S}, p_\alpha)$-sequence) if there exist four sequences $\{u_n\}$, $\{v_n\}$, $\{x_n\}$ and $\{y_n\}$ in A such that

$$\begin{aligned} &x_n\mathcal{S}y_n,\\ d(u_n, Tx_n) &= d(A, B),\\ d(v_n, Ty_n) &= d(A, B),\\ &a_n = p(u_n, v_n) > 0, \text{ (resp. } a_n = p_\alpha(u_n, v_n) > 0),\\ &b_n = p(x_n, y_n) > 0, \text{ (resp. } b_n = p_\alpha(x_n, y_n) > 0), \end{aligned}$$

for all $n \in \mathbb{N}$.

Remark 2.1. From the above definition it is clear that every $(T, \mathcal{S}, p)$-sequence is a $(T, \mathcal{S}, p_\alpha)$-sequence, but the converse need not be true.

The following definition from [1] is adapted to our setting.

Definition 2.3. A mapping $T : A \to B$ is $\mathcal{S}$-nondecreasing if for all $u, v, x, y \in A$

$$\left.\begin{array}{c} x\mathcal{S}y \\ d(u, Tx) = d(A, B) \\ d(v, Ty) = d(A, B) \end{array}\right\} \Rightarrow u\mathcal{S}v.$$

Mapping T is $\mathcal{S}$-strictly-increasing-continuous if $Tx_n \to Tz$ for every $\mathcal{S}$-strictly-increasing sequence $\{x_n\} \subseteq A$ converging to $z \in A$.

Definition 2.4. A mapping $T : A \to B$ is an $(\mathcal{A}, \mathcal{S}, p)$-contraction with respect to a function $\varrho : D \times D \to \mathbb{R}$ if the following conditions are satisfied:

($\mathcal{A}_1$) $ran(p) := \{p(x,y) : x, y \in X\} \subseteq D$;

($\mathcal{A}_2$) if $\{x_n\} \subseteq A$ is a sequence such that

$$x_n\mathcal{S}^*x_{n+1}, d(x_{n+1}, Tx_n) = d(A,B), \text{ and}$$
$$\varrho(p_\alpha(x_{n+1}, x_{n+2}), p_\alpha(x_n, x_{n+1})) > 0 \text{ for all } n \in \mathbb{N},$$

then $p_\alpha(x_n, x_{n+1}) \to 0$.

($\mathcal{A}_3$) if $\{(a_n, b_n)\} \subseteq A \times A$ is a $(T, \mathcal{S}, p_\alpha)$-sequence such that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L \geq 0$, $L < a_n$ and $\varrho(a_n, b_n) > 0$ for all $n \in \mathbb{N}$, then $L = 0$;

($\mathcal{A}_4$) for all $u, v, x, y \in A$

$$\left.\begin{array}{r} u\mathcal{S}^*v, x\mathcal{S}^*y \\ d(u,Tx) = d(A,B) \\ d(v,Ty) = d(A,B) \end{array}\right\} \Rightarrow \varrho(p_\alpha(u,v), p_\alpha(x,y)) > 0.$$

Sometimes, we will also consider the following property:

($\mathcal{A}_5$) if $\{(a_n, b_n)\}$ is a $(T, \mathcal{S}, p)$-sequence or $(T, \mathcal{S}, p_\alpha)$-sequence such that $b_n \to 0$ and $\varrho(a_n, b_n) > 0$ for all $n \in \mathbb{N}$, then $a_n \to 0$.

Lemma 2.1. *If $\varrho(t,s) \leq s - t$ for all $t, s \in D \cap (0, \infty)$, then ($\mathcal{A}_5$) holds.*

Proof. Let $\{(a_n, b_n)\}$ be a $(T, \mathcal{S}, p)$-sequence or a $(T, \mathcal{S}, p_\alpha)$-sequence such that $b_n \to 0$ and for all $n \in \mathbb{N}$, $\varrho(a_n, b_n) > 0$. Then, by Definition 2.2, $a_n > 0$ and $b_n > 0$ for all $n \in \mathbb{N}$, so from $0 < \varrho(a_n, b_n) \leq b_n - a_n$ we get $0 < a_n < b_n$ for all $n \in \mathbb{N}$, which yields $a_n \to 0$.

Theorem 2.1. *Let (X, d) be a metric space endowed with a w_0-distance p and a transitive binary relation $\mathcal{S}$, and let $A, B \subseteq X$ be two nonempty subsets such that A_0 is $(\mathcal{S}, d)$-strictly-increasing-complete. Let $T : A \to B$ be an $\mathcal{S}$-nondecreasing $(\mathcal{A}, \mathcal{S}, p_\alpha)$-contraction with respect to $\varrho : D \times D \to \mathbb{R}$ such that $T(A_0) \subseteq B_0$. Assume that there exist $x_0, x_1 \in A$ such that $x_0\mathcal{S}x_1$ and $d(x_1, Tx_0) = d(A,B)$, and that one of the following conditions holds:*

- *T is $\mathcal{S}$-strictly-increasing-continuous;*
- *(X, d) is $\mathcal{S}$-strictly-increasing-regular and ($\mathcal{A}_5$) holds;*
- *(X, d) is $\mathcal{S}$-strictly-increasing-regular and $\varrho(t,s) \leq s - t$ for all $t, s \in D \cap (0, \infty)$.*

Then there exists a sequence $\{x_n\} \subseteq A_0$ converging to a best proximity point of T, such that $d(x_{n+1}, Tx_n) = d(A,B)$ for all $n \in \mathbb{N}$.

Proof. Let $x_0, x_1 \in A$ be two points such that $x_0 \mathcal{S} x_1$ and $d(x_1, Tx_0) = d(A, B)$. Since $T(A_0) \subseteq B_0$ and $x_1 \in A_0$, there exists $x_2 \in A_0$ such that $d(x_2, Tx_1) = d(A, B)$. But then we have $x_1 \mathcal{S} x_2$ because T is $\mathcal{S}$-nondecreasing. Continuing this process, we construct a sequence $\{x_n\} \subseteq A_0$ such that $d(x_{n+1}, Tx_n) = d(A, B)$ and $x_n \mathcal{S} x_{n+1}$ for all $n \in \mathbb{N}$. Also, we have

$$x_n \mathcal{S} x_m \text{ for all } n, m \in \mathbb{N} \text{ such that } m > n \tag{1}$$

because $\mathcal{S}$ is transitive. If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then x_{n_0} is clearly a best proximity point of T. Hence, we can assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$, so we have

$$x_n \mathcal{S}^* x_{n+1} \text{ for all } n \in \mathbb{N}. \tag{2}$$

Since T is an $(\mathcal{A}, \mathcal{S}, p_\alpha)$-contraction, by (1) and $(\mathcal{A}_4)$ we get

$$\varrho(p_\alpha(x_{n+1}, x_{n+2}), p_\alpha(x_n, x_{n+1})) > 0$$

for all $n \in \mathbb{N}$, so from $(\mathcal{A}_2)$ we see that $p_\alpha(x_n, x_{n+1}) \to 0$.

Next we shall show that $\{x_n\}$ is $\mathcal{S}$-strictly-increasing. In view of (1), suppose that there exist $n_0, m_0 \in \mathbb{N}$ with $m_0 > n_0$ such that $x_{n_0} = x_{m_0}$. If $p_0 = m_0 - n_0 > 0$, then $x_{n_0} = x_{n_0+kp_0}$ for all $k \in \mathbb{N}$. Therefore, the sequence $\{p_\alpha(x_n, x_{n+1})\}$ contains the constant subsequence

$$\{p_\alpha(x_{n_0+kp_0}, x_{n_0+kp_0+1}) = p_\alpha(x_{n_0}, x_{n_0+1}) > 0\}_{k \in \mathbb{N}}$$

which is in contradiction with $p_\alpha(x_n, x_{n+1}) \to 0$. Hence, $x_n \neq x_m$ for all $n \neq m$, so $x_n \mathcal{S}^* x_m$ for all $m, n \in \mathbb{N}$ with $m > n$.

Now let us prove that $\{x_n\}$ is a Cauchy sequence in A_0. To do so, by Lemma 1.1(iii) it suffices to show that

$$\lim_{n,m \to \infty} p_\alpha(x_n, x_m) = 0. \tag{3}$$

If (3) is not true, then by Lemma 1.2 there exist $\varepsilon_0 > 0$ and two subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ such that

$$p_\alpha(x_{n_k}, x_{m_k-1}) \leq \varepsilon_0 < p_\alpha(x_{n_k}, x_{m_k})$$

for all $k \in \mathbb{N}$, and

$$\lim_{k \to \infty} p_\alpha(x_{n_k}, x_{m_k}) = \lim_{k \to \infty} p_\alpha(x_{n_k-1}, x_{m_k-1}) = \varepsilon_0.$$

If $L := \varepsilon_0 > 0$, $a_k := p_\alpha(x_{n_k}, x_{m_k})$ and $b_k := p_\alpha(x_{n_k-1}, x_{m_k-1})$, then we see that $\{(a_k, b_k)\}$ is a $(T, \mathcal{S}, p_\alpha)$-sequence such that $\lim_{k \to \infty} a_k = \lim_{k \to \infty} b_k = L$. Since $L = \varepsilon_0 < a_k$ and

$$\varrho(a_k, b_k) = \varrho(p_\alpha(x_{n_k}, x_{m_k}), p_\alpha(x_{n_k-1}, x_{m_k-1})) > 0$$

for all $k \in \mathbb{N}$, by $(\mathcal{A}_3)$ we get $\varepsilon_0 = L = 0$, a contradiction, so (3) holds.

Thus, is an $\mathcal{S}$-strictly-increasing Cauchy sequence in A_0. Since A_0 is $(\mathcal{S}, d)$-strictly-increasing-complete, there exists $x \in A_0$ such that $x_n \to x$. Now consider the following three cases.

1. If T is $\mathcal{S}$-strictly-increasing-continuous, then we have $Tx_n \to Tx$, so $d(A,B) = d(x_{n+1}, Tx_n) \to d(x,Tx)$, i.e. $d(x,Tx) = d(A,B)$.

2. If (X,d) is $\mathcal{S}$-strictly-increasing-regular and $(\mathcal{A}_5)$ holds, we have that

$$x_n \mathcal{S} x \text{ for all } n \in \mathbb{N} \tag{4}$$

because $\{x_n\}$ is $\mathcal{S}$-strictly-increasing and $x_n \to x$. But T is $\mathcal{S}$-nondecreasing, so

$$Tx_n \mathcal{S} Tz \text{ for all } n \in \mathbb{N}. \tag{5}$$

Since $x \in A_0$ and $T(A_0) \subseteq B_0$, there exists $z \in A_0$ such that $d(z,Tx) = d(A,B)$. Let $a_n := p_\alpha(x_{n+1}, z)$ and $b_n := p_\alpha(x_n, x)$ for all $n \in \mathbb{N}$. If $a_n = p_\alpha(x_{n+1}, z) = 0$ for infinitely many $n \in \mathbb{N}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$p_\alpha(x_{n_k+1}, z) = 0 \text{ for all } k \in \mathbb{N}.$$

Therefore, we have $x_{n_k+1} = z$ for all $k \in \mathbb{N}$, which means that $z = x$ because $x_n \to x$.

Hence, we can assume that there exists $n_0 \in \mathbb{N}$ such that $p_\alpha(x_{n+1}, z) = 0$ for all $n \geq n_0$. There exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $b_{n_k} = p_\alpha(x_{n_k}, x) > 0$ for all $k \in \mathbb{N}$. Because, if that is not the case, there exists $N \in \mathbb{N}$ such that $p_\alpha(x_n, x) = 0$ for all $n \in \mathbb{N}$, so $p_\alpha(x_n, x_{n+1}) \leq p_\alpha(x_n, x) + p_\alpha(x, x_{n+1}) = 0$, i.e. $x_n = x_{n+1}$ for all $n > N$ - a contradiction. For brevity, without loss of generality, we can identify $\{x_{n_k}\}$ with the whole sequence $\{x_n\}$.

Then we also have that $x_n \neq x$ and $x_{n+1} \neq z$ for all $n \geq n_0$, so by (4) and (5) we get $x_n \mathcal{S}^* x$ and $x_{n+1} \mathcal{S}^* z$ for all $n \geq n_0$. From $(\mathcal{A}_4)$ it follows that

$$\varrho(a_n, b_n) = \varrho(p_\alpha(x_{n+1}, z), p_\alpha(x_n, x)) > 0 \text{ for all } n \geq n_0.$$

By (3), we get that for all $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $p_\alpha(x_n, x_m) < \varepsilon$ for all $m > n \leq n_\varepsilon$. Then for a fixed $n \geq \max\{n_0, n_\varepsilon\}$, by using the lower semicontinuity of p we get

$$p(x_n, x) \leq \liminf_{m\to\infty} p(x_n, x_m) \leq \liminf_{m\to\infty} p_\alpha(x_n, x_m) < \varepsilon$$

which yields that

$$p(x_n, x) \to 0. \tag{6}$$

Similarly, we can obtain that also $p(x, x_n) \to 0$, so by (6) $b_n = p_\alpha(x_n, x) \to 0$.

Since $\{(a_n, b_n)\}_{n \geq n_0}$ is a $(T, \mathcal{S}, p_\alpha)$-sequence, applying $(\mathcal{A}_5)$ yields that $a_n = p_\alpha(x_{n+1}, z) \to 0$, i.e. $p(x_{n+1}, z) \to 0$, which with (6) gives $z = x$ by Lemma 1.1(i).

3. If (X,d) is $\mathcal{S}$-strictly-increasing-regular and $\varrho(t,s) \leq s - t$ for all $t, s \in A \cap (0, \infty)$, this case reduces to the previous one by Lemma 2.1.

The following example illustrates Theorem 2.1:

Example 2.1. Let $t > 0$ be fixed. Let $X = \{0, t, 2t, 3t, 4t\}$ be endowed with usual metric $d(x,y) = |x - y|$, for all $x, y \in X$. Define $p : X \times X \to [0, \infty)$ by $p(x,y) = x + y$ which is a w-distance on X. Then $p_\alpha(x,y) = \max\{p(x,y), p(y,x)\} = x + y$, for each $x, y \in X$ is a w_0-distance on X.

Suppose that $A = \{0, t, 2t\}$ and $B = \{0, 3t, 4t\}$. Define $T : A \to B$ by $T0 = 0$, $Tt = 3t$ and $T2t = 4t$. Also we define $\varrho : [0,\infty) \times [0,\infty) \to \mathbb{R}$ by $\varrho(t,s) = 1$ for all $t, s \in [0,\infty)$. Clearly $d(A,B) = 0$, $A_0 = B_0 = \{0\}$.

Put the relation as $\leq$ which is the partial order on X. Note that $\leq$ is transitive.

The mapping $T : A \to B$ is an $(\mathcal{A}, \mathcal{S}, p)$-contraction with respect to a function ϱ; indeed, clearly $(\mathcal{A}_1)$ and $(\mathcal{A}_4)$ hold.

For $(\mathcal{A}_2)$, note that if the sequence $\{x_n\}$ in A satisfies in $x_n \mathcal{S}^* x_{n+1}$, $d(x_{n+1}, Tx_n) = d(A,B) = 0$ and $\varrho(p_\alpha(x_{n+1}, x_{n+2}), p_\alpha(x_n, x_{n+1})) = 1 > 0$, for all $n \in \mathbb{N}$, then there exists $k \in \mathbb{N}$ such that for all $n \geq k$, we have $x_n = 0$ and so $p_\alpha(x_n, x_{n+1}) \to 0$ as $n \to \infty$.

Now, we show that $(\mathcal{A}_3)$ holds. Suppose that $\{(a_n, b_n)\} \subseteq A \times A$ is a $(T, \leq, p_\alpha)$-sequence such that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L \geq 0$ and $L < a_n$ for all $n \in \mathbb{N}$.

From Definition 2.2, there exist four sequences $\{u_n\}$, $\{v_n\}$, $\{x_n\}$ and $\{y_n\}$ in A such that

$$\begin{aligned} x_n &\leq y_n, \\ d(u_n, Tx_n) &= d(A,B), \\ d(v_n, Ty_n) &= d(A,B), \\ a_n &= p_\alpha(u_n, v_n) = u_n + v_n > 0, \\ b_n &= p_\alpha(x_n, y_n) = x_n + y_n > 0, \end{aligned}$$

for all $n \in \mathbb{N}$. Since $d(A,B) = 0$, $u_n = Tx_n$ and $v_n = Ty_n$, for all $n \in \mathbb{N}$. Note that $b_n > 0$, for all $n \in \mathbb{N}$ and so $b_n = x_n + y_n \in \{t, 2t, 3t, 4t\}$, for all $n \in \mathbb{N}$; and hence $a_n \in \{3t, 4t, 6t, 7t, 8t\}$, for all $n \in \mathbb{N}$. Now, if $a_n = kt$, for all $n \in \mathbb{N}$ and for some $k \in \{3, 4, 6, 7, 8\}$, then $L = \lim_{n\to\infty} a_n = kt$, a contradiction, because $L < a_n$, for all $n \in \mathbb{N}$. Thus $L = 0$. Therefore T is an $(\mathcal{A}, \mathcal{S}, p)$-contraction with respect to a function ϱ.

Also, T is $\leq$-strictly-increasing-continuous; since if $\{x_n\}$ is a $\leq$-strictly-increasing sequence in A converges to z, then there exists N such that $x_n = z$ for all $n \geq N$; and so $Tx_n = Tz$ for all $n \geq N$ and hence $Tx_n \to Tz$ as $n \to \infty$.

Therefore all conditions of Theorem 2.1 are satisfied and note that the sequence $\{x_n\} = \{0\}$ in $A_0 = \{0\}$ converges to 0 which is the best proximity point of T and $d(x_{n+1}, x_n) = 0 = d(A,B)$, for all $n \in \mathbb{N}$.

We note that using a similar reasoning we can prove a version of Theorem 2.1 for $(\mathcal{A}, \mathcal{S}, p)$-contractions on a metric space with a w-distance p if we assume that $\mathcal{S}$ is symmetrical. In particular, we have:

Theorem 2.2. *Let (X,d) be a metric space with a w-distance p and a symmetric and transitive binary relation $\mathcal{S}$ and let A and B be two nonempty subsets of X such that A_0 is $(\mathcal{S}, d)$-strictly-increasing-complete. Also, let $T : A \to B$ be an $\mathcal{S}$-nondecreasing $(\mathcal{A}, \mathcal{S}, p)$-contraction with respect to $\varrho : D \times D \to \mathbb{R}$ such that $T(A_0) \subseteq B_0$. Suppose that there exist $x_0, x_1 \in A$ such that $x_0 \mathcal{S} x_1$ and $d(x_1, Tx_0) = d(A,B)$, and one of the conditions as in Theorem 2.1 holds. Then there exists a sequence $\{x_n\} \subseteq A_0$ such that $d(x_{n+1}, Tx_n) = d(A,B)$ for all $n \in \mathbb{N}$ which converges to a best proximity point of T $x \in A_0$.*

Proof. Following the same reasoning as in the proof of Theorem 2.1, we can construct an $\mathcal{S}$-strictly-increasing sequence $\{x_n\} \subseteq A_0$ such that for all $n \in \mathbb{N}$, $d(x_{n+1}, Tx_n) = d(A, B)$ and

$$\varrho(p(x_{n+1}, x_{n+2}), p(x_n, x_{n+1})) > 0.$$

But $\mathcal{S}$ is symmetrical, so we can also get

$$\varrho(p(x_{n+2}, x_{n+1}), p(x_{n+1}, x_n)) > 0 \text{ for all } n \in \mathbb{N}.$$

Hence, by $(\mathcal{A}_2)$ we have

$$p(x_n, x_{n+1}) \to 0 \text{ and } p(x_{n+1}, x_n) \to 0,$$

and we can apply Lemma 1.1 to prove by contradiction that $\{x_n\}$ is Cauchy, so the rest of the proof proceeds analogously to the proof of Theorem 2.1.

3. Best proximity points for $(\mathcal{A}, \gamma, p)$-contractions

Definition 3.1 ([1]). A sequence $\{x_n\} \subseteq X$ is γ-strictly-increasing if $\gamma(x_n, x_m) \geq 1$ and $x_n \neq x_m$ for all $m, n \in \mathbb{N}$ with $n < m$. A metric space (X, d) is γ-strictly-increasing-regular if for every γ-strictly-increasing sequence $\{x_n\}$ converging to $z \in X$ we have $\gamma(x_n, z) \geq 1$ for all $n \in \mathbb{N}$. A nonempty subset $Y \subseteq X$ is (γ, d)-strictly-increasing-complete if every d-Cauchy γ-strictly-increasing sequence converges to a point in Y.

Definition 3.2. Let $\{a_n\}$ and $\{b_n\}$ be two sequences in $[0, \infty)$. Then, $\{(a_n, b_n)\}$ is a (T, γ, p)-sequence (respectively, (T, γ, p_α)-sequence) if there exist four sequences $\{u_n\}$, $\{v_n\}$, $\{x_n\}$ and $\{y_n\}$ in A such that

$$\begin{aligned} \gamma(x_n, y_n) &\geq 1, \\ d(u_n, Tx_n) &= d(A, B), \\ d(v_n, Ty_n) &= d(A, B), \\ a_n = p(u_n, v_n) &> 0, \text{ (resp. } a_n = p_\alpha(u_n, v_n) > 0), \\ b_n = p(x_n, y_n) &> 0, \text{ (resp. } b_n = p_\alpha(x_n, y_n) > 0), \end{aligned}$$

for all $n \in \mathbb{N}$.

Definition 3.3. A mapping $T : A \to B$ is γ-admissible if for all $u, v, x, y \in A$

$$\left.\begin{aligned} \gamma(x, y) &\geq 1 \\ d(u, Tx) &= d(A, B) \\ d(v, Ty) &= d(A, B) \end{aligned}\right\} \Rightarrow \gamma(u, v) \geq 1.$$

Mapping T is γ-strictly-increasing-continuous if $Tx_n \to Tz$ for every γ-strictly-increasing sequence $\{x_n\} \subseteq A$ converging to $z \in A$.

Definition 3.4. A mapping $T : A \to B$ is an $(\mathcal{A}, \gamma, p)$-contraction with respect to a function $\varrho : D \times D \to \mathbb{R}$ if the following conditions are satisfied:

$(\mathcal{A}_1)$ $ran(p) := \{p(x, y) : x, y \in X\} \subseteq D$;

$(\mathcal{A}_2^\gamma)$ if $\{x_n\} \subseteq A$ is a sequence such that

$$x_n \neq x_{n+1}, \gamma(x_n, x_{n+1}) \geq 1, d(x_{n+1}, Tx_n) = d(A, B), \text{ and}$$
$$\varrho(p_\alpha(x_{n+1}, x_{n+2}), p_\alpha(x_n, x_{n+1})) > 0 \text{ for all } n \in \mathbb{N},$$

then $p_\alpha(x_n, x_{n+1}) \to 0$.

$(\mathcal{A}_3^\gamma)$ if $\{(a_n, b_n)\} \subseteq A \times A$ is a (T, γ, p_α)-sequence such that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L \geq 0$, $L < a_n$ and $\varrho(a_n, b_n) > 0$ for all $n \in \mathbb{N}$, then $L = 0$;

$(\mathcal{A}_4^\gamma)$ for all $u, v, x, y \in A$

$$\left.\begin{aligned} u \neq v, &\gamma(u, v) \geq 1 \\ x \neq y, &\gamma(x, y) \geq 1 \\ d(u, Tx) &= d(A, B) \\ d(v, Ty) &= d(A, B) \end{aligned}\right\} \Rightarrow \varrho(p_\alpha(u, v), p_\alpha(x, y)) > 0,$$

Sometimes, we will also consider the following property:

$(\mathcal{A}^{\gamma}{}_5)$ if $\{(a_n, b_n)\}$ is a (T, γ, p)-sequence or (T, γ, p_α)-sequence such that $b_n \to 0$ and $\varrho(a_n, b_n) > 0$ for all $n \in \mathbb{N}$, then $a_n \to 0$.

Corollary 3.1. Let (X, d) be a metric space endowed with a w_0-distance p and $\gamma : A \times A \to [0, \infty)$ be a transitive function, and let $A, B \subseteq X$ be two nonempty subsets such that A_0 is (γ, d)-strictly-increasing-complete. Let $T : A \to B$ be an γ-admissible $(\mathcal{A}, \gamma, p_\alpha)$-contraction with respect to $\varrho : D \times D \to \mathbb{R}$ such that $T(A_0) \subseteq B_0$. Assume that there exist $x_0, x_1 \in A$ such that $\gamma(x_0, x_1) \geq 1$ and $d(x_1, Tx_0) = d(A, B)$, and that one of the following conditions holds:

- T is γ-strictly-increasing-continuous;
- (X, d) is γ-strictly-increasing-regular and $(\mathcal{A}_5^\gamma)$ holds;
- (X, d) is γ-strictly-increasing-regular and $\varrho(t, s) \leq s - t$ for all $t, s \in D \cap (0, \infty)$.

Then there exists a sequence $\{x_n\} \subseteq A_0$ converging to a best proximity point of T, such that $d(x_{n+1}, Tx_n) = d(A, B)$ for all $n \in \mathbb{N}$.

Proof. Define the binary relation S_γ on A by $x\mathcal{S}_\gamma y$ if $\gamma(x, y) \geq 1$, for all $x, y \in A$. Now from Theorem 2.1 for $\mathcal{S}_\gamma$, we get the statement.

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