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Total Least Squares Fitting the Three-Parameter Inverse Weibull Density

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Abstract. The focus of this paper is on a nonlinear weighted total least squares fitting problem for the three-parameter inverse Weibull density which is frequently employed as a model in reliability and lifetime studies. As a main result, a theorem on the existence of the total least squares estimator is obtained, as well as its generalization in the l_q norm $(1 \leq q < \infty)$.

2010 Mathematics Subject Classifications: 65D10, 62J02, 62G07, 62N05

Key Words and Phrases: inverse Weibull density, total least squares, total least squares estimate, existence problem, data fitting

1. Introduction

The probability density function of the random variable T having a three-parameter inverse Weibull distribution (IWD) with location parameter $\alpha \geq 0$, scale parameter $\eta > 0$ and shape parameter $\beta > 0$ is given by

$$f(t;\alpha,\beta,\eta) = \begin{cases} \frac{\beta}{\eta}\left(\frac{\eta}{t-\alpha}\right)^{\beta+1} \mathrm{e}^{-\left(\frac{\eta}{t-\alpha}\right)^{\beta}} & t > \alpha \\ 0 & t \leq \alpha. \end{cases} \tag{1}$$

If $\alpha = 0$, the resulting distribution is called the two-parameter inverse Weibull distribution. This model was developed by Erto [6].

The IWD is very flexible and by an appropriate choice of the shape parameter β the density curve can assume a wide variety of shapes (see Fig. 1). The density function is strictly increasing on $(\alpha, t_m]$ and strictly decreasing on $[t_m, \infty)$, where $t_m = \alpha + \eta(1 + 1/\beta)^{-1/\beta}$. This implies that the density function is unimodal with the maximum value at t_m. This is in contrast to the standard Weibull model where the shape is either decreasing (for $\beta \leq 1$) or unimodal (for $\beta > 1$). When $\beta = 1$, the IWD becomes an inverse exponential distribution;

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when $\beta = 2$, it is identical to the inverse Rayleigh distribution; when $\beta = 0.5$, it approximates the inverse Gamma distribution. That is the reason why the IWD is a frequently used model in reliability and lifetime studies (see e.g. Cohen and Whitten [5], Lawles [18], Murthy *et al.* [21], Nelson [22]).

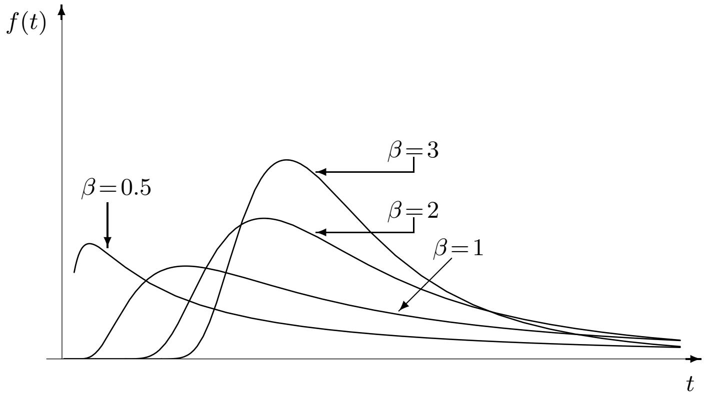


Figure 1: Plots of the inverse Weibull density for some values of β and by assuming $\alpha = 0$ and $\eta = 1.2$

In practice, the unknown parameters α, β and η of the three-parameter inverse Weibull density (1) are not known in advance and must be estimated from a random sample $t_1, \ldots, t_n$ consisting of n observations of the three-parameter inverse Weibull random variable T. There is no unique way to estimate the unknown parameters and many different methods have been proposed in the literature (see e.g. Abbasi *et al.* [1], Lawless [18], Marušić *et al.* [20], Murthy *et al.* [21], Nelson [22], Silverman [26], Smith and Naylor [27, 28], Tapia and Thompson [29]).

A very popular method for parameter estimation is the least squares method. The nonlinear weighted ordinary least squares (OLS) fitting problem for the three-parameter inverse Weibull density is considered by Marušić *et al.* [20]. In this paper we consider the nonlinear weighted total least squares (TLS) fitting problem for the three-parameter inverse Weibull density function. The structure of the paper is as follows. In Section 2 we briefly describe the TLS method and present our main result (Theorem 1) which guarantees the existence of the TLS estimator for the three-parametric inverse Weibull density. Its generalization in the l_q norm ($1 \leq q < \infty$) is given in Theorem 2. All proofs are given in Section 3.

2. The TLS Fitting Problem for the Three-parameter Inverse Weibull Density

Both the OLS and the TLS method require the initial nonparametric density estimates $\hat{f}$ which need to be as good as possible (see e.g. Silverman [26], Marušić *et al.* [20]). Suppose we are given the points (t_i, y_i), $i = 1, \ldots, n$, $n > 3$, where

$$0 < t_1 < t_2 < \ldots < t_n$$

are observations of the nonnegative three-parameter inverse Weibull random variable T and $y_i := \hat{f}(t_i)$ are the respective density estimates.

The goal of the OLS method (see e.g. [2, 3, 8, 11, 13, 14, 19, 25]) is to choose the unknown parameters of density function (1) such that the weighted sum of squared distances between the model and the data is as small as possible. To be more precise, let $w_i > 0$, $i = 1, \dots, n$, be the data weights which describe the assumed relative accuracy of the data. The unknown parameters α, β and η have to be estimated by minimizing the functional

$$S(\alpha,\beta,\eta) = \sum_{i=1}^{n} w_i [f(t_i;\alpha,\beta,\eta) - y_i]^2$$

on the set

$$\mathscr{P} := \{(\alpha,\beta,\eta) \in \mathbb{R}^3 : \alpha \geq 0; \beta,\eta > 0\}.$$

A point $(\alpha^\star,\beta^\star,\eta^\star) \in \mathscr{P}$ such that $S(\alpha^\star,\beta^\star,\eta^\star) = \inf_{(\alpha,\beta,\eta)\in\mathscr{P}} S(\alpha,\beta,\eta)$ is called the OLS estimator, if it exists. As we have already mentioned, this problem has been solved by Marušić *et al*. [20].

In the OLS approach the observations t_i of the independent variable are assumed to be exact and only the estimates y_i of the density (dependent variable) are subject to random errors. Unfortunately, this assumption does not seem to be very realistic in practice, and many errors (sampling errors, human errors, modeling errors and instrument errors) prevent us from knowing t_i exactly. In such situation, when also the observations of the independent variable contains errors, it seems reasonable to estimate the unknown parameters so that the weighted sum of squares of all errors is minimized. This approach, known as the *total least squares* (TLS) method, is a natural generalization of the OLS method (see e.g. [7]). In the statistics literature, the TLS approach is known as *errors-in-variables regression* or *orthogonal distance regression*, and in numerical analysis it was first considered by Golub and Van Loan [9].

The TLS method can be described as follows. Let $w_i, p_i > 0$, $i = 1, \dots, n$, be some weights. If we assume that y_i contains unknown additive error ε_i and that t_i has unknown additive error δ_i, then the mathematical model becomes

$$y_i = f(t_i + \delta_i;\alpha,\beta,\eta) + \varepsilon_i, \quad i = 1,\dots,n.$$

The unknown parameters α, β and η of density function (1) have to be estimated by minimizing the weighted sum of squares of all errors, i.e. by minimizing the functional (see e.g. [4, 7, 10, 17, 24])

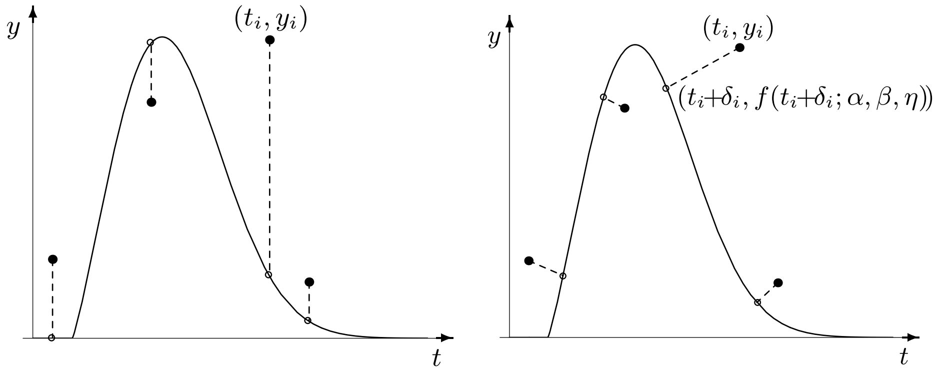
$$T(\alpha,\beta,\eta,\boldsymbol{\delta}) = \sum_{i=1}^{n} w_i [f(t_i+\delta_i;\alpha,\beta,\eta) - y_i]^2 + \sum_{i=1}^{n} p_i \delta_i^2 \tag{2}$$

on the set $\mathscr{P} \times \mathbb{R}^n$. A point $(\alpha^\star,\beta^\star,\eta^\star)$ in $\mathscr{P}$ is called the *total least squares estimator* (TLS estimator) of the unknown parameters (α,β,η) for the three-parameter inverse Weibull density, if there exists $\boldsymbol{\delta}^\star \in \mathbb{R}^n$ such that

$$T(\alpha^\star,\beta^\star,\eta^\star,\boldsymbol{\delta}^\star) = \inf_{(\alpha,\beta,\eta,\boldsymbol{\delta})\in\mathscr{P}\times\mathbb{R}^n} T(\alpha,\beta,\eta,\boldsymbol{\delta}).$$

Numerical methods for solving the nonlinear TLS problem are described in Boggs *et al*. [4] and Schwetlick and Tiller [24]. As in the case of the OLS approach, before the iterative minimization of the sum of squares it is still necessary to ask whether the TLS estimator exists. In the case of nonlinear TLS problems it is still extremely difficult to answer this question (see e.g. [3, 7, 12, 15–17]).

The difference between the OLS and the TLS approach is illustrated in Fig. 2. Geometrically, if $w_i = p_i$ for all $i = 1, \dots, n$, minimization of functional T corresponds to minimization of the weighted sum of squares of distances from data points to the model curve.



(a) OLS (b) TLS

Figure 2: The difference between the OLS and TLS approaches

Our main existence result for the TLS problem for the three-parameter inverse Weibull density is given in the next theorem.

Theorem 1. *Let the points* (t_i, y_i), $i = 1, \dots, n$, $n > 3$, *be given, such that* $0 < t_1 < t_2 < \dots < t_n$ *and* $y_i > 0$, $i = 1, \dots, n$. *Furthermore, let* $w_i, p_i > 0$, $i = 1, \dots, n$, *be some weights. Then there exists a point* $(\alpha^\star, \beta^\star, \eta^\star, \boldsymbol{\delta}^\star) \in \mathscr{P} \times \mathbb{R}^n$ *such that*

$$T(\alpha^\star, \beta^\star, \eta^\star, \boldsymbol{\delta}^\star) = \inf_{(\alpha,\beta,\eta,\boldsymbol{\delta}) \in \mathscr{P} \times \mathbb{R}^n} T(\alpha, \beta, \eta, \boldsymbol{\delta}),$$

i.e. the TLS estimator exists.

The proof is given in Section 3. The following total l_q norm ($q \geq 1$) generalization of Theorem 1 holds true.

Theorem 2. *Suppose* $1 \leq q < \infty$. *Let the points and weights be the same as in Theorem 1. Define*

$$T_q(\alpha, \beta, \eta, \boldsymbol{\delta}) := \sum_{i=1}^{n} w_i \left| f(t_i + \delta_i; \alpha, \beta, \eta) - y_i \right|^q + \sum_{i=1}^{n} p_i |\delta_i|^q. \tag{3}$$

Then there exists a point $(\alpha_q^\star, \beta_q^\star, \eta_q^\star, \boldsymbol{\delta}_q^\star) \in \mathscr{P} \times \mathbb{R}^n$ *such that*

$$T_q(\alpha_q^\star, \beta_q^\star, \eta_q^\star, \boldsymbol{\delta}_q^\star) = \inf_{(\alpha,\beta,\eta,\boldsymbol{\delta}) \in \mathscr{P} \times \mathbb{R}^n} T_q(\alpha, \beta, \eta, \boldsymbol{\delta}).$$

The proof of this theorem is omitted as it is similar to that of Theorem 1. It suffices to replace the l_2 norm with the l_q norm. Thereby all parts of the proof remain the same.

3. Proof of Theorem 1

Before starting the proof of Theorem 1, we need some preliminary results.

Lemma 1. *Suppose we are given data* (w_i, t_i, y_i), $i \in I := \{1,\dots,n\}$, $n > 3$, *such that* $0 < t_1 < t_2 < \ldots < t_n$ *and* $y_i > 0$, $i \in I$. *Let* $w_i, p_i > 0$, $i \in I$, *be some weights. Given any real number* q, $1 \le q < \infty$, *and any nonempty subset* I_0 *of* I, *let*

$$\Sigma_{I_0} := \sum_{i\in I\setminus I_0} w_i y_i^q + \sum_{i\in I_0} p_i |t_i - \tau_0|^q,$$

where

$$\tau_0 \in \operatorname*{argmin}_x \sum_{i=1}^n p_i |t_i - x|^q.$$

Then there exists a point in $\mathscr{P} \times \mathbb{R}^n$ *at which functional* T_q *defined by* (3) *attains a value less than* Σ_{I_0}.

Summation $\sum\limits_{i\in I_0}$ is to be understood as follows: The sum over those indices $i \le n$ for which $i \in I_0$. If there are no such indices, the sum is empty; following the usual convention, we define it to be zero. Summation $\sum\limits_{i\in I\setminus I_0}$ has similar meanings.

It is easy to verify that $t_1 \le \min_{i\in I_0} t_i \le \tau_0 \le \max_{i\in I_0} t_i \le t_n$. Note that for the case when $q = 2$, τ_0 is a well known weighted arithmetic mean, and for the case when $q = 1$, τ_0 is a weighted median of the data (see e.g. Sabo and Scitovski [23]).

Proof. Since τ_0 is an element of the closed interval $[t_1, t_n]$, there exists $r \in \{1,\dots,n\}$ such that

$$\tau_0 \in (t_{r-1}, t_r],$$

where $t_0 = 0$ by definition. Let us first choose real y_0 such that

$$0 < y_0 < \min_{i\in I} y_i \tag{4}$$

and then define functions $\alpha, \beta, \eta : (0,1) \to \mathbb{R}$ by:

$$\begin{aligned}
\beta(b) &:= \tau_0 y_0 \frac{\mathrm{e}^b}{b}\\
\eta(b) &:= \tau_0 b^{1/\beta(b)},\\
\alpha(b) &:= \tau_0 - \eta(b) b^{-1/2\beta(b)} = \eta(b)\big[b^{-1/\beta(b)} - b^{-1/2\beta(b)}\big].
\end{aligned}$$

Clearly, functions β and η are positive. Furthermore, by using the inequality $b^{-1/\beta(b)} - b^{-1/2\beta(b)} > 0$, which holds for every $b \in (0,1)$, it is easy to show that function α is also positive. Thus, we have showed that $(\alpha(b), \beta(b), \eta(b)) \in \mathscr{P}$ for all $b \in (0,1)$. Let us now associate with each real $b \in (0,1)$ a three-parametric inverse Weibull density function

$$f(t;\alpha(\beta),\beta(b),\eta(b)) = \begin{cases} \frac{\beta(b)}{t-\alpha(b)}\Big(\frac{\eta(b)}{t-\alpha(b)}\Big)^{\beta(b)} \mathrm{e}^{-\left(\frac{\eta(b)}{t-\alpha(b)}\right)^{\beta(b)}} & t > \alpha(b)\\ 0 & t \le \alpha(b). \end{cases} \tag{5}$$

This function has maximum at the point

$$\alpha(b)+\eta(b)(1+1/\beta(b))^{-1/\beta(b)}=\tau_0-\varepsilon(b),$$

where

$$\varepsilon(b):=\eta(b)\Big[b^{-1/2\beta(b)}-\Big(1+\frac{1}{\beta(b)}\Big)^{-1/\beta(b)}\Big].$$

It is strictly increasing on $(\alpha(b),\tau_0-\varepsilon(b)]$ and strictly decreasing on $[\tau_0-\varepsilon(b),\infty)$. Furthermore, by a straightforward calculation, it can be verified that

$$f(\tau_0+\alpha(b);\alpha(b),\beta(b),\eta(b))=y_0, \tag{6}$$

$$\lim_{b\to 0}\beta(b)=\infty, \tag{7}$$

$$\lim_{b\to 0}\eta(b)=\tau_0, \tag{8}$$

$$\lim_{b\to 0}\alpha(b)=0. \tag{9}$$

Now we are going to show that

$$\lim_{b\to 0} f(t;\alpha(b),\beta(b),\eta(b))=0, \quad t\neq\tau_0. \tag{10}$$

First, in view of (8) and (9), we obtain

$$\lim_{b\to 0}\Big(\frac{\eta(b)}{t-\alpha(b)}\Big)=\frac{\tau_0}{t}.$$

If $\tau_0<t$, then from (7) and (9) it follows readily that $\lim_{b\to 0}\mathrm{e}^{-\left(\frac{\eta(b)}{t-\alpha(b)}\right)^{\beta(b)}}=1$ and $\lim_{b\to 0}\beta(b)\left(\frac{\eta(b)}{t-\alpha(b)}\right)^{\beta(b)}=0$, and therefore

$$\lim_{b\to 0} f(t;\alpha(b),\beta(b),\eta(b))=\lim_{b\to 0}\Big[\frac{\beta(b)}{t-\alpha(b)}\Big(\frac{\eta(b)}{t-\alpha(b)}\Big)^{\beta(b)}\mathrm{e}^{-\left(\frac{\eta(b)}{t-\alpha(b)}\right)^{\beta(b)}}\Big]=0.$$

If $\tau_0>t$, then there exists a sufficiently great $k_0\in\mathbb{N}$ such that

$$\mathrm{e}<\Big(\frac{\eta(b)}{t-\alpha(b)}\Big)^{k_0}$$

for every sufficiently small $b>0$. Now, by using the inequality $x<\mathrm{e}^x$ $(x\geq 0)$ we obtain

$$\beta(b)<\mathrm{e}^{\beta(b)}<\Big(\frac{\eta(b)}{t-\alpha(b)}\Big)^{k_0\beta(b)}, \quad b\approx 0,$$

and therefore, for any $b\approx 0$ we have

$$0<f(t;\alpha(b),\beta(b),\eta(b))=\frac{\beta(b)}{t-\alpha(b)}\Big(\frac{\eta(b)}{t-\alpha(b)}\Big)^{\beta(b)}\mathrm{e}^{-\left(\frac{\eta(b)}{t-\alpha(b)}\right)^{\beta(b)}}$$

$$< \frac{1}{t-\alpha(b)}\Big(\frac{\eta(b)}{t-\alpha(b)}\Big)^{(k_0+1)\beta(b)} \mathrm{e}^{-\big(\frac{\eta(b)}{t-\alpha(b)}\big)^{\beta(b)}}.$$

Since

$$\lim_{b\to 0}\Big(\frac{\eta(b)}{t-\alpha(b)}\Big)^{(k_0+1)\beta(b)} \mathrm{e}^{-\big(\frac{\eta(b)}{t-\alpha(b)}\big)^{\beta(b)}} = 0,$$

then from the above-mentioned inequality it follows that

$$\lim_{b\to 0} f(t;\alpha(b),\beta(b),\eta(b)) = 0, \quad t > \tau_0.$$

Thus, we proved the desired limits (10).

Note that

$$\begin{aligned} f(\tau_0;\alpha(b),\beta(b),\eta(b)) &= \frac{\beta(b)}{\tau_0-\alpha(b)}\Big(\frac{\eta(b)}{\tau_0-\alpha(b)}\Big)^{\beta(b)} \mathrm{e}^{-\big(\frac{\eta(b)}{\tau_0-\alpha(b)}\big)^{\beta(b)}} \\ &= \frac{\beta(b)}{\tau_0-\alpha(b)}\sqrt{b}\,\mathrm{e}^{-\sqrt{b}} = \frac{\tau_0 y_0\,\mathrm{e}^{b-\sqrt{b}}}{(\tau_0-\alpha(b))\sqrt{b}}, \end{aligned}$$

from where taking the limit as $b \to 0$ it follows that

$$\lim_{b\to 0} f(\tau_0;\alpha(b),\beta(b),\eta(b)) = \infty. \tag{11}$$

Due to (9), (10) and (11), we may suppose that b is sufficiently small, so that

$$0 < \alpha(b) < t_1 \tag{12}$$

$$0 < f(t_i;\alpha(b),\beta(b),\eta(b)) < y_i, \quad \text{if } t_i \neq \tau_0 \tag{13}$$

$$f(\tau_0;\alpha(b),\beta(b),\eta(b)) > \max_{i\in I} y_i. \tag{14}$$

Let us now show that for each $i \in I_0$ and for every $b \in (0,1)$ there exists a unique number $\tau_i(b)$ such that (see Figure 3)

$$\begin{cases} t_i < \tau_i(b) < \tau_0 - \varepsilon(b) < \tau_0, & \text{if } t_i < \tau_0 \\ t_i < \tau_i(b) < t_i + \varepsilon(b), & \text{if } t_i = \tau_0 \\ \tau_0 < \tau_i(b) < t_i, & \text{if } t_i > \tau_0 \end{cases} \tag{15}$$

and

$$f(\tau_i(b);\alpha(b),\beta(b),\eta(b)) = y_i. \tag{16}$$

First, since the function $t \mapsto f(t_i;\alpha(b),\beta(b),\eta(b))$ has maximum at the point $\tau_0 - \varepsilon(b)$ and it is strictly increasing on $(\alpha(b), \tau_0 - \varepsilon(b)]$ and strictly decreasing on $[\tau_0 - \varepsilon(b), \infty)$, by using (4), (6), (13) and (14) we obtain

$$\begin{cases} f(t_i;\alpha(b),\beta(b),\eta(b)) < y_i < f(\tau_0-\varepsilon(b);\alpha(b),\beta(b),\eta(b)), & \text{if } t_i < \tau_0 \\ f(t_i;\alpha(b),\beta(b),\eta(b)) < y_i < f(\tau_0;\alpha(b),\beta(b),\eta(b)), & \text{if } t_i > \tau_0 \\ f(t_i+\varepsilon(b);\alpha(b),\beta(b),\eta(b)) < y_i < f(t_i;\alpha(b),\beta(b),\eta(b)), & \text{if } t_i = \tau_0. \end{cases}$$

The existence of the desired numbers $\tau_i(b)$, $i \in I_0$, follows from the well-known Intermediate Value Theorem which states that a continuous real function assumes all intermediate values on a closed interval, while uniqueness follows from monotonicity.

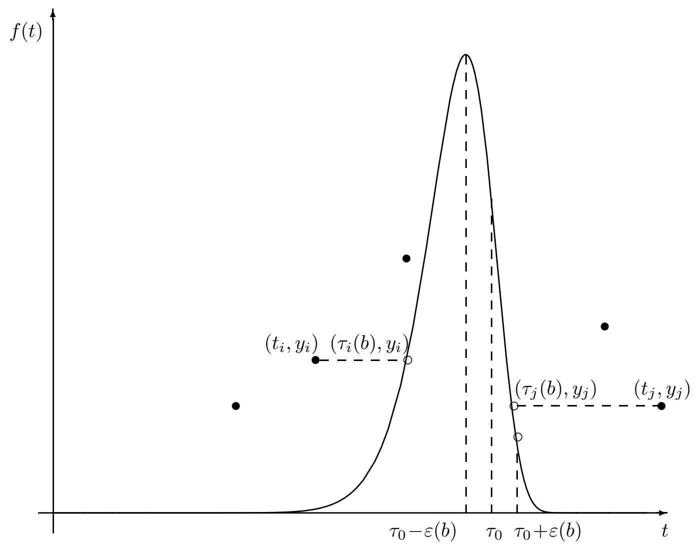


Figure 3: $i, j \in I_0$, $t_i < \tau_0$, $t_j > \tau_0$; $0 < \delta_i(b) = \tau_i(b) - t_i < \tau_0 - \varepsilon(b) - t_i < \tau_0 - t_i$; $\tau_0 - t_j < \tau_j(b) - t_j = \delta_j(b) < 0$

Setting

$$\delta_i(b) := \begin{cases} \tau_i(b) - t_i, & \text{if } i \in I_0 \\ 0, & \text{if } i \in I \backslash I_0 \end{cases}, \tag{17}$$

(16) becomes

$$f(t_i + \delta_i(b); \alpha(b), \beta(b), \eta(b)) = y_i, \quad i \in I_0. \tag{18}$$

Note that only one of the following two cases can occur:

(i) $|I_0| = 1$, or

(ii) $|I_0| > 1$.

Case (i): $|I_0| = 1$. In this case we have $\tau_0 = t_r$. It follows from (15) that $0 < \delta_r(b) < \varepsilon(b)$. Without loss of generality, in addition to (12)-(14) we may suppose that b is sufficiently small, so that

$$t_{r-1} + \varepsilon(b) < \frac{t_{r-1} + t_r}{2} < t_r - \varepsilon(b)$$

and

$$f((t_{r-1} + t_r)/2; \alpha(b), \beta(b), \eta(b)) < \min_{i \in I} y_i.$$

Due to these two additional assumptions and the fact that the function $t \mapsto f(t_i; \alpha(b), \beta(b), \eta(b))$ is strictly increasing on $(\alpha(b), t_r - \varepsilon(b)]$ and strictly decreasing on $[t_r - \varepsilon(b), \infty)$, we deduce:

If $t_i < t_r$, then

$$\begin{aligned} 0 <& f(t_i;\alpha(b)-\delta_r(b),\beta(b),\eta(b)) = f(t_i+\delta_r(b);\alpha(b),\beta(b),\eta(b)) \\ <& f(t_i+\varepsilon(b);\alpha(b),\beta(b),\eta(b)) \le f(t_{r-1}+\varepsilon(b);\alpha(b),\beta(b),\eta(b)) \\ <& f((t_{r-1}+t_r)/2;\alpha(b),\beta(b),\eta(b)) < \min_{i\in I} y_i \le y_i, \end{aligned} \tag{19}$$

whereas if $t_i > t_r$, then

$$\begin{aligned} 0 <& f(t_i;\alpha(b)-\delta_r(b),\beta(b),\eta(b)) = f(t_i+\delta_r(b);\alpha(b),\beta(b),\eta(b)) \\ <& f(t_i;\alpha(b),\beta(b),\eta(b)) < y_i. \end{aligned} \tag{20}$$

Thus, it follows from (18), (19) and (20) that, for every $b \in (0,1)$,

$$\begin{aligned} T_q(\alpha(b)-\delta_r(b),\beta(b),\eta(b),\mathbf{0}) =& \sum_{i=1}^{n} w_i \big| f(t_i;\alpha(b)-\delta_r(b),\beta(b),\eta(b)) - y_i \big|^q \\ <& \sum_{\substack{i=1 \\ i\neq r}}^{n} w_i y_i^q = \Sigma_{I_0} \end{aligned}$$

Case $|I_0| > 1$. Note that only one of the following two subcases can occur:

(i) $\tau_0 \neq t_i$ for all $i \in I_0$, or

(ii) $\tau_0 = t_r$ for some $r \in I_0$.

Subcase (i): In this subcase, it follows from (13), (15), (17) and (18) that, for every $b \in (0,1)$,

$$\begin{aligned} T_q(\alpha(b),\beta(b),\eta(b),\boldsymbol{\delta}(b)) =& \sum_{i\in I\setminus I_0} w_i \big| f(t_i;\alpha(b),\beta(b),\eta(b)) - y_i \big|^q + \sum_{i\in I_0} p_i |\delta_i(b)|^q \\ <& \sum_{i\in I\setminus I_0} w_i y_i^q + \sum_{i\in I_0} p_i |t_i - \tau_0|^q = \Sigma_{I_0}. \end{aligned}$$

Subcase (ii): Assume that $\tau_0 = t_r$ for some $r \in I_0$. Let index $s \in I_0$ be such that $t_s < \tau_0$. Then by (15), for every $b \in (0,1)$,

$$0 < \delta_r(b) < \varepsilon(b) \text{ and } 0 < \delta_s(b) < t_r - \varepsilon(b) - t_s$$

and therefore

$$p_s|\delta_s(b)|^q + p_r|\delta_r(b)|^q < p_s|t_r - \varepsilon(b) - t_s|^q + p_r\varepsilon^q(b).$$

It can be easily shown that the above right-hand side is less than

$$p_s|t_r - t_s|^q$$

whenever b is small enough. Therefore, for every small enough b we have

$$T_q(\alpha(b),\beta(b),\eta(b),\boldsymbol{\delta}(b)) = \sum_{i\in I\setminus I_0} w_i \big| f(t_i;\alpha(b),\beta(b),\eta(b)) - y_i \big|^q$$

$$+ p_r|\delta_r(b)|^q + p_s|\delta_s(b)|^q + \sum_{i\in I_0\setminus\{r,s\}} p_i|\delta_i(b)|^q$$
$$< \sum_{i\in I\setminus I_0} w_i y_i^q + \sum_{i\in I_0} p_i|t_i-\tau_0|^q = \Sigma_{I_0}.$$

This completes the proof of the lemma. □

Proof of Theorem 1.

Proof. Since functional T is nonnegative, there exists

$$T^\star := \inf_{(\alpha,\beta,\eta,\boldsymbol{\delta})\in\mathscr{P}\times\mathbb{R}^n} T(\alpha,\beta,\eta,\boldsymbol{\delta}).$$

To complete the proof it should be shown that there exists a point $(\alpha^\star,\beta^\star,\eta^\star,\boldsymbol{\delta}^\star)\in\mathscr{P}\times\mathbb{R}^n$ such that $T(\alpha^\star,\beta^\star,\eta^\star,\boldsymbol{\delta}^\star)=T^\star$.

Let $(\alpha_k,\beta_k,\eta_k,\boldsymbol{\delta}^k)$ be a sequence in $\mathscr{P}\times\mathbb{R}^n$, such that

$$T^\star = \lim_{k\to\infty} T(\alpha_k,\beta_k,\eta_k,\boldsymbol{\delta}^k) = \lim_{k\to\infty}\Big[\sum_{i\in I} w_i\big(f(t_i+\delta_i^k;\alpha_k,\beta_k,\eta_k)-y_i\big)^2 + \sum_{i\in I} p_i(\delta_i^k)^2\Big]$$
$$= \lim_{k\to\infty}\Big\{\sum_{t_i+\delta_i^k\le\alpha_k} w_i y_i^2 + \sum_{t_i+\delta_i^k>\alpha_k} w_i\Big[\frac{\beta_k}{\eta_k}\Big(\frac{\eta_k}{t_i+\delta_i^k-\alpha_k}\Big)^{\beta_k+1} \mathrm{e}^{-\left(\frac{\eta_k}{t_i+\delta_i^k-\alpha_k}\right)^{\beta_k}} - y_i\Big]^2$$
$$+ \sum_{i\in I} p_i(\delta_i^k)^2\Big\}. \tag{21}$$

where $I=\{1,\dots,n\}$. The summation $\sum\limits_{t_i+\delta_i^k\le\alpha_k}$ (or $\sum\limits_{t_i+\delta_i^k>\alpha_k}$) is to be understood as follows: The sum over those indices $i\le n$ for which $t_i+\delta_i^k\le\alpha_k$ (or $t_i+\delta_i^k>\alpha_k$). If there are no such points t_i, the sum is empty; following the usual convention, we define it to be zero.

There is no loss of generality in assuming that all sequences $(\alpha_k),(\beta_k),(\eta_k),(\delta_1^k),\dots,(\delta_n^k)$ are monotone. This is possible because the sequence $(\alpha_k,\beta_k,\eta_k,\delta_1^k,\dots,\delta_n^k)$ has a subsequence $(\alpha_{l_k},\beta_{l_k},\eta_{l_k},\delta_1^{l_k},\dots,\delta_n^{l_k})$, such that all its component sequences are monotone; and since $\lim_{k\to\infty} T(\alpha_{l_k},\beta_{l_k},\eta_{l_k},\boldsymbol{\delta}^{l_k}) = \lim_{k\to\infty} T(\alpha_k,\beta_k,\eta_k,\boldsymbol{\delta}^k)=T^\star$.

Since each monotone sequence of real numbers converges in the extended real number system $\bar{\mathbb{R}}$, define

$$\alpha^\star := \lim_{k\to\infty}\alpha_k,\quad \beta^\star := \lim_{k\to\infty}\beta_k,\quad \eta^\star := \lim_{k\to\infty}\eta_k,\quad \boldsymbol{\delta}^\star := \lim_{k\to\infty}\boldsymbol{\delta}^k = (\delta_1^\star,\dots,\delta_n^\star).$$

Note that $0\le\alpha^\star,\beta^\star,\eta^\star\le\infty$, because $(\alpha_k,\beta_k,\eta_k)\in\mathscr{P}$. Also note that $\delta_i^\star\in\mathbb{R}$ for each $i=1,\dots,n$. Indeed, if $|\delta_i^\star|=\infty$ for some i, then it would follow from (21) that $T^\star=\infty$, which is impossible.

To complete the proof it is enough to show that $(\alpha^\star,\beta^\star,\eta^\star)\in\mathscr{P}$, i.e. that $0\le\alpha^\star<\infty$ and $\beta^\star,\eta^\star\in(0,\infty)$. The continuity of the functional T will then imply that $T^\star=\lim_{k\to\infty}T(\alpha_k,\beta_k,\eta_k,\boldsymbol{\delta}^k)=T(\alpha^\star,\beta^\star,\eta^\star,\boldsymbol{\delta}^\star)$.

It remains to show that $(\alpha^\star, \beta^\star, \eta^\star) \in \mathscr{P}$. The proof will be done in five steps. In step 1 we will show that $\alpha^\star < t_n$. In step 2 we will show that $\beta^\star \neq 0$. The proof that $\eta^\star \neq \infty$ will be done in step 3. In step 4 we prove that $\eta^\star \neq 0$. Finally, in step 5 we show that $\beta^\star \neq \infty$.

Step 1. If $\alpha^\star \geq t_n$, from (21) it follows that $T^\star = \sum_{i=1}^n w_i y_i^2 + \sum_{i\in I} p_i {\delta_i^\star}^2$. Since according to Lemma 1 (for $q = 2$ and $I_0 = \{1\}$) there exists a point in $\mathscr{P} \times \mathbb{R}^n$ at which functional T attains a value smaller than Σ_{I_0} and since $\Sigma_{I_0} < \sum_{i=1}^n w_i y_i^2 + \sum_{i\in I} p_i {\delta_i^\star}^2$, this means that in this way ($\alpha^\star \geq t_n$) functional T cannot attain its infimum. Thus, we have proved that $\alpha^\star < t_n$.

Before continuing the proof, let us introduce some notation and make one remark. First let us define

$$I_0 := \begin{cases} I_{\alpha^\star}, & \text{if } I_{\alpha^\star} \neq \emptyset \\ \{1\}, & \text{otherwise} \end{cases}$$

where $I_{\alpha^\star} := \{i \in I : t_i + \delta_i^\star = \alpha^\star\}$. Let us note that Lemma 1 with $q = 2$ implies that

$$T^\star < \sum_{i\in I\setminus I_0} w_i y_i^2 + \sum_{i\in I_0} p_i (t_i - \tau_{I_0})^2 =: \Sigma_{I_0}, \tag{22}$$

where $\tau_{I_0} = \frac{\sum_{i\in I_0} p_i t_i}{\sum_{i\in I_0} p_i}$.

By taking an appropriate subsequence of $(\alpha_k, \beta_k, \eta_k, \boldsymbol{\delta}^k)$, if necessary, we may assume that if $t_i + \delta_i^\star < \alpha^\star$, then $t_i + \delta_i^k < \alpha_k$ for every $k \in \mathbb{N}$. Similarly, if $t_i + \delta_i^\star > \alpha^\star$, we may assume that $t_i + \delta_i^k > \alpha_k$ for every $k \in \mathbb{N}$. Due to this, now it is easy to show that from (21) it follows that

$$\begin{aligned} T^\star \geq & \sum_{t_i+\delta_i^\star<\alpha^\star} w_i y_i^2 + \lim_{k\to\infty} \Big\{ \sum_{t_i+\delta_i^\star>\alpha^\star} w_i \Big[\frac{\beta_k}{\eta_k} \Big(\frac{\eta_k}{t_i + \delta_i^k - \alpha_k} \Big)^{\beta_k+1} \mathrm{e}^{-\left(\frac{\eta_k}{t_i+\delta_i^k-\alpha_k}\right)^{\beta_k}} - y_i \Big]^2 \Big\} \\ & + \sum_{i\in I} p_i {\delta_i^\star}^2. \end{aligned} \tag{23}$$

Step 2. If $\beta^\star = 0$, then by using the inequality $x < \mathrm{e}^x$ $(x \geq 0)$ we obtain

$$0 < \frac{\beta_k}{\eta_k} \Big(\frac{\eta_k}{t_i + \delta_i^k - \alpha_k} \Big)^{\beta_k+1} \mathrm{e}^{-\left(\frac{\eta_k}{t_i+\delta_i^k-\alpha_k}\right)^{\beta_k}} < \frac{\beta_k}{t_i + \delta_i^k - \alpha_k}, \quad \text{if } \ t_i + \delta_i^\star > \alpha^\star,$$

wherefrom it follows readily that

$$\lim_{k\to\infty} \Big[\frac{\beta_k}{\eta_k} \Big(\frac{\eta_k}{t_i + \delta_i^k - \alpha_k} \Big)^{\beta_k+1} \mathrm{e}^{-\left(\frac{\eta_k}{t_i+\delta_i^k-\alpha_k}\right)^{\beta_k}} \Big] = 0, \quad \text{if } \ t_i + \delta_i^\star > \alpha^\star.$$

Now, from (23) it follows that

$$T^\star \geq \sum_{i\in I\setminus I_0} w_i y_i^2 + \sum_{i\in I_0} p_i {\delta_i^\star}^2$$

$$= \sum_{i\in I\setminus I_0} w_i y_i^2 + \sum_{i\in I_0} p_i(t_i - \alpha^\star)^2$$
$$\geq \sum_{i\in I\setminus I_0} w_i y_i^2 + \sum_{i\in I_0} p_i(t_i - \tau_{I_0})^2 = \Sigma_{I_0}, \tag{24}$$

which contradicts (22). Therefore, in this way ($\beta^\star = 0$) functional T cannot attain its infimum. Thus, we have proved that $\beta^\star \neq 0$.

The last inequality in (24) follows directly from a well-known fact that the quadratic function $x \mapsto \sum_{i\in I_0} p_i(t_i - x)^2$ attains its minimum $\sum_{i\in I_0} p_i(t_i - \tau_{I_0})^2$ at point τ_{I_0}.

Step 3. Let us show that $\eta^\star \neq \infty$. We prove this by contradiction. Suppose on the contrary that $\eta^\star = \infty$. Without loss of generality, we may then assume that if $t_i + \delta_i^\star > \alpha^\star$, then $\mathrm{e} < \frac{\eta_k}{t_i+\delta_i^k-\alpha_k}$ for all $k \in \mathbb{N}$. Then from the inequality $x < \mathrm{e}^x$ $(x \geq 0)$ it follows that if $t_i+\delta_i^\star > \alpha^\star$, then

$$\beta_k < \mathrm{e}^{\beta_k} < \Big(\frac{\eta_k}{t_i + \delta_i^k - \alpha_k}\Big)^{\beta_k}, \quad k \in \mathbb{N}.$$

Thus, if $t_i + \delta_i^\star > \alpha^\star$, then

$$0 < \frac{\beta_k}{\eta_k}\Big(\frac{\eta_k}{t_i + \delta_i^k - \alpha_k}\Big)^{\beta_k+1} \mathrm{e}^{-\big(\frac{\eta_k}{t_i+\delta_i^k-\alpha_k}\big)^{\beta_k}} = \frac{\beta_k}{t_i + \delta_i^k - \alpha_k}\Big(\frac{\eta_k}{t_i + \delta_i^k - \alpha_k}\Big)^{\beta_k} \mathrm{e}^{-\big(\frac{\eta_k}{t_i+\delta_i^k-\alpha_k}\big)^{\beta_k}}$$
$$< \frac{1}{t_i + \delta_i^k - \alpha_k}\Big(\frac{\eta_k}{t_i + \delta_i^k - \alpha_k}\Big)^{2\beta_k} \mathrm{e}^{-\big(\frac{\eta_k}{t_i+\delta_i^k-\alpha_k}\big)^{\beta_k}}. \tag{25}$$

Furthermore, since $\lim_{k\to\infty} \big(\frac{\eta_k}{t_i+\delta_i^k-\alpha_k}\big) = \infty$ and $\beta^\star \neq 0$, we have $\lim_{k\to\infty} \big(\frac{\eta_k}{t_i+\delta_i^k-\alpha_k}\big)^{\beta_k} = \infty$ and therefore $\lim_{k\to\infty} \big(\frac{\eta_k}{t_i+\delta_i^k-\alpha_k}\big)^{2\beta_k} \mathrm{e}^{-\big(\frac{\eta_k}{t_i+\delta_i^k-\alpha_k}\big)^{\beta_k}} = 0$, so that from (25) it follows that

$$\lim_{k\to\infty} \Big[\frac{\beta_k}{\eta_k}\Big(\frac{\eta_k}{t_i + \delta_i^k - \alpha_k}\Big)^{\beta_k+1} \mathrm{e}^{-\big(\frac{\eta_k}{t_i+\delta_i^k-\alpha_k}\big)^{\beta_k}}\Big] = 0, \text{ if } t_i + \delta_i^\star > \alpha^\star.$$

Putting the above limits into (23), we immediately obtain

$$T^\star \geq \sum_{i\in I\setminus I_0} w_i y_i^2 + \sum_{i\in I} p_i \delta_i^{\star 2} \geq \Sigma_{I_0},$$

which contradicts (22). Hence we proved that $\eta^\star \neq \infty$.

So far we have shown that $\alpha^\star < t_n$, $\beta^\star \neq 0$ and $\eta^\star \neq \infty$. By using this, in the next step we will show that $\eta^\star \neq 0$.

Step 4. Let us show that $\eta^\star \neq 0$. To see this, suppose on the contrary that $\eta^\star = 0$. Then only one of the following two cases can occur:

(i) $\eta^\star = 0$ and $\beta^\star \in (0, \infty)$, or

(ii) $\eta^\star = 0$ and $\beta^\star = \infty$.

Now, we are going to show that functional T cannot attain its infimum in either of these two cases, which will prove that $\eta^\star \neq 0$.
Case (i): $\eta^\star = 0$ and $\beta^\star \in (0, \infty)$. In this case we would have

$$\begin{aligned}
&\lim_{k\to\infty} \frac{\beta_k}{\eta_k}\Big(\frac{\eta_k}{t_i + \delta_i^k - \alpha_k}\Big)^{\beta_k+1} \mathrm{e}^{-\left(\frac{\eta_k}{t_i+\delta_i^k-\alpha_k}\right)^{\beta_k}} \\
&= \lim_{k\to\infty} \frac{\beta_k}{t_i + \delta_i^k - \alpha_k}\Big(\frac{\eta_k}{t_i + \delta_i^k - \alpha_k}\Big)^{\beta_k} \mathrm{e}^{-\left(\frac{\eta_k}{t_i+\delta_i^k-\alpha_k}\right)^{\beta_k}} \\
&= 0, \text{ if } t_i + \delta_i^\star > \alpha^\star
\end{aligned}$$

and hence from (23) it would follow that

$$T^\star \geq \sum_{t_i+\delta_i^\star\neq\alpha^\star} w_i y_i^2 + \sum_{i\in I} p_i {\delta_i^\star}^2 \geq \Sigma_{I_0}$$

which contradicts assumption (22).
Case (ii): $\eta^\star = 0$ and $\beta^\star = \infty$. Since $\eta_k \to 0$, there exists a real number $L > 1$ and sufficiently great $k_0 \in \mathbb{N}$ such that if $t_i + \delta_i^\star > \alpha^\star$ and $k > k_0$, then $\eta_k/(t_i + \delta_i^k - \alpha_k) < 1/L$. Without loss of generality, we may assume that $k_0 = 1$. Thus, if $t_i + \delta_i^\star > \alpha^\star$, then

$$\begin{aligned}
0 &< \frac{\beta_k}{\eta_k}\Big(\frac{\eta_k}{t_i + \delta_i^k - \alpha_k}\Big)^{\beta_k+1} \mathrm{e}^{-\left(\frac{\eta_k}{t_i+\delta_i^k-\alpha_k}\right)^{\beta_k}} = \frac{\beta_k}{t_i + \delta_i^k - \alpha_k}\Big(\frac{\eta_k}{t_i + \delta_i^k - \alpha_k}\Big)^{\beta_k} \mathrm{e}^{-\left(\frac{\eta_k}{t_i+\delta_i^k-\alpha_k}\right)^{\beta_k}} \\
&< \frac{1}{t_i + \delta_i^k - \alpha_k}\Big(\frac{\beta_k}{L^{\beta_k}}\Big) \mathrm{e}^{-\left(\frac{\eta_k}{t_i+\delta_i^k-\alpha_k}\right)^{\beta_k}}. \qquad (26)
\end{aligned}$$

Furthermore, since

$$\lim_{k\to\infty} \Big(\frac{\beta_k}{L^{\beta_k}}\Big) = 0 \text{ and } \lim_{k\to\infty} \mathrm{e}^{-\left(\frac{\eta_k}{t_i+\delta_i^k-\alpha_k}\right)^{\beta_k}} = 1,$$

from (26) it follows that

$$\lim_{k\to\infty} \Big[\frac{\beta_k}{\eta_k}\Big(\frac{\eta_k}{t_i + \delta_i^k - \alpha_k}\Big)^{\beta_k+1} \mathrm{e}^{-\left(\frac{\eta_k}{t_i+\delta_i^k-\alpha_k}\right)^{\beta_k}}\Big] = 0, \text{ if } t_i + \delta_i^\star > \alpha^\star.$$

Finally, from (23) we obtain $T^\star \geq \sum_{t_i+\delta_i^\star\neq\alpha^\star} w_i y_i^2 + \sum_{i\in I} p_i {\delta_i^\star}^2 \geq \Sigma_{I_0}$, which contradicts assumption (22). This means that in this case functional T cannot attain its infimum.

Thus, we have proved that $\eta^\star \neq 0$.

Step 5. It remains to show that $\beta^\star \neq \infty$. We prove this by contradiction. Suppose that $\beta^\star = \infty$. Arguing as in case (ii) from step 4, it can be shown that

$$\lim_{k\to\infty}\Big[\frac{\beta_k}{t_i+\delta_i^k-\alpha_k}\Big(\frac{\eta_k}{t_i+\delta_i^k-\alpha_k}\Big)^{\beta_k} e^{-\left(\frac{\eta_k}{t_i+\delta_i^k-\alpha_k}\right)^{\beta_k}}\Big]=0, \text{ if } 0<\frac{\eta^\star}{t_i+\delta_i^\star-\alpha^\star}<1. \tag{27}$$

If $\frac{\eta^\star}{t_i+\delta_i^\star-\alpha^\star} > 1$, then there exists a sufficiently great $k_0 \in \mathbb{N}$ such that $e < \big(\frac{\eta_k}{t_i+\delta_i^k-\alpha_k}\big)^{k_0}$. Now, by using the inequality $x < e^x$ $(x \geq 0)$ we obtain

$$\beta_k < e^{\beta_k} < \Big(\frac{\eta_k}{t_i+\delta_i^k-\alpha_k}\Big)^{k_0\beta_k}, \quad k\in\mathbb{N},$$

and therefore

$$\begin{aligned} 0 &< \frac{\beta_k}{t_i+\delta_i^k-\alpha_k}\Big(\frac{\eta_k}{t_i+\delta_i^k-\alpha_k}\Big)^{\beta_k} e^{-\left(\frac{\eta_k}{t_i+\delta_i^k-\alpha_k}\right)^{\beta_k}} \\ &< \frac{1}{t_i+\delta_i^k-\alpha_k}\Big(\frac{\eta_k}{t_i+\delta_i^k-\alpha_k}\Big)^{(k_0+1)\beta_k} e^{-\left(\frac{\eta_k}{t_i+\delta_i^k-\alpha_k}\right)^{\beta_k}}. \end{aligned} \tag{28}$$

Since $\lim_{k\to\infty}\big(\frac{\eta_k}{t_i+\delta_i^k-\alpha_k}\big)^{\beta_k} = \infty$, we have that

$$\lim_{k\to\infty}\Big(\frac{\eta_k}{t_i+\delta_i^k-\alpha_k}\Big)^{(k_0+1)\beta_k} e^{-\left(\frac{\eta_k}{t_i+\delta_i^k-\alpha_k}\right)^{\beta_k}} = 0$$

and therefore from (28) it follows that

$$\lim_{k\to\infty}\Big[\frac{\beta_k}{t_i+\delta_i^k-\alpha_k}\Big(\frac{\eta_k}{t_i+\delta_i^k-\alpha_k}\Big)^{\beta_k} e^{-\left(\frac{\eta_k}{t_i+\delta_i^k-\alpha_k}\right)^{\beta_k}}\Big]=0, \text{ if } \frac{\eta^\star}{t_i+\delta_i^\star-\alpha^\star}>1. \tag{29}$$

From (23), (27) and (29) we would obtain $T^\star \geq \sum_{t_i+\delta_i^\star\neq\alpha^\star} w_i y_i^2 + \sum_{i\in I} p_i {\delta_i^\star}^2 \geq \Sigma_{I_0}$, which contradicts (22). Thus, we have proved that $\beta^\star \neq \infty$ and completed the proof. □

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