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Power Graphs of Cyclic and Dihedral Groups: Structure and Extremal Parameters

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Abstract. The (undirected) power graph $\mathcal{P}(G)$ of a finite group G has vertex set G and an edge $\{x, y\}$ whenever one of x, y is a positive power of the other. Power graphs were introduced in directed form by Kelarev–Quinn and, in the undirected group setting, by Chakrabarty–Ghosh–Sen, and have since been studied widely.
In this paper, we give an explicit and computation-friendly treatment of $\mathcal{P}(C_n)$ and $\mathcal{P}(D_{2n})$ from a unified perspective. For the cyclic group C_n, we show that adjacency is governed by divisibility of element orders, identify $\mathcal{P}(C_n)$ as a blow-up of the comparability graph of the divisor lattice, and derive closed formulas for degrees and edge counts. Exact expressions are obtained for the clique number and chromatic number as the maximum totient-weight of a divisor chain, and for the independence number as the width of the divisor poset.
For the dihedral group D_{2n}, we establish a sharp structural decomposition: $\mathcal{P}(D_{2n})$ is obtained from $\mathcal{P}(C_n)$ by attaching n pendant leaves at the identity. This yields direct transfer principles for several invariants and, in particular, an exact formula for the independence number

$$\alpha(\mathcal{P}(D_{2n})) = n + W'(n),$$

where $W'(n)$ denotes the width of the divisor poset of n with the element 1 removed. We conclude with algorithmic remarks showing that the main parameters can be computed efficiently from the prime factorization of n.

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Key Words and Phrases: Power graph, cyclic group, dihedral group, divisor lattice, comparability graph, clique number, chromatic number, independence number

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1. Introduction

1.1. Background and motivation

Graphs canonically associated with algebraic objects often encode subtle structural information while remaining amenable to explicit computation. Classical examples include Cayley graphs, commuting graphs, and various subgroup-lattice graphs, each designed to translate algebraic data into combinatorial language. The *power graph* is another natural construction: it records the containment relations among cyclic subgroups by declaring two elements adjacent whenever one is a positive power of the other. Equivalently, edges reflect comparability of the cyclic subgroups generated by vertices. The directed version goes back to Kelarev–Quinn [1], while the undirected version for semigroups (and hence for groups) was formalized by Chakrabarty–Ghosh–Sen [2]. Since then, power graphs have been investigated from many perspectives, including isomorphism problems, metric invariants, extremal questions, and forbidden subgraph phenomena; see Cameron–Ghosh [3] and the surveys [4–8].

A recurring theme in this area is that power graphs lie at an interface between group theory and order theory. Indeed, the power relation is inherently "poset-like": if $x = y^m$ then $\langle x\rangle \subseteq \langle y\rangle$, and adjacency in the undirected power graph is precisely the symmetrization of this containment relation. Consequently, many graph-theoretic parameters (clique number, chromatic number, independence number, diameter, degree distribution) can often be expressed in terms of subgroup structure, element orders, and associated lattices. This makes explicit computations feasible for families of groups with well-controlled subgroup lattices, and it motivates the present focus on cyclic and dihedral groups, two fundamental and ubiquitous examples.

This paper concentrates on the families of cyclic groups C_n and dihedral groups D_{2n}, which serve as basic testing grounds for general phenomena. For cyclic groups, the subgroup structure is completely determined by divisors of n: for each $d \mid n$ there is a unique subgroup of order d, and the number of elements of order d is $\varphi(d)$ (see, e.g., [9, 10]). As a result, the power relation admits a particularly clean reformulation: adjacency can be described purely by divisibility among element orders. This reduces many questions about $\mathcal{P}(C_n)$ to combinatorial questions on the divisor poset (or divisor lattice) of n. In particular, one can view $\mathcal{P}(C_n)$ as a "blow-up" of a comparability graph arising from this poset, so that several invariants become accessible through standard tools from posets and perfect graph theory.

For dihedral groups D_{2n}, the geometry of the presentation (rotations and reflections) produces an even more transparent power-graph structure. The rotation subgroup $\langle r\rangle \cong C_n$ is normal and contributes a copy of the cyclic power graph. Reflections have order 2 and interact with the power relation in a rigid way, which leads to a decomposition of $\mathcal{P}(D_{2n})$ into a cyclic "core" together with a simple attachment of reflection vertices. This separation allows one to transfer a range of invariants from $\mathcal{P}(C_n)$ to $\mathcal{P}(D_{2n})$ almost mechanically and to obtain closed formulas with minimal additional work. From a broader viewpoint, the cyclic–dihedral case illustrates a general strategy: identify a structured subgroup (here the rotations) that already controls the main complexity, and treat the remaining elements via a clean graph-theoretic attachment.

1.2. Contributions

Our first goal is to present a unified and self-contained account of $\mathcal{P}(C_n)$ that emphasizes explicit formulas and a poset/comparability-graph perspective. In particular:

- we describe adjacency in $\mathcal{P}(C_n)$ purely by divisibility of element orders;
- we express $\omega(\mathcal{P}(C_n))$ and $\chi(\mathcal{P}(C_n))$ as a maximum *totient-weighted* chain sum in the divisor poset of n;
- we show $\alpha(\mathcal{P}(C_n))$ equals the width of the divisor poset (hence is computable via coefficients of $\prod_i(1 + x + \cdots + x^{a_i})$);
- we give explicit formulas for degrees and edge counts in terms of order strata.

A key technical point is giving a complete proof that $\chi = \omega$ by using perfectness: $\mathcal{P}(C_n)$ is a blow-up of a comparability graph, comparability graphs are perfect, and replication preserves perfectness [11, 12].

Our second goal is to treat dihedral groups D_{2n} by an exact decomposition theorem:

$$\mathcal{P}(D_{2n}) \cong \mathcal{P}(C_n) \text{ with } n \text{ pendant leaves attached at the identity.}$$

This yields clean transfer statements for degrees, cliques, chromatic number, triangles, and edges. It also gives an explicit independence formula

$$\alpha(\mathcal{P}(D_{2n})) = n + W'(n),$$

where $W'(n)$ is the width of the divisor poset with the minimum element 1 removed.

1.3. Organisation of the paper

The paper is organised as follows. In Section 2 we fix notation and record basic facts about power graphs and cyclic/dihedral groups. In Section 3 we summarise the main theorems and the invariants we compute. Detailed proofs for cyclic groups appear in Section 4, including a complete proof of $\chi(\mathcal{P}(C_n)) = \omega(\mathcal{P}(C_n))$. The dihedral case is handled in Section 5 via the structural decomposition and its consequences. Finally, Section 6 provides worked examples, Section 7 discusses computation from prime factorizations, and Section 8 lists concluding remarks and open directions.

2. Preliminaries and setup

We collect notation and basic facts used throughout.

2.1. Power graphs

Let G be a finite group with identity element e. The (undirected) *power graph* $\mathcal{P}(G)$ is the simple graph with vertex set $V(\mathcal{P}(G)) = G$, where two distinct vertices $x, y \in G$ are adjacent if and only if one is a positive power of the other, i.e.,

$$\{x, y\} \in E(\mathcal{P}(G)) \quad \Longleftrightarrow \quad (\exists\, m \in \mathbb{N} \text{ with } x = y^m) \;\text{ or }\; (\exists\, m \in \mathbb{N} \text{ with } y = x^m).$$

Equivalently, x and y are adjacent if and only if $x \in \langle y \rangle$ or $y \in \langle x \rangle$, which in turn is equivalent to comparability of the cyclic subgroups $\langle x \rangle$ and $\langle y \rangle$ under inclusion. We write $\langle x \rangle$ for the cyclic subgroup generated by x, and $\operatorname{ord}(x) = |\langle x \rangle|$ for the order of x.

Two basic consequences of the definition will be used repeatedly. First, for every $x \in G$ we have $x^{\operatorname{ord}(x)} = e$, so e is adjacent to every vertex of $\mathcal{P}(G)$ (except itself). Hence $\mathcal{P}(G)$ is connected. Second, since every vertex is at distance at most 1 from e, we immediately obtain

$$\operatorname{diam}(\mathcal{P}(G)) \leq 2, \qquad \operatorname{rad}(\mathcal{P}(G)) = 1 \;\text{ if } |G| \geq 2,$$

and $\operatorname{diam}(\mathcal{P}(G)) \in \{0, 1, 2\}$ depending on whether $|G| = 1$, $\mathcal{P}(G)$ is complete, or neither.

2.2. Cyclic groups and divisor data

For $n \geq 1$, we write C_n for the cyclic group of order n. We will frequently use the standard structure theorem for cyclic groups: for each divisor $d \mid n$ there exists a unique subgroup of C_n of order d, and the elements of order d are precisely the generators of that subgroup. In particular, the number of elements of order d equals Euler's totient function $\varphi(d)$; see [9, 10].

We also use the divisor poset of n. Let

$$\operatorname{Div}(n) := \{d \in \mathbb{N} : d \mid n\}$$

equipped with the partial order given by divisibility. If

$$n = \prod_{i=1}^{t} p_i^{a_i} \quad \text{with distinct primes } p_i \text{ and exponents } a_i \geq 1,$$

then every divisor $d \mid n$ corresponds uniquely to an exponent vector $(e_1, \ldots, e_t)$ with $0 \leq e_i \leq a_i$ via

$$d = \prod_{i=1}^{t} p_i^{e_i}.$$

Under this correspondence, divisibility becomes coordinatewise comparison:

$$\prod_i p_i^{e_i} \;\Big|\; \prod_i p_i^{f_i} \quad \Longleftrightarrow \quad e_i \leq f_i \;\text{ for all } i,$$

so $\operatorname{Div}(n)$ is (noncanonically) isomorphic to the product of chains $\prod_{i=1}^{t}\{0, 1, \ldots, a_i\}$.

2.3. Dihedral groups

For $n \geq 2$, we write D_{2n} for the dihedral group of order $2n$, presented by

$$D_{2n} = \langle r, s \mid r^n = e,\ s^2 = e,\ srs = r^{-1} \rangle.$$

The subgroup $\langle r \rangle$ is the rotation subgroup and is cyclic of order n, hence $\langle r \rangle \cong C_n$. Every element of D_{2n} can be written uniquely in the form r^k or sr^k for $0 \leq k < n$. The elements sr^k are the reflections; using the relation $srs = r^{-1}$ one checks

$$(sr^k)^2 = sr^k sr^k = s(r^k s)r^k = s(sr^{-k})r^k = e,$$

so each reflection has order 2. We will exploit the dichotomy between rotations and reflections to describe $\mathcal{P}(D_{2n})$ by a clean decomposition into a cyclic "core" and a simple attachment of reflection vertices.

3. Main results

We now summarise the main statements proved later.

For cyclic groups, Theorem 1 establishes a clean adjacency criterion in $\mathcal{P}(C_n)$: two vertices are adjacent if and only if their element orders are comparable under divisibility. This immediately yields a completeness criterion: $\mathcal{P}(C_n)$ is complete exactly when n is a prime power (Corollary 1), a condition already implicit in early work on power graphs [3, 4].

We then relate the main colouring and clique invariants of $\mathcal{P}(C_n)$ to a single divisor-chain optimisation. Define

$$\Lambda(n) := \max \left\{ \sum_{d \in \mathcal{C}} \varphi(d) :\ \mathcal{C} \subseteq \{d : d \mid n\} \text{ is a chain under divisibility} \right\}.$$

Using this, we prove that

$$\omega(\mathcal{P}(C_n)) = \Lambda(n) \quad \text{and} \quad \chi(\mathcal{P}(C_n)) = \Lambda(n),$$

see Theorems 2 and 3. The chromatic statement uses a complete perfectness argument (Lemmas 1 to 3).

For the independence number, Theorem 4 identifies $\alpha(\mathcal{P}(C_n))$ with the width $W(n)$ of the divisor poset. We also give a practical way to compute $W(n)$ from the prime factorisation of n using coefficients of a product polynomial (Theorem 4).

Turning to dihedral groups, Theorem 5 shows that $\mathcal{P}(D_{2n})$ is obtained from $\mathcal{P}(C_n)$ by attaching n pendant leaves at the identity. As a consequence, the clique number and chromatic number are unchanged from the cyclic situation (Corollary 2). Finally, writing $W'(n)$ for the width of the divisor poset with 1 removed, we prove

$$\alpha(\mathcal{P}(D_{2n})) = n + W'(n)$$

in Theorem 6.

4. Power graphs of cyclic groups

4.1. Adjacency and completeness

Theorem 1 (Adjacency in $\mathcal{P}(C_n)$)**.** *Let* $G = C_n = \langle g \rangle$*. For* $x, y \in G$ *with* $\mathrm{ord}(x) = d_x$ *and* $\mathrm{ord}(y) = d_y$*, the following are equivalent:*

(a) $\{x, y\}$ *is an edge of* $\mathcal{P}(G)$*;*

(b) $d_x \mid d_y$ *or* $d_y \mid d_x$*;*

(c) $\langle x \rangle \subseteq \langle y \rangle$ *or* $\langle y \rangle \subseteq \langle x \rangle$*.*

Proof. $(a) \Rightarrow (c)$: If $x = y^m$ for some $m \in \mathbb{N}$, then $x \in \langle y \rangle$, hence $\langle x \rangle \subseteq \langle y \rangle$. The case $y = x^m$ is symmetric.

$(c) \Rightarrow (b)$: If $\langle x \rangle \subseteq \langle y \rangle$, then $|\langle x \rangle| \mid |\langle y \rangle|$, i.e. $\mathrm{ord}(x) \mid \mathrm{ord}(y)$. The other containment is symmetric.

$(b) \Rightarrow (a)$: Assume $d_x \mid d_y$. In C_n there is a unique subgroup of every order dividing n [9, 10]. Let $H_y = \langle y \rangle$, so $|H_y| = d_y$. Since $d_x \mid d_y$, the subgroup H_y has a unique subgroup of order d_x, namely $\langle y^{d_y/d_x} \rangle$. But $\langle x \rangle$ is also a subgroup of order d_x, so by uniqueness

$$\langle x \rangle = \langle y^{d_y/d_x} \rangle \subseteq \langle y \rangle.$$

Hence $x \in \langle y \rangle$, so $x = y^m$ for some $m \in \mathbb{N}$, giving adjacency. The case $d_y \mid d_x$ is symmetric.

Corollary 1 (Completeness criterion)**.** $\mathcal{P}(C_n)$ *is complete if and only if* n *is a prime power.*

Proof. If $n = p^a$, then the divisors of n form a chain under divisibility, so any two non-identity elements have comparable orders and are adjacent by Theorem 1; since the identity is adjacent to all vertices, the graph is complete.

If n has two distinct prime divisors $p \neq q$, choose x of order p and y of order q. Then $p \nmid q$ and $q \nmid p$, so x, y are not adjacent by Theorem 1; hence the graph is not complete.

4.2. Clique number as a totient-weighted chain maximum

Definition 1 (Weighted chain functional)**.** *Define*

$$\Lambda(n) \ := \ \max\Big\{ \sum_{d \in \mathcal{C}} \varphi(d) \ : \ \mathcal{C} \subseteq \{d : d \mid n\} \text{ is a chain under divisibility} \Big\}.$$

Theorem 2 (Clique number)**.** *For* $G = C_n$*,*

$$\omega(\mathcal{P}(G)) = \Lambda(n).$$

Proof. Lower bound. Fix a divisor chain $\mathcal{C} = \{d_1 \mid d_2 \mid \cdots \mid d_k\}$. For each d_i, let $V_{d_i} = \{x \in C_n : \mathrm{ord}(x) = d_i\}$, so $|V_{d_i}| = \varphi(d_i)$. If $x \in V_{d_i}$ and $y \in V_{d_j}$ with $i \le j$, then $d_i \mid d_j$, hence x and y are adjacent by Theorem 1. Also, within a fixed V_{d_i}, any two elements generate the same unique subgroup of order d_i and are powers of each other. Thus $\bigcup_{i=1}^{k} V_{d_i}$ is a clique of size $\sum_{i=1}^{k} \varphi(d_i)$, so $\omega(\mathcal{P}(C_n)) \ge \Lambda(n)$.

Upper bound. Let K be a clique in $\mathcal{P}(C_n)$ and consider the set of orders appearing:

$$\mathcal{O}(K) = \{\mathrm{ord}(x) : x \in K\}.$$

If $\mathcal{O}(K)$ contained two incomparable divisors d, e, then choosing $x, y \in K$ with orders d, e would contradict Theorem 1. Hence $\mathcal{O}(K)$ is a chain. For each $d \in \mathcal{O}(K)$, at most $\varphi(d)$ vertices of order d can occur. Therefore

$$|K| \le \sum_{d \in \mathcal{O}(K)} \varphi(d) \le \Lambda(n),$$

so $\omega(\mathcal{P}(C_n)) \le \Lambda(n)$.

4.3. Chromatic number and perfectness (fully expanded)

Definition 2 (Comparability graph and replication)**.** *Let $(P, \preceq)$ be a poset. Its* comparability graph $\mathrm{Comp}(P)$ *has vertex set P and an edge between distinct $u, v \in P$ iff $u \preceq v$ or $v \preceq u$.*

Given a graph Γ and a vertex $v \in V(\Gamma)$, replicating *v means replacing v by a clique K_m (for some $m \ge 1$) such that every new vertex has exactly the same neighbours outside the clique as v had in Γ.*

Lemma 1 (Poset model for $\mathcal{P}(C_n)$)**.** *Let P be the set of positive divisors of n ordered by divisibility, and let $\Gamma_0 = \mathrm{Comp}(P)$. Then $\mathcal{P}(C_n)$ is obtained from Γ_0 by replicating each divisor-vertex $d \in P$ into a clique of size $\varphi(d)$.*

Proof. Partition C_n into order layers $V_d = \{x \in C_n : \mathrm{ord}(x) = d\}$ for $d \mid n$. Then $|V_d| = \varphi(d)$. By Theorem 1, if $x \in V_d$ and $y \in V_e$ with $d \ne e$, then x is adjacent to y iff $d \mid e$ or $e \mid d$, matching adjacency of d, e in $\mathrm{Comp}(P)$. Within each layer V_d, all vertices are pairwise adjacent, so each V_d is a clique. This is exactly the replication construction.

Lemma 2 (Comparability graphs are perfect)**.** *If P is a finite poset, then* $\mathrm{Comp}(P)$ *is a perfect graph.*

Proof. For any induced subposet $Q \subseteq P$, the induced subgraph $\mathrm{Comp}(P)[Q]$ is $\mathrm{Comp}(Q)$. A clique in $\mathrm{Comp}(Q)$ is exactly a chain in Q, so $\omega(\mathrm{Comp}(Q))$ equals the maximum chain size. A proper coloring of $\mathrm{Comp}(Q)$ partitions Q into antichains, and conversely any antichain partition gives a coloring. By Mirsky's theorem (dual Dilworth), the minimum number of antichains in a partition equals the maximum chain size. Hence $\chi(\mathrm{Comp}(Q)) = \omega(\mathrm{Comp}(Q))$ for every induced subgraph, i.e. $\mathrm{Comp}(P)$ is perfect; see [11, Ch. 7].

Lemma 3 (Replication preserves perfectness (Lovász)). *If Γ is perfect and Γ' is obtained from Γ by replicating a vertex, then Γ' is perfect.*

Proof. This is the replication lemma of Lovász [12]; see also [11, Ch. 6].

Theorem 3 (Chromatic number). *For $G = C_n$,*

$$\chi(\mathcal{P}(G)) = \omega(\mathcal{P}(G)) = \Lambda(n).$$

Proof. Let P be the divisor poset of n and $\Gamma_0 = \mathrm{Comp}(P)$. By Lemma 2, Γ_0 is perfect. By Lemma 1, $\mathcal{P}(C_n)$ is obtained from Γ_0 by a sequence of vertex replications. By Lemma 3, perfectness is preserved under each replication, hence $\mathcal{P}(C_n)$ is perfect. Therefore $\chi(\mathcal{P}(C_n)) = \omega(\mathcal{P}(C_n))$. Finally, Theorem 2 gives $\omega(\mathcal{P}(C_n)) = \Lambda(n)$.

4.4. Independence number and divisor poset width

Let $W(n)$ denote the width (maximum antichain size) of the divisor poset of n.

Theorem 4 (Independence number). *For $G = C_n$,*

$$\alpha(\mathcal{P}(G)) = W(n).$$

Moreover, if $n = \prod_{i=1}^{t} p_i^{a_i}$ then $W(n)$ equals the maximum coefficient of

$$F_n(x) = \prod_{i=1}^{t} (1 + x + \cdots + x^{a_i}),$$

a standard consequence of Sperner theory for products of chains [13, Ch. 3].

Proof. By Theorem 1, two distinct vertices $x, y \in C_n$ are adjacent iff $\mathrm{ord}(x)$ and $\mathrm{ord}(y)$ are comparable by divisibility; in particular, vertices of the same order are adjacent. Hence an independent set can contain at most one vertex from each order layer, and the set of orders appearing must be an antichain. Therefore $\alpha(\mathcal{P}(C_n)) \leq W(n)$.

Conversely, let $\mathcal{A}$ be an antichain of divisors of size $W(n)$. For each $d \in \mathcal{A}$ choose an element $x_d \in C_n$ of order d. If $d \neq e$ are in $\mathcal{A}$, then d and e are incomparable, so x_d and x_e are non-adjacent by Theorem 1. Thus $\{x_d : d \in \mathcal{A}\}$ is an independent set of size $W(n)$, proving $\alpha(\mathcal{P}(C_n)) \geq W(n)$.

4.5. Degrees and edges

Proposition 1 (Edge count in $\mathcal{P}(C_n)$).

$$|E(\mathcal{P}(C_n))| = \sum_{d|n} \binom{\varphi(d)}{2} + \sum_{\substack{d,e|n \\ d|e,\ d<e}} \varphi(d)\, \varphi(e).$$

Proof. For each $d \mid n$, the $\varphi(d)$ vertices of order d form a clique, contributing $\binom{\varphi(d)}{2}$ edges. If $d < e$ with $d \mid e$, then every vertex of order d is adjacent to every vertex of order e by Theorem 1, contributing $\varphi(d)\varphi(e)$ edges between the layers. Summing over all such pairs gives the formula.

Proposition 2 (Degrees in $\mathcal{P}(C_n)$)**.** *If $x \in C_n$ has order d, then*

$$\deg(x) = \Big(\sum_{e \mid d} \varphi(e)\Big) - 1 + \sum_{\substack{e \mid n \\ d \mid e,\, e \neq d}} \varphi(e) = (d-1) + \sum_{\substack{e \mid n \\ d \mid e,\, e \neq d}} \varphi(e).$$

The identity has degree $n-1$.

Proof. A vertex y is adjacent to x iff $\operatorname{ord}(y)$ divides $\operatorname{ord}(x)$ or is a multiple of it (Theorem 1). The subgroup $\langle x \rangle$ has size d, contributing exactly $d-1$ neighbours other than x itself. For each divisor $e \mid n$ with $d \mid e$ and $e \neq d$, every element of order e is adjacent to x, contributing $\varphi(e)$ neighbours. Summing gives the formula. For $x = e$, the identity is adjacent to every non-identity element, so $\deg(e) = n - 1$.

5. Power graphs of dihedral groups

5.1. Structure and immediate consequences

Theorem 5 (Structural decomposition of $\mathcal{P}(D_{2n})$)**.** *Let $G = D_{2n} = \langle r, s \mid r^n = s^2 = e,\, srs = r^{-1} \rangle$, with rotation subgroup $R = \langle r \rangle \cong C_n$ and reflections $\mathcal{R} = \{sr^k : 0 \leq k < n\}$. Then:*

(a) The induced subgraph on R is isomorphic to $\mathcal{P}(C_n)$.

(b) Each reflection has degree 1, adjacent only to the identity e.

(c) There are no edges between any reflection and any nonidentity rotation, nor between two distinct reflections.

Equivalently,

$$\mathcal{P}(D_{2n}) \cong \Big(\mathcal{P}(C_n)\Big) \text{ with } n \text{ pendant leaves attached at } e.$$

Proof. (a) Any power of a rotation is a rotation, and the power relation among rotations agrees with the power relation in $C_n \cong \langle r \rangle$, so the induced subgraph on R is $\mathcal{P}(C_n)$.

(b) If $x = sr^k$ is a reflection, then $x^2 = e$ and $\langle x \rangle = \{e, x\}$. Thus x is adjacent to e (since $e = x^2$) and to no other vertex, so $\deg(x) = 1$.

(c) If $y \in R \setminus \{e\}$ is a nonidentity rotation, then every positive power of y lies in R, so it cannot equal a reflection. Conversely, the only positive powers of a reflection are x and e, so a nonidentity rotation cannot be a power of a reflection. Distinct reflections cannot be powers of one another because the only possible values of x^m are e or x. Hence there are no such edges.

Corollary 2. *For all* $n \geq 2$,

$$\omega(\mathcal{P}(D_{2n})) = \omega(\mathcal{P}(C_n)) = \Lambda(n), \qquad \chi(\mathcal{P}(D_{2n})) = \chi(\mathcal{P}(C_n)) = \Lambda(n).$$

Moreover, $\operatorname{diam}(\mathcal{P}(D_{2n})) = 2$ *(and* $\operatorname{rad}(\mathcal{P}(D_{2n})) = 1$*).*

Proof. By Theorem 5, reflections have degree 1 and cannot appear in any clique of size ≥ 3, so maximum cliques lie in R and $\omega(\mathcal{P}(D_{2n})) = \omega(\mathcal{P}(C_n)) = \Lambda(n)$ by Theorem 2.

For chromatic number, color R optimally with $\Lambda(n)$ colors using Theorem 3. Each reflection is adjacent only to e, so each reflection may be colored with any color different from the color of e. Thus no extra colors are needed and $\chi(\mathcal{P}(D_{2n})) = \Lambda(n)$.

Since e is adjacent to all vertices, rad $= 1$ and diam ≤ 2. For $n \geq 2$ there exist two distinct reflections, which are not adjacent; their distance is 2 via e, hence diam $= 2$.

5.2. Independence number (expanded proof)

Let $W'(n)$ denote the width of the divisor poset of n with the minimum element 1 removed.

Theorem 6. *For* $G = D_{2n}$ *with* $n \geq 2$,

$$\alpha(\mathcal{P}(D_{2n})) = n + W'(n).$$

Proof. Lower bound. By Theorem 5, the set of all n reflections is independent, and reflections have no edges to nonidentity rotations. Let $\mathcal{A}$ be an antichain in the divisor poset with 1 removed, of size $W'(n)$. For each $d \in \mathcal{A}$ choose a rotation $x_d \in R$ of order d. Since $\mathcal{A}$ is an antichain, the chosen rotations are pairwise non-adjacent (by Theorem 1 inside $R \cong C_n$). Hence we obtain an independent set of size $n + W'(n)$.

Upper bound. Let S be any independent set. Since e is adjacent to all vertices, any maximum independent set omits e (for $n \geq 2$ there are independent sets of size 2). Then S splits into reflections and nonidentity rotations. The reflection part has size at most n. The rotation part can contain at most one element from each order layer, and the set of orders must be an antichain in the divisor poset with 1 removed. Hence the rotation part has size at most $W'(n)$. Therefore $|S| \leq n + W'(n)$, giving equality.

5.3. Edges, triangles, and degrees

Proposition 3. *For* $G = D_{2n}$,

$|E(\mathcal{P}(D_{2n}))| = |E(\mathcal{P}(C_n))| + n$, *and every triangle of* $\mathcal{P}(D_{2n})$ *lies in the rotation part* R.

Proof. By Theorem 5, each reflection contributes exactly one edge to e, and there are no other edges involving reflections. Thus the edge set is the disjoint union of the cyclic-part edges and these n leaf-edges. Since reflections have degree 1, no triangle can involve a reflection, so every triangle lies in R.

Proposition 4. *In $\mathcal{P}(D_{2n})$, the identity has degree $2n-1$, each reflection has degree 1, and each nonidentity rotation has the same degree as in $\mathcal{P}(C_n)$ given by Proposition 2.*

Proof. Immediate from Theorem 5.

6. Examples

Example 1. *Let $n = 12 = 2^2 \cdot 3$. Divisors and totients:*

$$1(1),\ 2(1),\ 3(2),\ 4(2),\ 6(2),\ 12(4).$$

A chain $1 \mid 3 \mid 6 \mid 12$ gives $\sum \varphi(d) = 1+2+2+4 = 9$, so

$$\omega(\mathcal{P}(C_{12})) = \chi(\mathcal{P}(C_{12})) = 9$$

by Theorems 2 and 3. An antichain of maximum size is $\{3,4,6\}$, hence $\alpha(\mathcal{P}(C_{12})) = 3$ by Theorem 4. For the dihedral group,

$$\alpha(\mathcal{P}(D_{24})) = 12 + W'(12) = 15$$

by Theorem 6.

7. Algorithmic notes: computing $\Lambda(n)$ and $W(n)$

Let $n = \prod_{i=1}^{t} p_i^{a_i}$.

- **Computing $W(n)$ and $W'(n)$.** The coefficients of

$$F_n(x) = \prod_{i=1}^{t} (1 + x + \cdots + x^{a_i})$$

can be computed by repeated convolution (dynamic programming). Then $W(n)$ is the maximum coefficient of $F_n(x)$. Since 1 is comparable with every divisor, no antichain of size ≥ 2 can contain 1, so for every $n \geq 2$ one has $W'(n) = W(n)$.

- **Computing $\Lambda(n)$.** View $\Lambda(n)$ as a maximum-weight chain problem in the divisor poset. Let $w(d) := \varphi(d)$. Define, for each divisor $d \mid n$,

$$\mathrm{DP}(d) := w(d) + \max\{\mathrm{DP}(e) : e \mid d,\ e < d\},$$

with the convention that the maximum over the empty set is 0 (so $\mathrm{DP}(1) = w(1) = 1$). Then $\Lambda(n) = \max_{d \mid n} \mathrm{DP}(d)$.

8. Conclusion and open problems

We presented an explicit treatment of power graphs for cyclic and dihedral groups, emphasizing (i) the divisor-poset structure behind $\mathcal{P}(C_n)$, and (ii) the pendant-leaf decomposition of $\mathcal{P}(D_{2n})$. This yielded exact formulas for ω, χ, α, degrees, and edge counts, and reduced several dihedral invariants to the cyclic case.

Open directions.

- Spectral parameters (eigenvalues, energy) of $\mathcal{P}(D_{2n})$ using the leaf-attachment decomposition.
- Automorphism groups $\mathrm{Aut}(\mathcal{P}(C_n))$ and $\mathrm{Aut}(\mathcal{P}(D_{2n}))$, in the spirit of [3, 14].
- More explicit descriptions (or closed forms) for optimal chains achieving $\Lambda(n)$ from the prime factorization of n.

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